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GRADATIONS IN EUCLID:

BOOKS I. AND II.

AN INTRODUCTION TO PLANE GEOMETRY, ITS USE AND APPLICATION;

WITH

An Explanatory Preface;

REMARKS ON GEOMETRICAL REASONING, AND ON ARITHMETIC AND ALGEBRA APPLIED TO GEOMETRY;

PRACTICAL RESULTS AND EXERCISES.

BY

HENRY GREEN, A.M.

FIFTH THOUSAND.

"GEOMETRY IS, PERHAPS, OF ALL THE PARTS OF MATHEMATICS, THAT WHICH OUGHT TO BE TAUGHT FIRST: IT APPEARS TO ME VERY PROPER TO INTEREST CHILDREN, PROVIDED IT IS PRESENTED TO THEM PRINCIPALLY IN RELATION TO ITS APPLICATIONS, WHETHER ON PAPER OR ON LAND."

Lacroix, p. 306.

JOHN HEYWOOD, EXCELSIOR BUILDINGS, JOHN DALTON STREET, MANCHESTER;
AND 18, PATERNOSTER SQUARE, LONDON, E.C.
AGENTS: SIMPKIN, MARSHALL, & Co.

——— "Geometrie,
Through which a man hath the sleight
Of length, of brede, of depth, of height."

GOWER, *Con. A.*, b. viii.

PREFACE,

RESPECTING THE GRADATIONS IN EUCLID'S PLANE GEOMETRY, SKELETON PROPOSITIONS, ETC.

THE chief aim of the Author or Compiler of the Gradations in Euclid, with Skeleton Propositions, &c., for Written Examinations, has been to furnish a *useful book* to those who have not much time for the pursuit of Geometrical Studies, and to whom, therefore, the Practical Application of whatever they learn is of great importance. He is, however, persuaded that those who have both time and full opportunity, either in Public Schools or in Colleges, for attaining proficiency in the Higher Mathematics, will find an Introduction, such as is given in this work, very suitable to prepare them more thoroughly to appreciate Geometrical Truths, and to take an interest in them as the ground-work of accurate science.

The INTRODUCTION is of general use to all Students of Geometry: it contains—a brief account of the Gradual Growth of Geometry and of the Elements of Euclid; the Signs and Contractions that may be employed; and some Remarks on the Nature of Geometrical Reasoning—on the Application of Arithmetic and Algebra to Geometry—on Incommensurable Quantities—and on Written and Oral Examinations. Several of the subjects treated of pre-suppose, indeed, that the Learner has a clear understanding of Fractions, common and decimal—of the extraction of the Square Root—and of the introductory principles of Algebra: but this knowledge is indispensable for those who would really master the Elements of Geometry.

The Editions of *Euclid* by POTTS and BLAKELOCK have shown the advantages of printing separately and distinctly the parts of a Proposition and of its Demonstration: it is a plan which undoubtedly gives very valuable help to Learners in attaining a more exact acquaintance with the Principles on which Geometry, as a science, is founded. No argument is here needed to prove the importance of being able to estimate the force

and certainty of demonstrable truths:—it is the first condition of success, and the sure means of proficiency in geometrical and in all other studies.

The GRADATIONS IN EUCLID endeavour to carry out the Plan to a greater extent, and with increased distinctness. The Propositions throughout are separated into successive steps; and in the margin, between the vertical lines, direct references are made to the Reasons—the Definitions, Axioms, or preceding Propositions—on which such Construction or Demonstration depends. The method of printing, which has been adopted, also gives a clearer view both of the whole Proposition and of its parts; and familiarises the mind to an orderly and systematic arrangement—so important an auxiliary to all sound progress. By following out a plan of this kind, Learners can scarcely fail to form a distinct conception of what they have to do or to prove, and of the means by which their purpose is to be accomplished.

The EXPLANATORY NOTES direct the Learner's attention to several points of interest connected with the Definitions and Propositions; and to many of the Propositions is appended an account of the chief PRACTICAL USES to which they may be applied. This is valuable for many reasons, but principally that the Learner may at once see, not simply the theoretical and abstract truths of Geometry, but their direct utility in various ways. There are very many persons who, from studying only the Common Editions of Euclid, which treat exclusively of the Theory of Geometry, never attain to a perception of its importance, and never realise the full advantages of geometrical studies. Indeed, they have read and demonstrated the whole work without perceiving any value in their laborious pursuit except as an exercise of the memory and mental powers in a very dry course of reasoning. It is the *main object* of our Gradations in Euclid *to combine Theory and Practice*, and, as soon as a geometrical truth has been established, *to point out its use and application*. The Author is thoroughly persuaded that this *immediate* Combination of Theory and its Application, not only awakens and maintains a livelier interest, but, in fact, leads to a more scholar-like understanding of both, than when they are studied separately, or at wide intervals of time.

The Uses to which the Propositions may be applied are very numerous; and we have given only a *portion* of them—more in the way of example, and to point out in some instances the progress of geometrical discovery,

than with the view of exhausting the subject. The various works on Practical Mathematics will supply what may be wanting in this respect. For the full developement of the Uses and Applications of the First and Second Books of Euclid, geometrical principles not worked out in these books must occasionally be introduced; and though it is not strictly logical to employ truths that have not really been established, as the ground-work of further reasoning—now and then, in this part of the work, *Lemmas*, or truths borrowed from another part of the subject, will be adopted as the foundation of new truths.

The PRACTICAL RESULTS will, it is presumed, be instructive to the Learner in various ways, but especially as exhibiting a synoptical view of all the Problems contained in Euclid's Elements of Plane Geometry He will not, indeed, have arrived at the means of demonstrating the Problems which lie out of the First and Second Books; but, inasmuch as their construction depends almost entirely on these two books, he will possess, for *practical purposes*, a knowledge of the methods by which, the regular Geometrical Figures, not being sections of a Cone, are to be constructed.

The APPENDIX, containing GEOMETRICAL ANALYSIS and GEOMETRICAL EXERCISES, appeared to the Writer needed for the completion of his plan—*i. e.*, for comprising a systematic teaching of Geometry, as far as the First and Second Books furnish the means of doing it. The Appendix, and a Key to the Exercises, will each be published separately from the Gradations.

The SKELETON PROPOSITIONS, &c., *for pen-and-ink examinations*, are arranged and will be published in two Series—one with the references in the margin—the other without those references. The *first* Series is intended for beginners; the *second*, for those who may be reasonably supposed to be prepared for a strict examination. The two Series will be found well adapted to test the Progress of the Learner, and to ascertain how far his knowledge of geometrical principles, and his power to apply them, really extend. The object is, in the *first* Series, to furnish the Learner, step by step, with the truths from which other truths are to be evolved, but to leave him to work out the results, and from the results, as they arise, to aim at more advanced conclusions: in the *second* Series, where there are no references supplied in the margin, the object is to make the examinations strict and thorough, yet so as to be conducted on one uniform plan. This uniformity will be found greatly to assist Examiners'

when they compare the examination papers together for the purpose of deciding on their respective merits.

The Skeleton Propositions may be used either simultaneously with the Gradations in Euclid, or after the First and Second Books have been read in any of the usual editions of SIMSON'S *Euclid*, as a recapitulation of the ground already gone over: if used *simultaneously*, the Learner must first study the Definitions and Propositions in their order, and then, laying the printed book aside, reduce his knowledge to a written form, as the references indicate which are given in the vertical columns of the Skeleton Propositions; but if used as a *recapitulatory exercise*, a course in some respects different is recommended.

In the *recapitulatory exercise*, the following plan is recommended for adoption:—*first*, that the Learner should give in writing a statement of the meaning of various Geometrical Terms, of the nature of Geometrical Reasoning, and of the application of Algebra and Arithmetic to Geometry; *secondly*, that he should fill in—*not by copying from any book, but from the stores of his own mind and thought*, trained by previous study of the GRADATIONS in Euclid, or of some similar work—the Definitions, Postulates, and Axioms of which the leading words are printed; and *thirdly*, that he should proceed to take the Propositions in order, and write out the proofs at large, as the printed forms and references in the margin indicate: this should be done systematically in all the Propositions, beginning with those truths already established which are required for the Construction and Demonstration, and then taking in order the Exposition, the Data and Quæsita, or the Hypothesis and Conclusion, the Construction with its methods, and the Demonstration with its proofs, separated from each other, and given, step by step, in regular progression.

For the *thorough Examinations*, that Series of the Skeleton Propositions must be used which contains the General Enunciation only, without any references printed in the margin. The Spaces for the Exposition, Construction, and Demonstration are retained, and also the vertical lines within which the Learners themselves are to place the references; but this is done simply for the purpose of securing a uniformity of plan in the written examinations, and for the convenience of Examiners. The Student under examination should also be required to write out the Propositions, &c.,* needed in the Construction and Demonstration, and to supply the

* This is required because the repetition of truths and principles gains for them a more permanent residence in the mind.

references to the various geometrical truths by which the steps of the Proposition are established.

No figures, or diagrams, are given in either of the Series of Skeleton Propositions; as it is more conducive to the Learner's sound progress that he be left entirely to himself to construct these.

The Uses and Applications of the Propositions, at least *in a brief way*, —and where requisite, the Algebraical and Arithmetical Illustrations,— should not be neglected: it is in these that the practical advantage of abstract truths is rendered apparent.

It is imperative that the Teacher should revise each Proposition after it has been written out, and note the misapprehensions and inaccuracies before the Learner proceeds to the following Proposition. In Self-Tuition, the Learner must consult the Gradations, and by them correct the already filled-up Skeleton; but he must be faithful to himself, and to his own improvement, by not consulting the Gradations *as a Key*, until he has first worked out and written down his own conception of what the Demonstration demands. He will thus build up *for* himself and *of* himself; he will make the dead bones of the Skeleton Propositions live, clothe them with flesh and sinews, and round them off in all their proper proportions.

A course of this kind followed faithfully through two books of the Elements of Geometry, will scarcely fail to render the Student competent by himself to master the other books of Euclid, and, should he desire it, by the same means. He will have learned the value of method and exactness; and, expert in these, he will attain a solid and durable knowledge of geometrical principles.

At the present day nearly every edition of Euclid's Elements must be, more or less, a compilation, in which the Author draws freely on the labours of his predecessors. The Gradations are, in a great degree, of this character; and an open acknowledgment will suffice, once for all, to repel any charge of intentionally claiming what belongs to others. It is affectation to pretend to great originality on a subject which has, like Geometry, for so many centuries exercised men's minds. If by the methods employed in the following pages, the Study of Geometry among all classes of the community be rendered more interesting and more practically useful, the objects of the Editor will be accomplished. He desires no worthier calling than to be a fellow-labourer with the many excellent

and talented Masters to whom the responsibility is entrusted of training the young in sound learning.

In commendation of the study of Geometry, Principal Hill, of Harvard University, in his *First Lessons* on the subject, p. 8, declares "Geometry is the most useful of all the Sciences. To understand Geometry will be a great help in learning all other Sciences; and no other Science can be learned unless you know something of Geometry. To study it will make your eye quicker in seeing things, and your hand steadier in doing things. You can draw better, write better, cut out clothes, make boots and shoes, work at any mechanical trade, or learn any art the better for understanding Geometry."

The principal editions of Euclid to which the Editor is under obligation, are those of POTTS and LARDNER, and of an old Writer who professes to give "the uses of each Proposition in all the parts of the Mathematicks." An exemplification and recommendation of the plan pursued in the Gradations, and in the Skeleton Propositions, may be found in the preface to LARDNER'S *Euclid*, and in a *Treatise on the Study and Difficulties of Mathematics*, p. 74, attributed to Professor DE MORGAN. He is also indebted to various persons—Schoolmasters and others—for valuable suggestions, which he takes this opportunity to acknowledge.

The work is longer than the Author at first contemplated; but he trusts that the additions—especially the Practical Results and the Exercises—will add considerably to its usefulness and value.

INTRODUCTION.

SECTION I.

GRADUAL GROWTH OF GEOMETRY AND OF THE ELEMENTS OF EUCLID.

GEOMETRY, *land-measuring*, as the word denotes (from *gee*, earth or land, and *metron*, a measure), was in its origin an Art, and not a Science: it embraced a system of rules, more or less complete, for performing the simpler operations of land-surveying; but these rules rested on no regularly-demonstrated principles,—they were the offspring rather of experiment and individual skill, than of scientific research. In the same way poems—even some of the noblest—were composed before the principles of poetry had been collected into a system; languages were spoken, long before a grammar had been compiled; and men reasoned and debated before they possessed either a logic or a rhetoric: so measurements were made, while as yet there was no accurate theory of measuring—no abstract speculations concerning space and its properties.

The points and lines of such a Geometry were necessarily visible quantities. A mark, which men could see, would be their point; a measuring-rod, or string, which they could handle, their line; a wall, or a hedge, or a mound of earth, their boundary. The first advance beyond this would be to identify the instruments which they used in measuring, with the lines and boundaries themselves: the finger's breadth, the cubit, the foot, and the pace, would become representatives of a certain length without reference to the shape. It was only as the ideas and perceptions of those who cultivated the art of measuring grew more refined and subtile, that an Abstract Geometry would be evolved, such as Mathematicians understand by the term, in which a point marks only position; a line, extension from point to point; and a surface, space enclosed by mathematical lines.

Geometry, thus understood, has been defined in general terms to be—"the Science of Space," or "the Science of Form." It

investigates the properties of lines, surfaces, and solids, and the relations which exist between them. *Plane Geometry* investigates the properties of space under the two aspects of length and breadth; *Solid Geometry*, under the three—of length, breadth, and thickness. It is the consideration of the Elements of Plane Geometry on which we are about to enter.

The Truths of Geometry, as a science, regularly as they are laid down and deduced in the Elements of Euclid, were not worked out by one mind, nor established in any systematic order. Some were discovered in one age, some in another; two or three propositions by one philosopher, and two or three by some one else. The collection of geometrical truths had thus a gradual growth, until it received comparative completion at the hands of Euclid of Alexandria.

Thales, who predicted the eclipse that happened B.C. 609, is said to have brought Geometry from Egypt, and to have established by demonstration Propositions 5, 15, and 26, of bk. i.; 31, iii.; and 2, 3, 4, and 5, of bk. iv. Pythagoras, born about 570 B.C., was the first who gave to Geometry a scientific form, and discovered Propositions 32 and 47 of bk. i.; Oenopides, a follower of Pythagoras, added the 12th and 23rd of bk. i.; and Eudoxas, B.C. 366, a friend of Plato, wrote the doctrine of proportion as developed in the fifth book of the Elements. These assertions may not rest on the firmest authority, yet they show, even if they are only surmises, that Geometry was regarded by the Greeks as a science of very gradual formation, receiving accessions from age to age, and from various countries. It was at first a set of rules, until philosophy investigated the principles on which the rules were founded, and out of the chaos created knowledge.

According to Proclus, EUCLID of Alexandria flourished in the reign of the first Ptolemy, B.C. 323-283. To him belongs the glory, for such it is, of having collected into a well-arranged system the scattered principles and truths of Geometry, and of having produced a work, which, after standing the test of above twenty centuries, seems destined to remain the Standard Geometry for ages to come.

Euclid's work comprises thirteen books, of which the first four and the sixth treat of Plane Geometry; the fifth, of the Theory of Proportion, applicable to magnitude in general; the seventh, eighth, and ninth are on Arithmetic; the tenth, on the Arithmetical

Characteristics of the divisions of a straight line; the eleventh and twelfth, on the Elements of Solids; and the thirteenth, on the Regular Solids. To the thirteen books by Euclid, Hypsicles of Alexandria, about A.D. 170, added the fourteenth and fifteenth books—also on the Regular or Platonic Solids.

In modern times it is not usual to read more than six books of Euclid's Elements. The seventh, eighth, ninth, and tenth books treat of Arithmetic and of the Doctrine of Incommensurables, and have no proper connection with the first six books; and the eleventh and twelfth books, comprehending the First Principles of Solid Geometry, are to a considerable degree superseded by other Treatises.

Of the Six Books, the *first* may be described in general terms as treating of the Geometry of Plane Triangles; the *second*, of Rectangles upon the parts into which a straight line may be divided; the *third* book, of those Properties of the Circle which can be deduced from the preceding books; the *fourth* book, of such regular and straight-lined figures as can be described in or about a circle; the *fifth*, of Proportion with regard to magnitude in general; and the *sixth*, of similar figures, and of Proportion as applied to Geometry.

Our proposed limits confine us, for the present at least, to the First and Second Books. The First Book, besides the Definitions, Postulates, and Axioms, contains forty-eight Propositions, of which fourteen are problems for giving power to construct various lines, angles, and figures; and thirty-four are theorems, being the expositions of new geometrical truths. Of these theorems, some may be regarded as simply subsidiary to the proof of others that are more important, and of wider and more general application. The Propositions to be ranked among those of high importance, are Props. 4, 8, and 26, containing the criteria of the equality of triangles; Prop. 32, proving that the three interior angles of every triangle are together equal to two right angles; Prop. 41, declaring that a parallelogram on the same base and of the same altitude as a triangle, is double of the triangle; Prop. 47, demonstrating that the square on the hypotenuse of a right-angled triangle, is equal to the sum of the squares on the base and perpendicular. It is on these Propositions of the first book—namely, 4, 8, and 26, 32, 41, and 47—that Geometry in its after applications mainly depends, and, therefore, *must they* be most thoroughly understood and mastered.

The Second Book treats of the properties of RIGHT-ANGLED PARALLELOGRAMS, contained by the parts of divided straight lines.

There are fourteen Propositions, of which Props. 11 and 14 are problems—the other twelve are theorems. Props. 12 and 13 give the Elements of Trigonometrical Analysis, or the Arithmetic of Sines, and are of great use in the Higher Geometry: the other Propositions may be classified according to the mode of dividing the line or lines; Prop. 1 relating to the rectangles formed by one undivided line and the parts of a divided line; Props. 2, 3, 4, 7, and 8, to the rectangles formed by a line and any two parts into which it may be divided; Props. 5 and 9, to the rectangles on a line divided equally and unequally; and Props. 6 and 10, to the rectangles formed on a line bisected and produced.

The English Translation of Euclid, published by Dr. Robert Simson, of Glasgow, in 1756, has nearly, in some form or other, superseded all others, and is considered the standard text of an English Euclid. As containing "the Elements of Geometry," it is "unexceptionable, but is not calculated to give the scholar a proper idea of the Elements of Euclid," as Euclid himself left them. Various alterations, additions, and improvements were made by Simson; but, "with the exception of the editorial fancy about the perfect restoration of Euclid, there is little to object to in this celebrated edition. It might, indeed, have been expected that some notice would have been taken of various points on which Euclid has evidently fallen short, of that formality of rigour which is tacitly claimed for him. We prefer," says De Morgan, "this edition very much to many which have been fashioned upon it—particularly to those which have introduced algebraical symbols into the demonstrations in such a manner as to confuse geometrical demonstration with algebraical demonstration."—(See the Article *Eucleides*, by De Morgan, in *Smith's Dictionary of Greek and Roman Biography*, Vol. II., pp. 63-74.)

In the face of such authority, it may seem bold to advocate the use of a Symbolical Notation; but, within certain limitations, the symbols of Arithmetic and of Algebra have a universal meaning, and may therefore be employed without any disadvantage,* and

* See Lacroix' *Essais*, ed. 1838, p. 227. "L'histoire des Mathématiques prouve aussi que c'est l'usage de plus en plus étendu des symboles arbitraires imaginés dans la vue d'abréger les expressions ou de mettre en évidence leur analogie, qui a contribué le plus à l'avancement de la science; en soulageant la mémoire et facilitant les combinaisons des relations données et des raisonnemens."

certainly without confusion in our ideas. The precaution needed is, that we take care not to depart from the strictly geometrical application.

For an outline of the origin and progress of the science of Geometry, the learner should consult the *Introduction to the Elements of Euclid*, edited by Robert Potts, M.A., Trinity College, Cambridge.

SECTION II.

SYMBOLICAL NOTATION AND ABBREVIATIONS THAT *MAY* BE USED.

Signs in Geometry possess an accuracy equal to that of words, and far greater clearness than the undivided paragraphs of *Simson*, or even than the demonstrations by *Potts*, so carefully broken up into clauses. Cambridge may not approve the use of certain symbols in Geometry which once she allowed, but *Lacroix*, in a passage just quoted, testifies to their advantage: thus fortified, we make no further apology for using them, especially after declaring their meanings.

One strong recommendation of them has thus been pointed out: "It is quite possible, and, in fact, frequently happens, that a boy gets up his Euclid lesson *by rote*, from the ordinary editions now in use;" but it is almost, if not quite, impossible to bring the memory simply into play when a proposition of Euclid is set forth in symbolical language: the reasoning powers must then be exercised, or the work will not be accomplished.

I.—*Arbitrary Signs common to Arithmetic, Algebra, and Geometry.*

∵	because.	+	*plus*, more, increased by.
∴	therefore.	−	*minus*, less, lessened by.
=	equals, *or* equal.	~	difference between.
≠	not equal.	:	ratio.
>	greater than.	:=:	equality of ratios.
≯	not greater than.	: :: :	proportion.
<	less than.	: : :	progression.
≮	not less than.		

II.—*Representative, or Geometrical Signs.*

· a point.	⊥ perpendicular to, or at right angles.
| straight line.	▱ parallelogram.
‖ parallel to.	□ square.
‖s parallels.*	▭ rectangle.
∠ angle.	⊙ circle.
△ triangle.	⊙ce circumference.

A *single* capital letter, in reference to a diagram, as A, or B, denotes the point A, or the point B.†

Two capital letters, also in reference to a diagram, as A B, or C D, denote the straight line A B, or C D, or the side of a triangle, or other figure.

Two capital letters, with the figure ^{2} just above to the right hand, as $A\ B^2$, denote, not the square of A B, but the square on the straight line A B.

Capital letters, with a point between them, as A B . C D, denote, not the product of A B multiplied by C D, but the rectangle formed by two of its sides meeting in a common point.

III.—*Abbreviations.*

Add......*Addendo*, by adding.	E. or Exp. .Exposition, or Particular enunciation.
App. .....Application.	Gen.....General enunciation.
Ax. .....Axiom.	H. or Hyp..Hypothesis.
Conc. .....Conclusion, inference.	H. 1 &c. ...Step 1 &c. of the Hyp.
Cor. .....Corollary.	Prob. .....Problem.
C. or Cons. .Construction.	Pr. or Prs. Proposition, or Propositions.
C. 1 &c. ...Step 1 &c. of the Construction.	Pst. or Psts.Postulate, or Postulates
Dat......*Datum*, or *data*.	*Quæs.* .....*Quæsitum*, or *Quæsita*.
Def. .....Definition.	Rec. .....Recapitulation.
D. or Dem. Demonstration.	Remk. .....Remark to be made.
D. 1 &c. ...Step 1 &c. of the Dem.	

* When an *s* is added to the sign, the plural is denoted.

† N.B. A *single* capital letter may also denote an angle, when the sign ∠, or word *angle* is used. Also, a parellelogram may be named by naming the single letters at the opposite angles.

Sch. .....*Scholium*, or *Scholia.*
Sim. .....Similarly.
S. or Sol. ...Solution.
Sub. .....*Subtrahendo*, by taking away.
Sup. .....Suppose.
Superp. ...*Superponendo*, by superposition.
Theor. .....Theorem.

Q.E.D., *quod erat demonstrandum*, which was the thing to be proved.
Q.E.F., *quod erat faciendum*, which was the thing to be done.

ad imposs. ad impossibile, reduced to an impossibility.
a fort. .....*a fortiori*, by a stronger argument or reason.
alt. .....altitude.
altr. .....alternate.
assum. .....*assumendo*, by assuming, adopting, or taking.
bis. .....bisect.
bisd.....bisected.
bisg.....bisecting.
com.....common.
con. sup. ...contrary supposition.
c. sc.....circumscribe.
d. sc. .....describe.
eq. ang. ...equiangular.
eq. lat. .....equilateral.
ex. abs. .....*ex absurdo*, by an absurdity.
ext. .....exterior.
fig. .....figure.
int. .....interior.
magn. .....magnitude.
opp. .....opposite.
par. ..... parallel.
parlm. .....parallelogram.
pos. .....position.
qu. ang. ...quadrangular.
qu. lat.....quadrilateral.
rad. .....radius.
rect. .....rectangle.
rectil. .....rectilineal
rectang. ...rectangular.
rem. .....remaining.
rt..... .....right.
sq. .....square.
st. .....straight.
uneq. .....unequal.
vert.. .....vertex.

SECTION III.

EXPLANATION OF SOME GEOMETRICAL TERMS.

A *Definition* (from *definire*, to set bounds to) is a short description of a thing by such of its properties as serve to distinguish it from all other things of the same kind.

A *Postulate* (postulatum, *a thing demanded*) is a self-evident problem, the admission of which is demanded without formal proof.

An *Axiom* (axioma, *a thing of worth*) is a self-evident theorem, or the assertion of a truth, which does not need demonstration: it is worthy of credit as soon as stated.

A *Proposition* (proponere, *to set forth*) is something proposed to be done, as a problem; or to be proved, as a theorem.

A *Problem* (probleema, *a thing proposed*) is a proposal to do a thing, to construct a figure, or to solve a question.

A *Theorem* (theoreema, *a subject of contemplation*) is the assertion of a geometrical truth, and requires demonstration.

The *Data* (datum, *a thing granted*) are the things granted in a problem;

The *Quæsita* (quæsitum, *a thing sought*) are the things sought for in it;

The *Hypothesis* (hypothesis, *a supposition*) is the supposition made in a theorem;

The *Conclusion* (concludere, *to infer*) is the consequence or inference deduced from it.

The *General Enunciation* (enunciare, *to speak out, declare*) of a proposition sets forth in general terms the conditions of the problem, or theorem, with what has to be done, or with what is inferred or concluded.

A *Diagram* (diagramma, *a drawing of lines*) is the drawing which represents a geometrical figure.

The *Exposition* (exponere, *to set forth*), or *Particular Enunciation*, sets forth the same conditions with an especial reference to a figure that has been drawn.

The *Solution* (solutio, *an unloosening, an explaining*) of a problem shows how the thing proposed may be done.

The *Construction* (constructio, *a putting together*) prepares, by the drawing of lines, &c., for the demonstration of a proposition.

The *Demonstration* (demonstrare, *to point out*) proves that the process indicated in the solution is sound, or that the conclusion deduced from an hypothesis is true—*i.e.*, in accordance with geometrical principles.

The *Recapitulation* (recapitulare, *to go over the main points again*), or *Conclusion*, is simply the repetition of the proposition, or general enunciation, as a fact, or as a truth, with the declaration Q.E.F., or Q.E.D.

A *Corollary* (corolla, *a little wreath, a deduction*) is an inference made immediately from a proposition.

A *Scholium* (scholion, *a comment*) is a note or explanatory observation.

A *Lemma* (leema, *a thing taken*) is a preparatory proposition borrowed from another part of the same subject, and introduced for the purpose of establishing a more important proposition.

The *Converse* (conversum, *a thing turned round*) of a proposition is when the hypothesis of a former proposition becomes the conclusion, or predicate, of the latter proposition; as in Props. 5 and 6, 18 and 19, 21 and 25, bk. i.

The *Contrary* of a proposition is when that which the proposition assumes is denied.

Direct Demonstration is when the very thing asserted is proved to be true.

Indirect Demonstration is when all other cases, or conditions, except the one in question, are proved not to be true, and the inference is made: therefore, the very thing in question must be true; the assumption being that one out of several, or many, must be right.

The *Position* only of a line is meant, when the line is said to be given.

The *Length* only of a line is meant, when the line is said to be finite.

The *Base* of a figure (basis, *a foundation*) is the side on which it appears to stand; but each side, in turn, with the position of the figure changed, may become the base.

The *Vertex* (vertex, *the top or crown of the head*) is the highest angular point of a figure: with a change of position in the figure, each angle may be named the vertical angle.

The *Subtend* (subtendere, *to stretch under*) of an angle is the side stretching across opposite to the angle.

The *Hypotenuse* (hupotenousa, *that which stretches under*) is the subtend to a right angle.

The *Perpendicular* (perpendiculum, *a plumb-line*) is the line forming with the base a right angle: lines are perpendicular to each other when at the point of junction they form a right angle.

A *Figure* is *applied* to a straight line when the line forms one of its boundaries.

The *Altitude* of a figure (altitudo, *height*) is the perpendicular distance from the side or angle opposite to the base, to the base itself, or to the base produced.

A *Diagonal* (diagonios, *from corner to corner*) is a straight line joining two opposite angular points.

The *Complement* of an angle (complementum, *that which fills up*) is what is wanted to make an acute angle equal to a right angle, or to 90°.

The *Supplement* of an angle (supplementum, *a filling up*) is what is wanted to make an angle equal to two right angles, or to 180°.

The *Explement* of an angle (explementum, *a filling*) is what is wanted to make an angle equal to four right angles, or to 360°.

The *Complements of a Parallelogram* (see Fig. to Definition A.), when the parallelogram is bisected by its diagonal, and subsidiary parallelograms are formed by two lines—one, parallel to one side, and the other, parallel to the other side, and both intersecting the diagonal,—the *complements of the parallelogram* are those subsidiary parallelograms through which the diagonal does not pass; and these, with the subsidiary parallelograms through which the diagonal does pass, fill up or complete the whole parallelogram.

The *Area* of a figure (area, *an open space*) is the quantity of surface contained in it, reckoned in square units, as square inches, square feet, &c.

A *locus* (locus, *place*) in Plane Geometry is a straight line, or a plane curve, every point of which, and none else, satisfies a certain condition.

SECTION IV.

NATURE OF GEOMETRICAL REASONING.

In Mechanics, Chemistry, and the kindred sciences we are so accustomed to experimental proof, and find it so very useful, that we readily suppose that experiment may be applied to Geometry equally well. Thus we draw a triangle, and at some

little distance from the angular points we cut off the angles; then, by experiment, we place the three angles together so that two of the sides of different angles may form a straight line, and with the third angle we fill up the unoccupied space between the other two: it will thus appear that the three angles which we cut off from the original triangle are together equal to two right angles. But the *appearance* is not a perfect and absolute proof. "Mathematical proofs," says *Locke*, vol. iv., p. 428, "like diamonds, are hard as well as clear, and will be touched with nothing but strict reasoning."

Now the Demonstrations in Euclid's Elements of Geometry consist of strict arguments or reasonings by means of which the assertions made in the propositions are proved to be absolutely true. Thus, in the 15th Prop., bk. i., the assertion is made that "the opposite, or vertical angles, formed by two intersecting lines, are equal;" and the demonstration shows by strict argument or reasoning, founded upon truths already admitted or proved, that the assertion must be received as true,—we cannot disbelieve it, if we would.

When fully stated, each argument contains both the thing which is proved, and the means by which the proof is established; and, as in the arrangement of the parts of an argument the means of proof usually precede the thing proved, they are named the *premisses* (premisse, *a thing which precedes or goes before*); and the thing proved is named the *conclusion*, or *inference*. Thus, in Prop. 1, bk. i., the premisses are—1st, things equal to the same thing are equal to each other; 2nd, the line A C, and also the line B C, are each equal to the same line A B; and 3rd, the inference, conclusion, or thing proved, is, that the line A C equals the line B C, or, adhering more strictly to the forms of reasoning, the two lines A C and B C are equal to each other.

Here in the premisses two things are laid down, or granted to be true: as—"things equal to the same thing are equal to each other,"—this is *one truth;* "the line A C equals the line A B, and the line B C also equals the same line A B,"—this is *another truth;* and from the two things thus declared to be true, there is made in the conclusion the necessary and unavoidable inference,—therefore, A C equals B C—*i. e.*, the two lines A C and B C are equal to each other.

The subject of the conclusion, "the two lines A C and B C," is is called the *minor* term; the predicate of the conclusion (*predi-*

catum, thing declared) "are equal to each other," is called the *major* term.

There are three things in the premisses,—the major and the minor terms, and the third term with which as with a standard the major and the minor terms are compared. This third term, being the *medium* of the comparison, is named the *middle* term : it enters into the major premiss as the subject, and into the minor premiss as the predicate. In the example given, "things equal to the same," is the *middle* term, being the subject of the major premiss—"things equal to the same are equal to each other," and the predicate of the minor premiss—"the lines A C and B C are each equal to the same line A B."

The premiss in which the major term—*i. e.*, the predicate of the conclusion—appears, is called the *major* premiss ; that in which the minor term, or subject of the conclusion, appears, is the *minor* premiss. Thus, in the argument already given :—

Major premiss,—because things equal to the same are equal to each other ;

Minor premiss,—and because the two lines A C and B C are each equal to the same line A B ;

Conclusion,—therefore, the two lines A C and B C are equal to each other.

Here, "*the two lines A C, B C*" is the subject, and "*equal to each other*" the predicate of the conclusion ;

"*The two lines A C, B C*" is the subject, and "*each equal to the same line A B,*" the predicate of the minor premiss ;

"*Things equal to the same line A B*" the subject, and "*equal to each other*" the predicate of the major premiss.

Thus, "*equal to each other*" is the *major* term ; and "*the two lines A C, B C,*" *the minor* term ; and "*things equal to the same line A B,*" the *middle* term of the argument.

In this mode of reasoning, it is seen that assertions are broadly made ; and we may ask, on what *evidence* are these assertions themselves to be received as true ?

The *first* kind of evidence is from the *definition* of the thing : thus, we define a triangle to be a figure bounded by three sides ; and if, of any figure placed before us, we can affirm that it has three sides exactly, the conclusion is inevitable, that this figure also is a triangle.

The *second* kind of evidence is from the *axioms*, or truths so plain that they need no proof: for example—we receive as undeniable, that, if equals be added to equals, the wholes are equal; and we argue, if to the line A D, or to its equal the line B C, we add another line E F, then the whole line made up of A D + E F, will equal the whole line made up of B C + E F.

The *third* kind of evidence is from the *hypothesis*, or supposition, which we make as the condition of our assertion: we declare, "in an isosceles triangle, the angles at the base are equal;" the very words, though not in the exact form of an hypothesis, directly imply the supposition, "if a triangle is isosceles," "then the angles at the base are equal." An isosceles triangle is here taken as the starting point of the reasoning; and though, for the demonstration of the inference, "the angles at the base are equal," it is necessary to draw various lines which are not mentioned in the hypothesis, the conclusion at which we arrive is altogether dependent on the hypothesis.

The *fourth* kind of evidence is *from proof already given;* for what has once been established, may afterwards be taken for granted. For instance, when we have once established the truth, that "the interior angles of every triangle are together equal to two right angles," and we afterwards come to a proposition in the demonstration of which we need this established truth, we do not again go through all the steps by which the equality of the sum of the interior angles of a triangle to two right angles has been proved; but, without going down again to the bottom of the ladder before we make a step higher, we start from the step we had already gained, and at once take up our position on a more advanced truth.

But the *foundation*, the *Principle of Geometrical Reasoning* is, hat from two propositions established or received as true, a third proposition, or inference, shall be made. Now, that this may be done, there must be something in common contained in both the propositions, with which common thing, the other two things are compared: we say, for example—

All the triangle is in the circle,
All the square is in the triangle;
therefore, All the square is in the circle:

the common term of comparison here is "the triangle," and our inference is correct.

But if we say—

All the triangle is in the circle,
All the square is in the circle,
and infer, All the square is in the triangle,—

this may be, or may not be true; for it may happen that only a part of the square is in the triangle. The fault of the apparent argument is, there is no proper term of comparison—no middle term which is at the same time the subject of the major premiss, and the predicate of the minor premiss. To have the argument sound, we must say—

Major P. All the triangle is in the circle,
Minor P. All the square is in the triangle;
therefore, All the square is in the circle.

When a connection is thus declared to exist between the premisses and the conclusion—that is, when reasons are stated and an inference made—this mode of argument receives the name of a *Syllogism* (syllogismos, *a collection*); for a Syllogism is a bringing together or collecting into one view the two steps of the reasoning on which a truth depends, and the truth itself; or as Whately, in his *Elements of Logic*, p. 52, defines a Syllogism, it is "an argument so expressed, that the conclusiveness of it is manifest from *the mere force of the expression, i.e.*, without considering the *meaning of the terms: e. g.*, in this Syllogism :—

"Every Y is X, or, Every man is mortal;
,, Z is Y, John is a man;
therefore, Z is X, John is mortal:

"the conclusion is inevitable, whatever terms, X, Y, and Z, respectively, are understood to stand for. And to this form all legitimate Arguments may ultimately be brought."

Every Syllogism is made up of three propositions, or assertions; of which, two are named the *premisses*, and the third, the *conclusion*. The Proposition which contains the *predicate* of the conclusion, is called the *major* premiss; and that which contains the *subject* of the conclusion, the *minor* premiss. The premisses are usually introduced by the word *because*, or by some similar word, and the conclusion by the word *therefore*.

The Syllogism is exhibited in *four forms*, or *figures*, distinguished from each other by the *situation* of the *Middle Term*, or Standard

of Comparison, with respect to the *extremes* of the conclusion—that is, with respect to the major or minor terms. "The *proper order* is to place the Major premiss *first*, and the Minor, *second;* but this does not *constitute* the Major and Minor premisses; for that premiss (wherever placed) is the Major which *contains the major term*, and the Minor, *the minor*."—(*Elements of Logic*, p. 57.) The major term, as we have before said, is the predicate, and the minor term the subject of the conclusion.

Taking X to represent the Major term, Z the Minor term, and Y the Middle term, we may now exhibit the four forms of the Syllogism, of which four forms one or the other is used in all legitimate reasoning.

I. The *First Form*, or figure, of the Syllogism, which is also the clearest and most natural, makes the Middle term the *subject of the major premiss*, and the *predicate of the minor*. Thus, in Prop. 1, bk. i., of Euclid's Elements :—

Major P.	Because Y = X,	or, because the line A B is equal to the line A C,
Minor P.	and Z = Y;	and the line B C is equal to the line A B;
Concl.	therefore, Z = X.	therefore, the line B C is equal to the line A C.

Here we may observe that B C, or Z, is the subject of the conclusion, and A C, or X, the predicate;

B C, or Z, the subject, and A B, or Y, the predicate of the minor premiss;

A B, or Y, the subject, and A C, or X, the predicate of the major premiss;

A C, or X, the major term; B C, or Z, the minor term; and A B, or Y, the middle term, or the standard of comparison.

II. In the *Second Form* of the Syllogism, the Middle term is the *predicate* of *both* premisses; as in Prop. 47, bk. i., of the Elements :—

Major P.	Because X = Y,	or, because the angle D B C is a right angle,
Minor P.	and Z = Y;	and the angle F B A is a right angle;
Concl.	therefore, Z = X.	therefore, the angle F B A = the angle D B C.

Here $\angle$ F B A, or Z, is the subject, and $\angle$ D B C, or X, the predicate of the conclusion;

∠ F B A, or Z, is the subject, and rt. angle, or Y, the predicate of the minor premiss;
∠ D B C, or X, is the subject, and rt. angle, or Y, the predicate of the major premiss;
∠ D B C, or X, is the major term; ∠ F B A, or Z, the minor term; and *right angle*, the middle term.

III. In the *Third Form* of the Syllogism, the Middle term is the *subject* of *both* premisses; as in Prop. 28, bk. i., of the Elements :—

Major P.	Because Y = X,	or, because	∠ E G B is equal to ∠ G H D,
Minor P.	and Y = Z;	and	∠ E G B is equal to ∠ A G H;
Concl.	therefore, Z = X.	therefore,	∠ A G H is equal to ∠ G H D.

Here Z, or ∠ A G H, is the subject, and X, or ∠ G H D, the predicate of the conclusion;
Y, or ∠ E G B, is the subject, and Z, or ∠ A G H, the predicate of the minor premiss;
Y, or ∠ E G B, is the subject, and X, or ∠ G H D, the predicate of the major premiss;
X, or ∠ G H D, represents the major term; Z, or ∠ A G H, the minor term, and Y, or ∠ E G B, the middle term.

IV. The *Fourth Form* of the Syllogism is the reverse of the first, and is the most unnatural of all: it places the Middle term the *predicate of the major premiss*, and the *subject of the minor*. Thus, in Prop. 26, case 2, bk. i., of the Elements :—

Major P.	Because X = Y,	or, because	∠ B H A is equal to ∠ E F D,
Minor P.	and Y = Z;	and	∠ E F D is equal to ∠ B C A;
Concl.	therefore, Z = X.	therefore,	∠ B C A is equal to ∠ B H A.

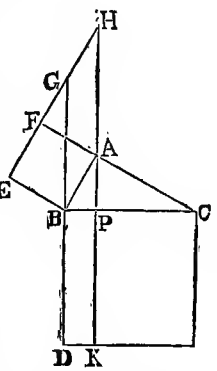
Here Z, or ∠ B C A, is the subject, and X, or ∠ B H A, the predicate of the conclusion;
Y, or ∠ E F D, is the subject, and Z, or ∠ B C A, the predicate of the minor premiss;
X, or ∠ B H A, is the subject, and Y, or ∠ E F D, the predicate of the major premiss;
and X, or ∠ B H A, is the major term; Z, or ∠ B C A, the minor; and Y, or ∠ E F D, the middle term.

N.B.—This fourth form is never employed except by some accidental awkwardness of expression. Prop. 26, case 2, bk. i., would be much better arranged if we reduced it to the second form, and said :—

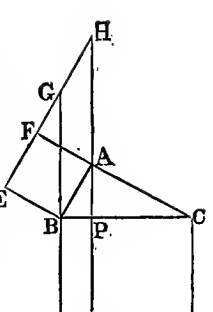
Major P.	Because X=Y,	or, because ∠BHA equals ∠EFD,
Minor P.	and Z=Y;	and ∠BCA equals ∠EFD;
Concl.	therefore, Z=X.	therefore, ∠BCA equals ∠BHA.

All legitimate arguments may be brought to one or the other of these four forms of the Syllogism ; and in reading Euclid's Elements it will be an improving exercise, occasionally, to present the arguments in the regular Syllogistic form. We subjoin an example from *The Study and Difficulties of Mathematics*, pp. 73, 74 : it is Prop. 47, bk. i., of the Elements. It will serve as a model for putting other propositions into the syllogistic form.

Exp. 1	Hyp.	Let A B C be a rt. angled triangle, B A C being the rt. angle ;
2	Concl.	the squares on A B and A C, sides about the rt. angle, together equal the square on B C, the side opposite the rt. angle.
Cons. 1	*by* 46. I.	Describe the squares on B C and B A ;
2	Pst. 2.	produce D B to meet E F, produced, if necessary, in . G ;
3	31. I.	and draw H K parallel to B D and through . A.



Dem. 1	*by* Def. 30.	I. *Conterminous sides of a square are at rt. angles to one another:*
	C. 1.	E B and B A are conterminous sides of a square ; ∴ E B and B A are at rt. angles.
2	*Sim.*	II. *By a similar Syllogism* we prove that sides D B and B C are at rt. angles, and also that sides G B and B C are at rt. angles.



DEM. 3	D. 1 & 2.	III. *Two rt. lines perpendicular to two other rt. lines make the same angle as those others:*
	D. 1 & 2.	the lines E B and B G, A B and B C, are two rt. lines perpendicular, &c.
		∴ the angle E B G is equal to the angle A B C.
4	Def. 30.	IV. *All the sides of a square are equal:*
	C. 1.	A B and B E are sides of a square;
5	Def. 10.	∴ A B and B E are equal sides.
	C.1&Hyp	V. *All right angles are equal:*
		∠ B E G and ∠ B A C are rt. angles;
6	26. I.	∴ ∠ B E G and ∠ B A C are equal angles.
		VI. *Two triangles having each two angles and the interjacent side equal, are equal in all respects:*
	D. 3, 4, 5	B E G and B A C are two triangles having the angles B E G and E B G respectively equal to the angles B A C and A B C, and the side E B equal to the side B A;
		∴ the triangles B E G and B A C are equal in all respects.
7	D. 6.	VII. The side B G is equal to the side B C,
	Def. 30.	and the side B C is equal to the side B D;
8	Def. A.	∴ the side B G is equal to the side B D.
		VIII. *A four-sided figure of which the opposite sides are parallel, is a parallelogram:*
	C. 1.	B G H A and B P K D are four-sided figures of which the opposite sides are parallel;
9	35. I.	∴ B G H A and B P K D are parallelograms.
		IX. *Parallelograms upon the same base and between the same parallels, are equal:*
	C. 3.	E B A F and B G H A are parallelograms on the same base, &c.;
		∴ E B A F and B G H A are equal parallelograms.

DEM. 10	36. I.	x. *Parallelograms on equal bases and between the same parallels, are equal:*
	C. 3.	B G H A and B D K P are parallelograms on equal bases, namely, B G = B D; ∴ the parallelograms B G H A and B D K P are equal.
11	D. 9.	xi. The parallelogram E B A F is equal to parallelogram B G H A;
	D. 10.	parallelogram B G H A is equal to parallelogram B D K P; ∴ E B A F, the square on A B, is equal to parallelogram B D K P.
12	*Sim.*	xii. *A similar argument, from the commencement,* proves that the square on A C is equal to the rectangle C P K.
13	Ax. 8.	xiii. The rectangles B K and C K are together equal to the square on A B;
	D. 11 & 12	the squares on sides B A and A C are together equal to the rectangles B K and C K; ∴ the squares on sides B A and A C are together equal to the square on A B.

INDUCTION is a species of argument in which that is inferred respecting a whole class which has been ascertained respecting several individuals of the class. Carried out completely, *Inductive reasoning* is that in which a universal proposition is proved by proving separately each of its particular cases: thus, the angles A, B, C, and D are all the angles in a certain figure A B C D: we prove that A is a right angle, B a right angle, C a right angle, and D a right angle; and we say—therefore, all the angles of the figure A B C D are right angles.

The argument "*a fortiori*," by the stronger reason, proves that a given predicate belongs in a greater degree to one subject than to another: as, A is greater than B; B greater than C; much more, *a fortiori*, is A greater than C. An example of this kind of argument occurs in Prop. 21, bk. i., of the *Elements:* thus—

> The angle B D C is greater than the angle C E D,
> and angle C E D is greater than the angle B A C;
> much more ∴ is angle B D C greater than angle B A C.

The "*reductio ad impossibile*," the reduction to an impossibility, is when the argument shows that any given assertion is impossible: as in the 14th Prop., bk. i., where it is supposed that both the line B E and the line B D are continuations of another line C B: the demonstration conducts to the conclusion that the less angle A B E equals the greater angle A B D; but this is an impossibility, for the less cannot equal the greater.

There is also the "*reductio ad adsurdum*," the reduction of an argument to an absurdity: it takes place when the conclusion, at which we arrive, involves something foolish, or utterly unreasonable: thus, in Prop. 7, bk. i., the angle B D C is proved, by the course of argument adopted, to be, first, equal to the angle B C D, and next, greater than the same angle B C D; but this is an absurdity. There is little real difference in Geometry between the impossible and the absurd.

"THE VALIDITY OF AN ARGUMENT depends upon two distinct considerations:—1, the truth of the relations assumed, or represented to have been proved before; 2, the manner in which these facts are combined so as to produce new relations; in which last, the *reasoning* properly consists. If either of these be incorrect in any single point, the result is certainly false." "The same thing holds good in every species of reasoning; and it must be observed, however different geometrical argument may be in form from that which we employ daily, it is not different in reality." "The commonest actions of our lives are directed by processes exactly identical with those which enable us to pass from one proposition of geometry to another. A porter, for example, who being directed to carry a parcel from the City to a street which he has never heard of, and who, on enquiry, finding it is in the Borough, concludes that he must cross the water to get at it, has performed an act of reasoning, differing nothing in kind from those by a series of which, did he know the previous propositions, he might be convinced that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the sides."—*On the Studies and Difficulties of Mathematics*, p. 76.

SECTION V.

THE APPLICATION OF ALGEBRA AND ARITHMETIC TO GEOMETRY.

ALGEBRAICAL GEOMETRY, or the Application of Algebra to Geometry, has two provinces: the one, when Algebra is employed for the investigation of geometrical theorems and problems, and as the great instrument of mathematical Analysis; the other, when the Notation and Methods of operation usually appropriated to Algebra are introduced to give expression to geometrical truths, and to furnish the formulas according to which the Practical Geometrician must solve the questions which come before him. For success in the use of it, the first kind requires a more extended acquaintance both with Geometry and with Algebra than beginners in either science can have attained: the second kind, therefore, although it is not fitted to enlarge the boundaries of strictly geometrical knowledge, is that of which we proceed to treat.

Arithmetical Geometry, or the Application of Arithmetic to Geometry, is never employed for purposes of investigation or of analysis: its object is, in cases of particular lines, surfaces, and solids, to express by numbers their properties—properties, the truth of which Geometry has already demonstrated, and for the numerical statement of which Algebra has given the requisite formula, or method of operation. All questions, as they are called, in Mensuration, are instances of the application of Arithmetical Geometry.

A special example will show the respective provinces of Geometry, Algebra, and Arithmetic: we assert, that the square on the hypotenuse (h) of a right-angled triangle is equal to the sum of the squares on the base (b) and on the perpendicular (a): Geometry demonstrates the truth of the assertion; Algebra either investigates and analyses that truth, or simply expresses it in the form of an equation, thus, $h^2 = a^2 + b^2$; and Arithmetic assuming a particular case, as that $h = 5$, $a = 4$, and $b = 3$, shows, that a square on the line of 5 equals the sum of the squares on the lines of 4 and 3 respectively; or that the square of 5 equals the sum of the squares of 4 and 3; 5×5, or 25, being equal to 4×4, or 16, added to 3×3, or 9. We shall afterwards see that no numbers, except 5, 4, and 3, or their equ multiples, can with perfect accuracy give an arithmetical expression to the geometrical truth.

The main Axioms of Geometry, Algebra, and Arithmetic are the same: thus, in Geometry we say,—Magnitudes equal to the same

magnitude are equal to each other; in Algebra, Quantities are equal when each is equal to the same quantity; and in Arithmetic, Numbers are equal to one another when they are equal to the same number: but the three variations in the mode of stating the axiom are expressions for one and the same universal truth,—Things equal to the same thing are equal to one another.

On the ground that a perfectly accurate expression of all geometrical truths cannot be given by numbers, *it is of great importance* in reasonings strictly geometrical, where we seek for absolute truth, that we should not confound a Geometrical Demonstration with the Algebraical or Arithmetical representation of it. Being accustomed to speak of one magnitude as containing another smaller magnitude a certain number of times, we hesitate not to say, that a line contains 10 linear units; or a rectangle, 25 square inches; or a solid, 6 cubic inches: but Plane Geometry, having to do with space generally, does not reason respecting any definitely-assigned quantity as expressed by concrete numbers, but about its universal properties.

A point, considered theoretically or mathematically, marks position, and not magnitude, and no succession or series of such points could make up a line; and a line mathematical, having extension only in one direction through space, and not being a part of space, no succession of lines—that is, of lengths without breadths—could form a surface; and a surface having extension only in two directions, length and breadth, no supposed piling up of surfaces could form a solid; for the first surface in the imaginary series being without thickness, and the second and the third, &c., being equally destitute of that property, no number of such surfaces could form a solid. Solids, therefore, though bounded by surfaces, are not made up of surfaces; surfaces, though bounded by lines, are not made up of lines; neither are lines, though they have countless points in them, made up of an aggregation of points.

But *a point and a line considered practically*, and as they were considered before Geometry, or land-measuring, became a theoretical Science, have in their very elements the property of extension; they do, and they must, occupy space. On the Atomic theory of Chemistry, there are ultimate atoms beyond which no actual division of an elementary body can proceed; so on the supposition that the points and lines employed in Mensuration possess visible properties, there are ultimate points and lines beyond which we

cannot carry our process of refinement. In practice the point and the line are both visible, and whatever is visible ceases to be a mathematical point, or a mathematical line: it has length and breadth to make an impression on the optic nerve, and is therefore a surface.

The word *monads*, applied to denote the elementary atoms by the aggregation of which mineral, vegetable, or animal substances are formed, may be introduced, not disadvantageously, into Practical Geometry. A point is the monad, or element of a line: we may make it as small as we please—the thousandth part, or the ten-millionth part of an inch; but however small, it really possesses extension: and so a line may become thinner and thinner, and finer and finer, but it possesses visible properties of breadth—in fact, it is a surface; and the monad of a line, or its least elementary part, if visible at all, must also be a surface.

We need not continue an argument of this kind. When Arithmetic is applied to Geometry, the line is made up of points, or parts—the point or part being some generally-recognised monad of length, or first and least element in the line: thus, when a line is made up of parts, each part containing one-tenth of an inch, then the unit, or monad of extension, is one-tenth of an inch. Again, for Surfaces made up of lines having visible breadth, the line that measures a surface in its length is made up of the units or monads of length; and the line which measures the same surface in its breadth is made up of the same units or monads, in this case called breadth. The monad of length and the monad of breadth, placed together at right angles to each other, constitute the monad or unit of Surface: it may be a square inch, or a square foot, or a square mile.

Thus, 100th of an inch, taken in one direction, may represent the *monad*, or elementary part *of length;*

And 100th of an inch, taken in a direction at right angles from one extremity of the monad of length, will represent the *monad of breadth;*

And the monad of length and the monad of breadth, depth, or height—for in a plane they are three words for one thing—thus placed, constitute the *monad*, or unit *of surface.*

Also, 100th of an inch, taken in a direction at right angles from the common point of junction of the two lines which represent the monad of surface, will represent the *monad of thickness;*

And the monad of length, the monad of breadth, and the monad

of thickness thus placed, and constituting extension in three directions, originate the *monad*, or unit *of solidity*.

Though, strictly speaking, the Monads, or least elementary parts of a line, are themselves lines—nay, from being visible, are actually surfaces,—yet, if such a monad be set in motion, it generates or traces out a longer line; and a line thus formed and set in motion generates a surface; and a surface also being moved, traces out a solid.

It is on this principle that a visible point may cover from view the line made up of such points; the visible line cover the surface made up of such lines; and the surface cover from view the solid made up of such surfaces.

In Plane Geometry, however, we have to do only with surfaces and with lines as constituting the boundaries of surfaces: consequently, we have to consider only two dimensions—the measurement of length and the measurement of breadth, and the combination of the two as indicating the measurement of surface.

Lines and Surfaces, as well as all other magnitudes, may be expressed by numbers; indeed, without numbers they can only be very imperfectly expressed. To do this, as we have seen, some Standard of length, or of surface, is assumed; as, a linear inch, or a square inch: the number of such linear inches, or square inches, expresses the magnitude of the line or of the surface:

Thus, the symbolical expression $\sqrt{2}$ denotes the diagonal of a square of which the side is 1: 4×3, or 12, the number of square units in a rectangle of which the sides contain 4 and 3 linear units respectively:

And generally, if we designate the number of linear units in one line by the algebraical symbol a, and the number in another line by b, the algebraical symbol $a\,b$ will denote the area of the rectangle formed by the lines a and b.

We proceed, therefore, to consider the meaning which we should attach to algebraical and numerical symbols when applied to lines and surfaces.

FIRST,—*of a Line.*

Any straight line whatever, as A B, or a, or 1, may be taken to represent the unit of length; to any longer line, as C D, or b, or 3, we apply A B, or a, or 1, as the measuring line or unit; and if

A B is contained an exact number of times, as 3 times, in C D, then we say, C D is equal to 3 times A B; thus, C D = 3 A B = 3 a = 1 × 3 = 3.

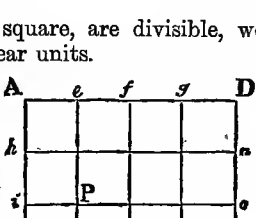
But A B may not be contained an exact number of times in C D, and, consequently, is not a measuring line for C D: we now seek for a third line, as E; and if this third line is contained an exact number of times, as twice, in A B, and an exact number of times, as seven times, in C D, then E is the common measuring line for both A B and C D, A B being equal to 2 E, and C D equal to 7 E. Thus, C D = $\frac{7}{2}$ of A B = $\frac{7}{2}$ or $3\frac{1}{2}$.

Incommensurable lines will afterwards be treated of; but we have already ascertained that a line, any line, may be represented by a letter, as a, b, &c., and that for the letters may be substituted a number either integral, or fractional, or mixed.

SECOND,—*of the Rectangle, of the Square, and of the Parallelogram.*

The sides of the rectangle, and of the square, are divisible, we will suppose, into an exact number of linear units.

Take two lines at right angles to each other, A B = 3 linear units, B C = 4 linear units. By drawing lines through the points A, h, and i, parallel to B C; and through the points k, l, m, and C, parallel to A B; the rectangle A B C D will be divided into squares, each equal to BiPk, and thus each representing a square unit, an inch, or a foot, as the case may be: in the upper row, AhnD, there are four square units; in the second row, $hion$, four square units; and in the third row, BioC, also, four square units. Now, there are as many rows as there are linear units in A B, and as many squares in each row as there are squares on B C, each of which is equal to the square BkPi; therefore, the whole number of square units in A B C D will be equal to the squares on B C, multiplied by the linear units in A B. If B C =



4 inches, feet, &c., and A B = 3 inches, feet, &c., the area of figure A B C D will contain 4 × 3, or 12 square inches, feet, &c.

Thus, the product of two numbers, representing the linear units in each side of the rectangle, expresses the number of square units in the rectangle itself, and is an Arithmetical representation of the *Area.*

Of course, this proof applies only to cases in which the two lines containing the rectangle have an exact measuring unit.

Now, generalising what has been proved, and for three linear units in A B substituting the algebraical symbol a, and for four linear units in B C the symbol b, the *Area* of any rectangle, as A B C D, will be represented by $a \times b$, or $a\,b$ square units; therefore, $a\,b$ will be the algebraical symbol for the area of a rectangle—*i.e.*, the units in the altitude multiplied by the units in the base, will give the square units in the whole rectangle.

"Hence it follows," says Potts (*Elements of Geometry*, p. 68), "that the term *rectangle* in Geometry corresponds to the term *product* in Arithmetic and Algebra; and that a similar comparison may be made between the products of the two numbers which represent the sides of rectangles, as between the areas of the rectangles themselves. This forms the basis of what are called Arithmetical or Algebraical proofs of Geometrical properties."

When the altitude and the base are equal, the surface is a square; and thus, a being equal to b, the square on a line is represented by $a\,a$, or a^2.

And as Parallelograms, of which Rectangles are one kind, upon the same base and of the same altitude are equal in Area, the *Area of any Parallelogram* may be found by multiplying the altitude into the base.

THIRD,—*of the Triangle and all other Rectilineal Figures.*

The Area of the triangle, by Prop. 41, bk. i., is equal to half the area of a parallelogram on the same or on an equal base, and between the same parallels; and we may therefore represent the *Area of the Triangle* by taking half the area of the rectangle which has the same base and altitude as the triangle: thus, if $a\,b$ is the algebraical expression for the area of the rectangle, $\frac{1}{2}\,a\,b$, or $\frac{a\,b}{2}$, will be the algebraical expression for the area of the triangle.

And, inasmuch as all rectilineal figures may be resolved into triangles by diagonals from the angular points, we are able to

express *the area of any rectilineal figure* by finding the area of each triangle separately, and then adding the areas together: $\frac{a\,b}{2} + \frac{a'\,b'}{2} + \frac{a''\,b''}{2}$, &c., according to the number of triangles into which the rectilineal figure is divided, will give the whole area. On this plan the altitude of each separate triangle into which the figure is divided, must be taken.

FOURTH,—*of the Regular Polygon and of the Circle.*

If from the centre of a regular polygon lines be drawn to the angular points, the polygon will be divided into equal triangles; and if the area of one of the triangles be found, that area multiplied by the number of triangles will give the *Area of the Regular Polygon.*

When we suppose a regular polygon to have an infinite number of sides, the sum of the sides will be equal to the circumference of a circle, the radius of which equals the altitude of each triangle in the polygon; and when that circumference is spread out into a straight line, it will represent the base of a triangle of which the altitude equals the radius of the circle; therefore, the *Area of a Circle* will equal half the circumference multiplied by the radius.

The diameter and circumference of a circle are incommensurable; but it has been ascertained that when the diameter is 1, the circumference is 3·1415926, &c.; and on this fact the methods are founded for calculating various problems in the Mensuration of Circles.

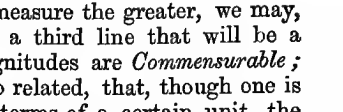
Those who desire to study more completely the Application of Algebrato Geometry, are referred to Waud's "*Treatise on Algebraical Geometry,*" or to Bourdon's "*Application de l'Algèbre à la Géométrie;*" and for the Application of Arithmetic to Geometry, they are recommended to "*A Treatise on Mensuration,*" and especially to the "*Appendix,*" by the Commissioners for National Education in Ireland: there is here a thoroughly scientific explanation of the Principles of Mensuration. For useful and improving exercises, the learner will find valuable assistance in "*Mensuration, Plane and Solid,*" by the Rev. J. Sidney Boucher, M.A. The work is well described as a "Series of Arithmetical Illustrations of the most important Practical Truths established by Geometry."

SECTION VI.

ON INCOMMENSURABLE QUANTITIES.

When of two lines the less does not measure the greater, we may, as we have seen (p. 25), often find a third line that will be a *Common Measure* of both,—such magnitudes are *Commensurable;* but when lines, or magnitudes, are so related, that, though one is capable of being represented in the terms of a certain unit, the other is not, those lines, or magnitudes, are *Incommensurable*, and are to be explained by the Principles on which the Theory of incommensurables in Arithmetic is founded :—

Thus, in the lines A B and C D, if, when A B is measured by a third line E, C D is not measured by that third line, the lines are incommensurable.



An instance of Incommensurable quantities may be given from the number 2. There is no number, either integral, or fractional, or mixed, which will express or measure the exact square root of 2. We may indeed employ the symbolical representations $\sqrt{2}$, or $\sqrt{3}$, or $\sqrt{50}$, &c., and we may reason about them, just as if their exact numerical values existed; but the numerical square root of such numbers we can never reach. In the same way there are many common fractions which cannot be expressed with perfect accuracy as decimal fractions: take the common fraction $\frac{4}{7}$, and attempt to find its exact value as a decimal fraction: we obtain ·571428571428, &c., *ad infinitum*, but never reach the absolutely true decimal representation of $\frac{4}{7}$

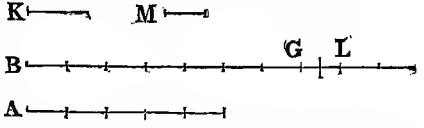
Among instances of incommensurable quantities or magnitudes, we may select the Diagonal and the Side of a Square. It is a Geometrical Principle, that the sum of the squares on the sides about a right angle is equal to the square on the side opposite the right angle. Take a square having 10 for its side: the square on one side is equal to 100; the square on the other side also equal to 100; and 100 + 100 = the square on the diagonal: but 200 has no exact square root: we express it symbolically, and say, $\sqrt{200}$ = the diagonal of a square of which the side = 10; but the exact representative numerical value of $\sqrt{200}$ cannot be found.

It is for a reason of this nature that in many instances the Arithmetical Illustration of a Geometrical Truth cannot be sub-

stituted for the Geometrical Demonstration; for the former in such instances only approximates to the Truth, the latter is the absolute Truth itself.

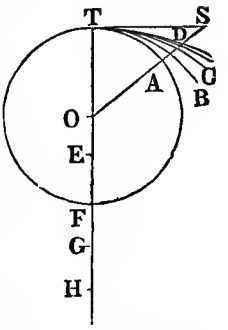
But the *Approximation* may be carried out to any assigned degree of exactness. If we are not satisfied with being so near the exact truth as the thousandth part of a unit, we may diminish the possible error to within the millionth, or billionth part, according to our fancy or the necessities of the case.

The proof of this we subjoin from the Article, "*Incommensurable*," in the *Penny Cyclopædia*, Vol. XII., p. 456.



"Let A and B be two incommensurable magnitudes; and let K be a third magnitude of the same kind, which may be as small as you please, only that it be given and known. Now, some aliquot part of A must be less than K; if not the hundredth, try the thousandth; if not the thousandth, try the millionth; and so on. Whatever K may be, it is possible to divide A into equal parts, each of which shall be less than K. Let M be such an aliquot part of A, and, having divided A into its parts, set off parts equal to M along B. Then A and B being incommensurable, B will not contain M, the measure of A, an exact number of times, but will lie between two multiples of M, say B G and B L. From this it is obvious that B does not differ from either B G or B L by so much as G L, and, therefore, not so much as K. But B G and B L are both commensurable with A, since all three are multiples of M. Here there are B G and B L, the first a little less than B, and the second a little greater, neither differing from B by so much as K, but both commensurable with A. Thus, it is also evident, that two whole numbers may be found which shall be as nearly as we please in the same ratio as two incommensurable quantities.

"The difficulty thus inherent in the application of arithmetic to concrete magnitude is not met with in practice, because no case can arise in which it is necessary to retain a magnitude so closely that no alteration, however small, can be permitted. But in exact reasoning, where any error, however small, is to be avoided, it is obvious that the arithmetic of commensurable magnitudes, and the arithmetic (if there be such a thing) of incommensurable magnitudes must not be confounded."



A clear example *of approximating nearer and nearer to a point which can never be reached,* is supplied by drawing a tangent to the diameter of a circle, and from the centre drawing a line until it shall cut the tangent,—this line cutting the tangent being named the *secant;* as in the circle T A F, T S being the tangent and O S the secant. If in T F, or T F produced, points be taken, E, F, G, H, &c., as centres of circles, to all of which T S shall be the common tangent; then each circle, as it cuts A S, shall approach nearer to S, as in the points B, C, and D; but no circle shall ever pass through the point S, inasmuch as T S being the common tangent, the circles cannot touch T S in any point except the point T; for if they did, a curved line and a straight line would coincide.

In a similar way, if £20 be subscribed annually for the purchase of books, and at the end of each year the books of the previous year be sold at half-price, and the proceeds of the sale added to a new subscription of £20, and if the two sums thus added be expended in the purchase of more books, and so on, year after year, the half of the value of the preceding year's books being yearly added to the fixed subscription of £20, there will never be £40 worth of books purchased in any one year; though in each successive year the expenditure will approach nearer and nearer to £40.

SECTION VII.

ON WRITTEN AND ORAL EXAMINATIONS, AND MEANS OF PROGRESS.

Written Examinations may be considered as the suitable test of accuracy,—oral examinations, of readiness: the one allows of the exercise of the reflective powers; the other brings into play quickness of perception and leads to promptness of action. The union of the two kinds, according to the nature of the subject under examination, should be aimed at in striving to ascertain progress, and actual knowledge and skill.

The advantages of Written Examinations in Geometry and in kindred subjects, are well pointed out by S. F. Lacroix in his Essay, "*On Teaching in general, and on the Teaching of Mathematics in particular.*" He says, p. 197, 198, "It has been proposed to substitute examination by writing, which gives to the candidate more time to collect his ideas,—which lessens the disadvantages of timidity,—and which, being carried on at the same time for all the pupils, permits the same questions to be asked of each, and renders their answers more suitable for comparison. This written examination may also be less troublesome for the Examiner, because, instead of the unremitting attention which he must give to oral answers, and the efforts of memory necessary to recall to his mind the impression which those answers make, he has only a labour capable of being divided and suspended when he experiences too much fatigue; and all the papers which serve asa basis for his judgment, are at the same time under his eye."

"It is principally on the applications of theories, that the questions of a written examination ought to run, and on calculations, altogether out of place in an oral examination."

For subjects not mathematical, however, a high place may be assigned to Oral Examinations. Of written examinations Lacroix afterwards says, p. 199, "But by this *written* examination alone we are never perfectly informed as to the readiness with which a scholar may express himself—a readiness which it is necessary to exercise and encourage, because it is useful at almost every moment of life, and because it is indispensable for men who will some day have projects to bring forward or to discuss in the presence of their companions or of their superiors, and it is only an oral examination which can make them appreciated in this respect."

The Advantages of Written Examinations in Geometry have suggested the "Skeleton Propositions;" and these advantages will, it is hoped, be increased by the aid which such outline propositions afford for training to method and exactness. Lest, however, the assistance given, by placing references in the margin, should be greater than is good for more advanced learners, *a Second Course of Examinations* is recommended—if, indeed, it be not absolutely necessary,—a Course in which no other aid is afforded to those under examination than the General Enunciations of the Propositions and a few vertical lines, within which learners are themselves to place the references to the truths already established, and on which the construction and the demonstration depend.

Among *means of Progress*, it is of great advantage to have stored

in the memory the VERY *words* of the Definitions, Postulates, and Axioms, and of the more important Propositions; and to associate with the words the *numbers*, as Definition 15, Axiom 8, Propositions 4, 8, 26, &c., of Book I.

But in the construction of Geometrical figures, and in the demonstration and application of Geometrical Truths, the *Reasoning Faculty* should be chiefly employed. A youth may repeat cleverly by rote every letter and every line of a long demonstration, and adduce the various proofs at the proper places, and yet be ignorant of the principles of the science. In the study of Geometry, no aids are so effectual as the determination and the endeavour thoroughly to understand the process of reasoning, and the nature and force of the argument.

As an instance of the method recommended to the learner, let him take that important Proposition, the 32nd of Book I. Having well considered the meaning of the words, let him, by reference to Prop. 31, and also to Prop. 29, Ax. 2; Prop. 13, and Ax. 1, recall to mind what is required for the construction of the figure, and what for the demonstration of the theorem. He may say to himself, "Here are several undoubted truths and facts: I have already proved them and accepted them as principles of Geometrical reasoning; and they are now given me that I may arrive at other truths. By using them, can I not demonstrate the inference or conclusion of this proposition?"

He may reply, that by thus exercising his judgment, he will do more than by any mere effort of memory, for really understanding and retaining Mathematical Truths.

For those who possess only the "Gradations of Euclid," and who yet wish to know the Plan proposed for the "Pen and Ink Examinations," an Example is now added of both Series,—of the one that has the references printed in the margin, and of the other without any references.

First,—with all the references by aid of which the Examination is to be conducted :—

PROP. 1.—PROB.

To describe an equilateral triangle upon a given finite straight line.

SOLUTION.—Psts. 3 and 1.

DEMONSTRATION.—Def. 15. and Ax. 1.

EXP.	1	*Datum.*
	2	*Quæs.*
CONS.	1	*by* Pst. 3.
	2	„ 3.
	3	„ 1.
	4	Sol.
DEM.	1	*by* C 1 & Def. 15
	2	C 2 & Def. 15
	3	D 1, 2 & Ax. 1
	4	D. 1, 2, 3.
	5	Recap.

USE AND APPLICATION.

Second,—without any of the references, to make the examination strict and thorough.

PROP. 1.—PROB.

To describe an equilateral triangle upon a given finite straight line.

SOLUTION.

DEMONSTRATION.

EXP.

CONS.

DEM.

USE AND APPLICATION.

The Skeleton Proposition when filled up will appear as below, symbols and contractions being allowed :—

PROP. 1.—PROB.

To describe an equilateral triangle on a given finite straight line.

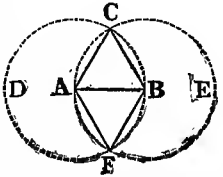
SOLUTION.—Psts. 3 and 1.—Pst. 3. A circle may be described from any centre at any distance from that centre.

Pst. 1. A st. line may be drawn from any one point to another.

DEMONSTRATION.—Def. 15, and Ax. 1.—Def. 15. A circle is a plane figure bounded by one continued line called its circumference, and having a certain point within it from which all st. lines drawn to the circumference are equal.

AX. 1. Magnitudes which are equal to the same, are equal to each other.

EXP. 1	*Dat.*	Given, the st. line A B;
2	*Quæs.*	on it to desc. an equil. △
CONS. 1	*by* Pst. 3.	From centre A with A B desc. ⊙ B C D;
2	„ 3.	from centre B with B A desc. ⊙ A C E;
3	„ 1.	Draw A C and B C;
4	Sol.	then △ A B C is the eq. lat. △ required.
DEM. 1	*by* C. 1, Def. 15	∵ A is centre of ⊙ B C D, ∴ A C = A B;
2	C. 2, Def. 15	∵ B „ ⊙ A C E, ∴ B C = A B;
3	D. 1, 2, Ax. 1	But A C, B C each = A B, ∴ A C = B C,
4	D. 1, 2, 3.	and ∴ A B = A C = B C;
5	Recap.	therefore, △ A B C is eq. lat. and on A B.



Q.E.F.

USE AND APPLICATION.—This problem may be applied to the measurement of inaccessible lines, by drawing on wood or brass an equil. triangle, and using the instrument, by placing it at . A, and along the st. line A B, so that .s C and B may be seen ; then, if it be carried along A B, until, at B, C and A can be seen along the edges of the instrument, the side A B will have been traversed, and side A B is equal to A C or B C. The side A B, being measured, will equal the other distances, A C or C B.

PART I.

GRADATIONS IN EUCLID.

ELEMENTS OF PLANE GEOMETRY,

BOOKS I. & II.

In using the letters that have reference to the Figures or Diagrams, *let it be remembered*:—

1°. That a *single* capital letter, as A, or B, denotes the point A, or the point B.

A *single* capital letter may also denote an *angle*, when the sign ∠ or the word *angle* is used with it;
And a *parallelogram* may be named by naming the letters at the opposite angles; as B K and C K, in the diagram on p. 17.

2°. *Two* capital letters, as A B, or C D, denote the *st. line* A B, or C D, or the *side* of a triangle or other st. lined figure.

But *two* capital letters, having the numeral ^{2} just above to the right hand, as AB^2, denote, not the square *of* st. line A B, but the square *on* st. line A B.

3°. Capital letters with a *point* between them, as A B. C D, denote, not the product of A B multiplied by C D, but the rectangle formed by two of its sides meeting in a common point.

GRADATIONS IN EUCLID.

BOOK I.

THE GEOMETRY OF PLANE TRIANGLES.

"In this First Booke," says Sir Henry Billingsley, who in 1576 published the earliest translation of Euclid into English, "is intreated of the most simple, sure, and first matters and groundes of Geometry; as, namely, Lynes, Angles, Triangles, Parallels, Squares, and Parallelogrammes. First of theyr definitions, shewyng what they are. After that it teacheth how to draw Parallel lynes, and how to forme diuersly figures of three sides, and foure sides, according to the varietie of their sides and Angles; and compareth them all with Triangles, and also together the one with the other. In it also is taught how a figure of any forme may be chaunged into a figure of an other forme. And for that it entreateth of these most common and general thynges, thys booke is more vniuersall than is the seconde, third, or any other, and therefore iustly occupieth the first place in order; as that without which the other bookes of Euclide which follow, and also the workes of others which haue written on Geometry, cannot be perceaued nor vnderstanded. And forasmuch as all the demonstrations and proofes of all the propositions in this whole booke, depende of these groundes and principles following, which by reason of their playnnes neede no greater declaration, yet to remoue all (be it neuer so litle) obscuritie, there are here set certayne short and manifest expositions of them."—*Billingsley's Euclid*, A.D. 1570.

DEFINITIONS.

1. A *Point* (punctum, *a small hole*) is that which has no parts, or which has no magnitude: it marks position. — THEON and PYTHAGORAS.

"A point is that of which there is no part."—EUCLID: or, "which cannot be parted or divided."—PROCLUS.

"A point is a monad having position."—PYTHAGORAS. A mathematical point cannot be drawn; for a visible point is, in fact, a surface.

2. A *Line* (linea, *a linen thread*) is length without breadth; or it is extension in one direction.

A mathematical line cannot be drawn; for whatever is visible must have breadth.

A line is *measured* by the number of units, or monads, of length contained in it; as, 5 inches; 9 feet; 13 miles.

3. The *extremities* (extremus, *outermost*), or *ends*, of a line are points.

4. A *right*, or *straight line*, is that which lies evenly between its extreme points.

"A straight line is the shortest distance between two points."—ARCHIMEDES, *adopted by* LEGENDRE.

"A straight line is that of which the extremity hidss all the rest, the eye being placed in the continuation of the line."—PLATO. Plato's line was thus a visible, not a mathematical line.

5. A *superficies* (super, *over;* facies, *a face*), or surface, is that which has only length and breadth: it is extension in two directions.

Such a surface is merely the *outside*, without any thickness.

6. The *extremities*, or *boundaries*, of a surface, are lines.

7. A *plane surface* is that in which any two points being taken, the straight line joining them lies wholly in that surface.—HERŌN THE ELDER.

"A plane surface is that which lies evenly, or equally, with the straight lines in it."—EUCLID.

"A plane surface is one whose extremities hide all the intermediate parts, the eye being placed in its continuation."—PLATO. "A plane surface is the smallest surface which can be contained between given extremities." "A plane surface is that to which a straight line may be applied in any manner of way."

An artisan with a straight line, or with a plane surface, tests the levelness or evenness of his work.

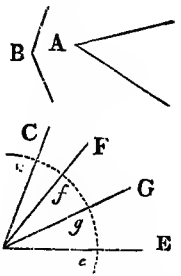
A plane surface is *measured* by the number of square units, or monads, of surface, contained within its boundaries; as, 4 square inches; 9 square feet; 13 square yards, &c.

8. A *plane angle* (angulus, *a corner*) is the inclination of two lines to each other in a plane, which meet together in the same point, but are not in the same straight line.

A plane angle is the opening of two lines from their point of meeting, or of intersection.

9. A *plane rectilineal angle* (rectus, *straight*, and linea, *a thread*) is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

A plane rectilineal angle is the opening of two straight lines from their point of meeting or of intersection, that point being the vertex. The magnitude of the angle is altogether independent of the length of the lines: angle B, in the figure, is greater than angle A. Where there are several angles meeting at a point, as at . D, each angle is distinguished by three letters, the letter at the point of meeting of the lines forming the angle being named in the middle; as, angles CDF, FDG, GDE.

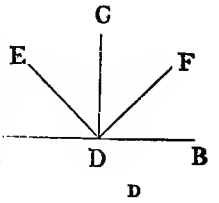


A plane rectilineal angle is *measured* by the number of degrees in the arc of a circle, of which the centre is at the angular point, and the circumference cuts the lines forming the angle: thus, the arc *ec* is the measure of the angle EDC; arc *eg*, of angle EDG, &c. A degree is the 360th part of the circumference of a circle and consequently varies with the size of the circle.

10. When a straight line standing on another straight line makes the adjacent angles equal to each other, each of these angles is called a *right angle;* and the straight line which stands on the other is called a *perpendicular* to it.

The angle CDA being equal to angle CDB, each of them is a right angle, and st. line CD is a perpendicular to st. line AB.

The *measure* of a right angle is always equal to an arc of 90°, *i.e.*, to the fourth part of the circumference of a circle: thus, if from · D an arc were drawn cutting the lines radiating from . D, the arc BC would measure the angle BDC, and the arc AC the angle ADC.



D

11. An *obtuse angle* (obtusus, *blunt*) is an angle greater than a right angle; as angle EDB, in the last figure.

The *measure* of an obtuse angle is always greater than 90°.

12. An *acute angle* (acutus, *sharp*) is an angle less than a right angle; as angle FDB.

The *measure* of an acute angle is always less than 90°.

13. A *term* (terma, *a limit*), or *boundary*, is the extremity of anything.

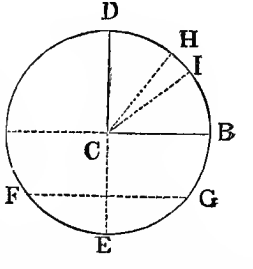
A boundary is the limit within which anything is contained.

14. A *figure* (figura, *shape, form*) is a surface enclosed by one or more boundaries.

"When all the points in a figure are also points in the same plane, the figure is called a plane figure."—HOSE.

15. A *circle* (circulus, *a ring*, or *hoop*) is a plane figure contained by one line, which is called the *circumference* (circumferre, *to carry round*), and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.

A circle is traced out by the motion of a st. line round one of its extremities: if the line BC revolve round the extremity C, the other extremity B will trace out the circumference of a circle. Any portion of a circumference is named *an arc* (arcus, *a bow*).



A circumference is *measured* by being divided into 360 equal parts, each part being called a degree. When the circumference is thus divided, and lines are drawn from the centre to each of these divisions, the angle formed by every adjacent pair of lines is called an angle of one degree.

"*We may thus compare angles together by comparing the number of degrees contained in the intercepted arcs of a circle described from the angular point as a centre.*" Thus, if the arc BD contains 90°, and the arc BH 45°, the angle BCD will be double of the angle BCH.

16. And the point (from which the equal lines are drawn) is called the *centre* (kentron, *a goad, a point*) of the circle; as C is the centre of the circle ADBE, having the st. lines CA, CD, CB, &c., all equal.

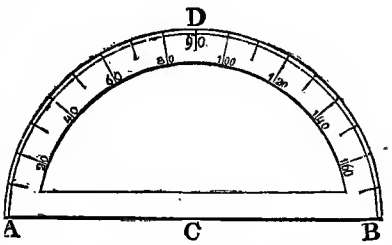
The straight lines from the centre to the circumference are *radii*.

17. A *diameter* of a circle (diametros, *a measure through*) is any straight line drawn through the centre, and terminated both ways by the circumference; as st. lines AB or DE, in the circle ADBE.

A diameter is double of a *radius* (radius, *the spoke of a wheel*).

18. A *semicircle* (semi, *half;* circulus, *a ring*) is the figure contained by the diameter and the part of the circumference cut off by the diameter; as ADBCA.

The centre of the diameter forming one boundary of the semicircle, is the same with the centre of the circle; and the semi-circumference being divided into 180 equal parts, and lines from the extremities of those equal parts being drawn to the centre, the arc opposite to each angle thus formed will be *the measure of an angle of one degree.*

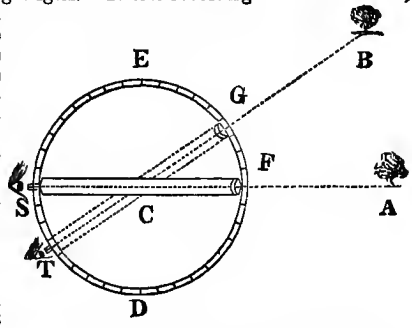


The *Semicircle* thus divided (or the *Protractor*, as it is called when the lines from the centre C are drawn to the edges of a rectangular figure), is employed *for the measurement of angles*, by laying the centre C at the angular point, and the edge CB, or AB, along one line of the angle, and then noting under what degree the other line passes: the arc between the two lines is the measure of the angle. Or, if it be required *to draw an angle of a fixed number of degrees*, the semicircle thus divided will enable us to do it; for, draw a st. line, and determine where the angular point in it is to be; then apply to that point the centre C, and lay the st. edge CB, or CA, along the line already drawn; note by a mark the degrees required, and join that mark to the angular point C by a straight line: the two st. lines will enclose the angle required.

N.B.—On the principle that heavy bodies, near the earth's surface, when left free to move, gravitate toward the earth's centre, the semicircle, or a skeleton of it, called a level, may be employed with a plummet suspended from D, or 90°, to *ascertain both the horizontal and vertical lines:* if the string by which the plummet is suspended covers the line at C, AB is in the horizontal line, and a line from D to C, or the string itself, will be the vertical line. An instrument of this kind is made use of by builders and others, and for purposes of levelling, &c.

In Surveying, *for the measurement of angles from any point to distant objects*, several instruments have been constructed: as the Sextant, an arc of 60°; the Quadrant, an arc of 90°; and the Theodolite, a whole circle of 360°, divided into two semicircles of 180° each.

The Theodolet, or Theodolite (a word of uncertain derivation), is an instrument for measuring angles. It has revolving on its centre C, a graduated index, on which is fixed a telescope S C F, with fine cross wires on the object glass F. There are also spirit levels attached, for ascertaining the true horizontal line. The observer sets the telescope S C F, so that an object at A may be seen from S, through the telescope: the theodolite remaining fixed, the telescope is turned round until, from T, the second object B can be covered by the intersection of the wires of the object-glass of the telescope: the number of degrees traversed by the telescope from S to T, or from F to G, is noted, and that number is the measure of the angle A C B.



19. A *segment* of a circle (segmentum, *a cutting, a slice*), is the figure contained by a straight line and the part of the circumference which it cuts off; as in Def. 15, the figure F G E F.

In a segment the straight line, as F G, is the *chord* (chordee, *a harp-string*); the part of the circumference cut off, as F E G, the *arc*.

A *Sector* (sector, *a cutter*) is any portion of a circle bounded by two radii and the arc which those radii intercept; as B C D B, B C I B.

20. *Rectilineal figures* are those which are bounded by right or straight lines.

21. *Trilateral figures* (trilaterus, *three-sided*), or *Triangles* (triangulus, *three-cornered*), are bounded by three straight lines.

All other rectilineal figures may be resolved into triangles, by joining some one angular point and the other angular points.

22. *Quadrilateral figures* (quadrilaterus, *four-sided*) are bounded by four straight lines.

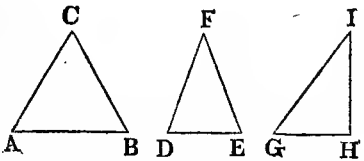
The *Diagonals* of a quadrilateral are the st. lines joining the opposite angles.

23. *Multilateral figures* (multilaterus, *many-sided*), or *Polygons* (polygonos, *of many angles*), are bounded by more than four right lines.

They are named from the number of angles: as *pentagon*, figure of five angles, or sides; *hexagon*, of six; *heptagon*, of seven; *octagon*, of eight, &c.

24. An *Equilateral Triangle* (equilaterus, *equal-sided*) is that which has three equal sides; as △ A B C, with sides, A B, B C, and C A equal.

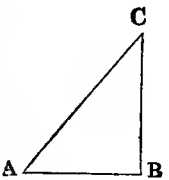
25. An *Isosceles Triangle* (isoskeles, *having equal legs*) is that which has two equal sides, or legs; namely, D F and E F, in △ F D E.



26. A *Scalene Triangle* (skaleenos, *of unequal sides*) is that which has three unequal sides; as in △ G H I, where sides I G, G H, H I, are unequal.

27. A *Right-angled Triangle* is that which has a right angle; as △ A B C, right-angled at ∠ B.

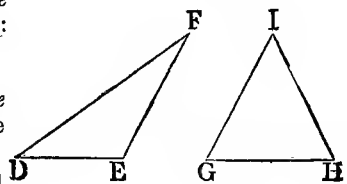
The side A C, opposite the right angle at B, is named the *hypotenuse* (hupotenousa, *that which stretches under*), or subtend: of the sides about the right angle, A B is named the *base*, and B C the *perpendicular*.



28. An *Obtuse-angled Triangle* is that which has an obtuse angle: as △ D E F, obtuse at ∠ E.

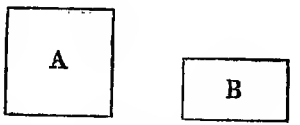
29. An *Acute-angled Triangle* is that which has three acute angles; as △ G H I.

The right and the obtuse-angled triangles have also two acute angles.



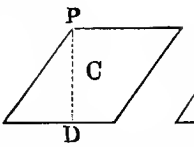
30. Of *quadrilaterals*—that is, of four-sided figures—a *square* has all its sides equal, and all its angles right angles; as fig. A.

31. An *oblong* (oblongus, *longer than broad*) is a figure which has all its angles right angles, but not all its sides equal; as fig. B.



The oblong is the same as the *rectangle* of Book II.

32. A *rhombus* (rhombos, *a turbot, or lozenge*) has all its sides equal, but its angles are not right angles; as fig. C.



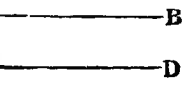
33. A *rhomboid* (rhomboeidees, *lozenge-like*) has its opposite sides equal to each other, but all its sides are not equal, nor are its angles right angles; as fig. D.

The term *parallelogram* may supersede that of rhomboid.
In figs. C and D, the perpendicular PD is the *altitude* of the figure.

34. All other four-sided figures, as fig. E, are called *Trapeziums* (trapezion, *a small four-legged table*).

The name of the figure is derived from the shape of some of the Greek tables, at which the master sat at the broad end, that he might be seen by all his guests.

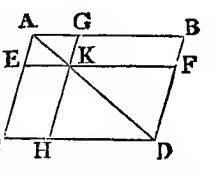
35. *Parallel straight lines* (paralleelos, *placed along opposite to each other*) are such as are in the same plane, and which being produced ever so far both ways, do not meet; as st. lines AB, CD.



The least objectionable of the many definitions proposed, from which we may reason respecting parallel lines, is that which Potts says (p. 50) "*simply expresses the conception of equidistance*:" thus, "Parallel lines are such as lie in the same plane, and which neither recede from, nor approach to, each other."

In the first Six Books of Euclid, all the lines are supposed to be in the same plane; the *test of parallelism* is, that two lines, being in the same plane, never meet, though indefinitely produced.

A. A *parallelogram* (paralleelogrammon, *a drawing with parallel sides*) is a four-sided figure, of which the opposite sides are parallel; as AB to CD, and AC to BD. The *Diameter*, or diagonal, AD, is the straight line joining two opposite angles: the *parallelograms about the diagonals, i. e.*, AEKG and KHDF, are those through which the diagonal passes; and the *Complements* are the two parallelograms, ECHK and GKFB, through which the diagonal does not pass.



Any figure with an equal number of equal sides, as four, six, eight, &c., will have its opposite sides parallel; but in the *Elements of Euclid* the name parallelogram is restricted to four-sided figures.

OBSERVATION.—"It is necessary to consider a solid—that is, a magnitude which has length, breadth, and thickness—in order to understand aright the definitions of a point, a line, and a superficies. A solid, or volume, considered apart from its physical properties, suggests the idea of the surfaces by which it is bounded; a surface, the idea of the line, or lines, which form its boundaries; and a finite line, the points which form its extremities. A solid is therefore bounded by surfaces; a surface is bounded by lines; and a line is terminated by two points. A point marks position only; a line has one dimension, length only, and defines distance; a superficies has two dimensions, length and breadth, and defines extension and a solid has three dimensions, length, breadth, and thickness, and defines some definite portion of space."

"It may also be remarked, that *two points* are sufficient to determine *the position of a straight line;* and *three points* not in the same straight line are necessary to fix *the position of a plane.*"—*Pott's Euclid,* p. 44.

POSTULATES.

1. Let it be granted that a straight line may be drawn from any one point to any other point;

2. That a terminated straight line may be produced to any length in a straight line;

3. And that a circle may be described from any centre at any distance from that centre.

The first and second Postulates concede the use of a ruler, but not of a scale; the third, that of the compasses, but not that a circle can be described round a given centre with a radius, or distance in the compasses of a given length.

Euclid himself gave *three other postulates,* which modern Editors of the *Elements* place as the tenth, eleventh, and twelfth Axioms. Those postulates were—4. Let it be granted that two straight lines cannot enclose a space; 5. That all right angles are equal; and 6. That when two lines are met or crossed by a third line, so that the two interior angles on the same side of it taken together are less than two right angles, the two lines so crossed shall meet, if continually produced.—*Smith's Biographical Dictionary,* "*Euclides,*" Vol. II., p. 66.

AXIOMS.

I. *The Seven Axioms, which apply to number and quantity as well as to magnitude,* were called by Euclid COMMON NOTIONS. They are—

1. Things which are equal to the same thing, are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the wholes are unequal.
5. If equals be taken from unequals, the remainders are unequal.
6. Things which are double of the same, are equal to one another.
7. Things which are halves of the same, are equal to one another.

II. *The Five Axioms, which apply especially to magnitude,* are peculiarly *Axioms of Geometry.*

8. Magnitudes which coincide with one another—that is, which exactly fill the same space—are equal.

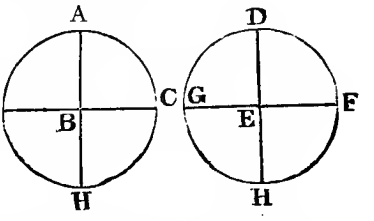
"*This Axiom is properly the definition of geometrical equality.*"

We prove the equality of two straight lines by placing them one upon the other, or by conceiving them so placed, and ascertaining that the extremities coincide: we prove the equality of two angles by placing vertex on vertex, and line on line, and then if the openings between the lines are equal, the other line of the one angle will coincide with the other line of the second angle, and the angles are equal.

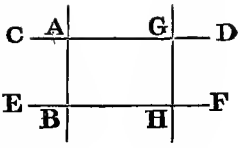
And so with respect to any figure: if all the parts and boundaries of one figure fall upon and cover all the parts and boundaries of another figure, the two figures are equal. The name of *Super-position* is given to this process, which may be either actual or mental: the one, when actually performed, being *the proof to the senses;* the other, though only conceived to be done, *being demonstration to the mind.*

9. The whole is greater than its part.
10. Two straight lines cannot enclose a space.
11. All right angles are equal to one another.

This may be illustrated by supposing that equal circles have been drawn from B and E, the vertices of the right angles: the fourth part of each circle, ABCA, or DEFD, or the arcs CA, FD will be the measure of each right angle; but the circles AGHC and DGHF are equal; therefore, the fourth parts of them must be equal; and, consequently, the angles measured by those equals must themselves be equal.

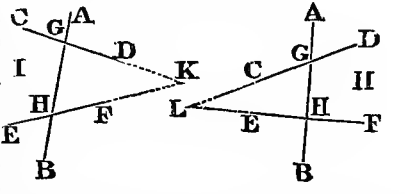


12. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these two straight lines, being continually produced, shall at length meet upon that side on which the angles are less than two right angles.



This is almost equivalent to saying, that *if two straight lines are parallel, all the perpendiculars enclosed between them shall be equal;* for if CD is parallel to EF, and the perpendicular GH less than the perpendicular AB, the lines CD and EF are nearer together towards D and F than towards C and E, contrary to the definition of parallel lines.

An illustration, by figures I. and II., will make the axiom plainer. I. The angles DGH and GHF are less than two right angles, and st. lines CD, EF meet in . K. II. The angles CGH and GHE, being less than two right angles, the st. lines CD and EF meet in . L—*i. e.*, on the side of AB on which the angles are less than two right angles.



Straight lines, like CD and EF (in fig. I.), *converge*, when they approach nearer and at last meet in a point, K: but st. lines, like LCD and LEF (in II.), *diverge*, when setting out from a point, L, they recede more and more.

When one st. line, AB, meets two other st. lines, CD and EF, there are four angles formed on the right hand of AB, and four on the left: of these angles, the *interior angles* are CGH, GHE, DGH, and GHF; the *exterior angles* are AGC, AGD, BHE, and BHF; the *opposite angles* are AGC to AHE, or BHF to BGD, &c.; the *adjacent angles*, AGC and AGD, or AGD and DGH, &c.; the *vertical angles*, those of which the vertex is at the same point, as ∠s CGH and AGD; and the *alternate angles*, every other one, as ∠s CGH and GHF, or ∠s DGH and GHE.

Instead of the twelfth axiom, Playfair adopts the following:—"*Two straight lines which intersect one another, cannot be both parallel to the same straight line.*"

PROPOSITIONS.

PROP 1.—PROB.

To describe an equilateral triangle upon a given finite straight line.

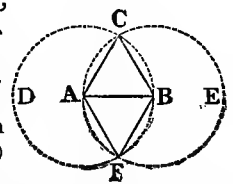
SOLUTION.—Pst. 3. A circle may be described from any centre at any distance from that centre.

Pst. 1. A straight line may be drawn from any one point to any other point.

DEMONSTRATION.—Def. 15. A circle is a plane figure contained by one line which is called the circumference, and is such that all straight lines drawn from a certain point within the figure (called the centre) to the circumferenee are equal to one another.

AX. 1. Magnitudes which are equal to the same magnitude, are equal to one another.

EXP. 1	*Datum.*	Given, the st. line AB;
3	*Quæs.*	on it to desc. an equil. △
CONS. 1	*by* Pst. 3.	From centre A with rad. AB desc. ⊙ BCD;
2	„ 3.	and from centre B with rad. BA desc. ⊙ ACE;
3	„ 1.	from the point C where the circles cut, draw st. lines CA and CB;
4	Sol.	then △ ABC shall be the equil. △ required.
DEM. 1	*by* C.1, Def.15	∵ . A is centre of ⊙ BCD, ∴ AC = AB;
2	C.2, Def.15	∵ . B „ ⊙ ACE, ∴ BC = BA:
3	D.1,2,Ax.1	But AC, BC each = AB, ∴ AC = BC.
4	D. 1, 2, 3	Thus AB, BC, and AC, sides of a triangle, are equal to one another.
5	Recap.	*Therefore, the* △ ABC *is equil. and on the given st. line* AB. Q.E.F.



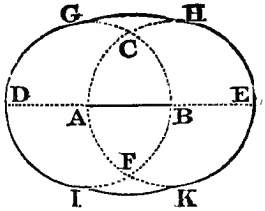
SCHOLIUM.—A second equil. triangle, AFB, may be drawn on the other side of the given st. line AB.

USE AND APPLICATION.—1. The only use to which Euclid applies this proposition, is in solving the next two problems, and problems 9, 10, and 11.

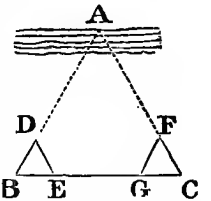
2. For merely practical purposes, it is sufficient, in describing an equil. triangle, to draw arcs intersecting in the common point C, or F, and to join

the points A, B, C. By an accommodation of this method, an isosceles triangle may be made : thus—from the extremities of the base draw arcs with a common radius equal to the equal sides of the triangle, and join the points A, B, and C.

3. A figure *approximating* to an *oval* may be drawn by describing, from the extremities of a given st. line AB, equal circles intersecting in .s C and F, and by taking the diameter AE, or BD, and from .s C and F drawing the arcs GH and IK to meet the arcs IG and KH ; the figure DGHEKID will approximate to an oval.



4. By drawing an equilateral triangle on wood or brass, an instrument, BDE, or CFG, may be made, with which an *inaccessible distance can be measured.* Let A be an object on the other side of a river ; at station B place the instrument so that A can be seen along the st. edge BD ; then, without changing the position of the instrument, look along the other st. edge BE, and set out a st. line of indefinite length as a continuation of BE ; carry the instrument along the st. line BC, with its edge BE upon the line BC, until by looking along the edge DE brought to coincide with CF, the same object A will appear in a straight line with CF : then A, B, and C will be the angular points of an equilateral triangle ; and as the sides are equal, by measuring from the station B to the station C, the *distance* from . B to A, or from . C to A, will be *ascertained.*



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Prop. 2.—Prob.

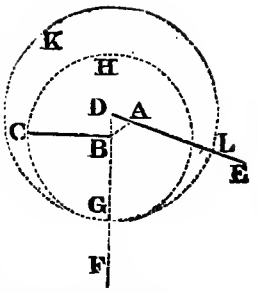
From a given point to draw a straight line equal to a given straight line.

Sol.—Pst. 1. A st. line may be drawn from any point to any other point.
P. 1. On a given st. line to desc. an equil. triangle.
Pst. 2. A terminated st. line may be produced to any length in a st. line.
Pst. 3. A circle may be drawn from any centre at any distance from that centre.

Dem.—Def. 15. All st. lines from the centre of the circle to the circumference, are equal.
Ax. 3. If equals be taken from equals, the remainders are equal.
Ax. 1. Things which are equal to the same thing, are equal each to the others.

EXP. 1	*Data.*	Given the point A, and the st. line BC;
2	*Quæs.*	from . A to draw a st. line = BC.
CONS. 1	*by* Pst. 1.	From . B the extremity of BC, draw a st. line to the point A;
2	P. 1.	on AB construct an eq. lat. △ BDA;
3	Pst. 2.	lengthen st. lines DB and DA indefinitely to E and F;
4	Pst. 3.	from centre B with rad. BC desc. the ⊙ CHG;
5	„	and from centre D with rad. DG desc. ⊙ GLK;
6	Sol.	then the st. line AL from A = the given st. line BC.
DEM. 1	*by* C. 4, 5.	∵ . B is the centre of ⊙ CGH, and . D of ⊙ GLK;
2	Def. 15.	∴ BG = BC, and DL = DG.
3	C. 2.	But the st. line DA = DB;
4	*Sub.*	and taking away these equals from the equals DL and DG,
5	Ax. 3.	the rem. AL = the rem. BG:
6	D. 2, Ax. 1	But BG being = BC, ∴ the rem. AL = BC.
7	Rec.	*Wherefore, from the given point* A *has been drawn a st. line* AL = *the given st. line* BC.

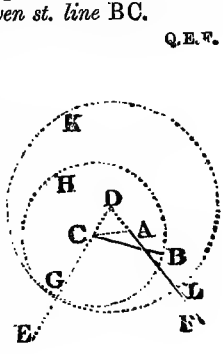
Q.E.F.



SCH.—When the given point is *out* of the given line, or of the given line produced, this problem admits of *eight* cases, each of which is a solution of the problem; but if the given point is *in* the given line, or in the given line produced, of only *four* cases.

It is very conducive to the learner's improvement, when the proposition admits of it, to vary the mode of solution: of the *eight* cases mentioned, we will take another, in which the given point A is joined to C, the other extremity of the st. line BC.

The same method will be pursued in the SOLUTION: join . s A and C, and on st. line AC construct an equil. triangle ADC; produce its sides



to E and F; and with CB as radius, desc. the circle GBH, and with DG as radius, the circle GLK; the st. line AL will be drawn equal to CB.

The DEMONSTRATION follows the same course as in the first case given above. The learner may solve some of the other cases for himself.

USE AND APP.—Practically the given distance BC will be taken in the compasses, or measured by a string, or some instrument, as a foot, or a yard, and a st. line of the required length be marked off from A, as AL.

PROP. 3—PROB.

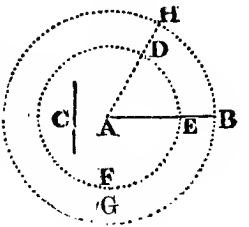
From the greater of two given st. lines to cut off a part equal to the less.

SOL.—P. 2. From a given point to draw a st. line equal to a given st. line.

Pst. 3. A ⊙ may be described from any centre at any distance from that centre.

DEM.—Def. 15. All straight lines from the centre to the circumference of a ⊙, are equal.

Ax. 1. Magnitudes which are equal to the same magnitude, are equal to each other.



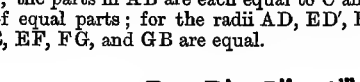
EXP. 1	*Data.*	Given the st. lines AB and C, AB being the greater of the two;
2	*Quæs.*	it is required to cut off from AB a part = the less C.
CONS. 1	*by* P. 2.	From the point A draw a st. line AD = to C;
2	Pst. 3.	and from centre A with radius AD desc. the ⊙ DEF;
3	Sol.	then the st. line AE cut off from AB, = C the less of the two given st. lines.
DEM. 1	*by* C. 2.	∵ A is the centre of the ⊙ FED,
2	Def. 15.	∴ the st. line AE = the st. line AD:
3	C. 1.	But the st. line AD = the given st. line C;
4	Ax. 1.	consequently, the st. line AE, cut off from AB, = the st. line C.
5	Rec.	*Therefore, from* AB, *the greater has been cut off,* &c. Q.E.F.

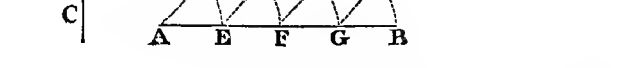
SCH.—A less line AD may be made equal to a greater AB, by describing a circle GBH with the radius AB, and producing AD until it meets the circle in H; then AH will equal AB.

USE AND APP.—This problem is performed *practically*, by transferring the distance C from . A on st. line AB.

2. The Problems 2 and 3 are of very frequent application, for in Geometry we are continually required to draw a st. line equal to a given st. line; or to take away from a greater line a part equal to the less; or to lengthen the less so that with the part produced it may equal the greater.

3. The 3rd Problem furnishes the means of constructing a *Scale of Equal Parts;* thus, Take a st. line AB of indefinite length towards B, and let C be the given st. line or part that is to be cut off from AB; from AB cut off a part equal to C, as AE; then again from EB another part equal to C, as EF; and so on; the parts in AB are each equal to C and to one another; and AB is a scale of equal parts; for the radii AD, ED′, FD″, GD‴, being each equal to C,—AE, EF, FG, and GB are equal.

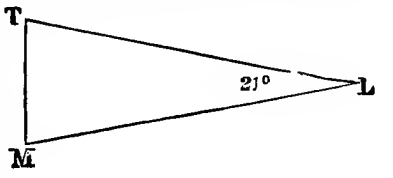




On the same principle we take a st. line KL, and from one extremity K set off on the line *ten* equal parts, as in KO; then from . O set along the st. line parts each equal to KO: if the parts between K and O are *tenths*, the parts 1, 2, 3, 4, 5 will be *units;* but if the parts between K and O are *units*, then the parts numbered 1, 2, 3, 4, 5 will be *tens*, namely, 10, 20, 30, 40, 50. By a scale of this kind the *comparative lengths* of st. lines may be readily measured.

For the advantageous use of a Scale of Equal Parts, we should understand the nature of *Representative Values.* A miniature, of not more than a square inch in surface, is representative of the human face; and a map, on a square of 12 inches, may be representative of a tract in the heavens that takes in distances which we can scarcely conceive. *The lines* in the miniature and in the map, are *in due proportion* to those in the face and in the expanse of heaven; and thus they possess *a representative value*,—they are not the actual distances, but they *stand* for them. The inch on a scale may indicate a mile, or a thousand miles of distance; hut if each portion of a mile, if each mile, or thousand miles, is given of the same relative size, then the map is a true representation; stretch out all its parts in an equal degree, and at last *by superposition* it would actually cover every point of the wide surface of which it is but the mark or outline.

By such a use of the Scale of Equal Parts, and of the Semicircle, the Scale for Angular Magnitude, we can construct figures that are a *true index* of the positions and real distances of cities, mountains, and seas, and in some respects of the constellations of heaven. For instance, if by actual observation and measurement it is ascertained that there are two towns each distant thirty-five miles from a third, and that the two, in reference to the third, are at an angle of 21° apart, a plan may easily be drawn which shall correctly shew their situation with respect to each other. From L draw a st. line LM of 35 from a scale of equal parts; at . L with a semicircle, or protractor, make an angle of 21°; and along LT from the same scale set another 35 : the points L M T will represent the situations and distances of the three towns.

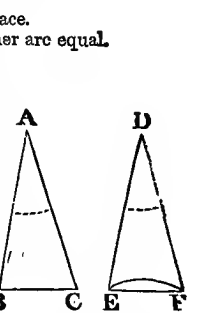


PROP. 4.—THEOR.—*(Important.)*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to each other, they shall likewise have their bases, or third sides, equal; and the two triangles shall be equal, and their angles shall be equal, each to each, viz., those to which the equal sides are opposite.

DEM.—Ax. 10. Two straight lines cannot enclose a space.
Ax. 8. Magnitudes which coincide with one another are equal.

EXP. 1	*by* Hyp. 1.	In the △s ABC, DEF, let side AB = side DE, and side AC = side DF;
2	„ 2.	Also let the included ∠ BAC = the included ∠ EDF;
3	Concl. 1.	then the base BC = the base EF;
4	„ 2.	and △ ABC = △ DEF;
5	„ 3.	also ∠ ABC = ∠ DEF; and ∠ ACB = ∠ DFE.



DEM. 1	*by Superp.*	For, applying △ ABC to △ DEF, so that . A is on . D, and st. line AB on DE;
2	D. 1, H. 1.	∵ AB coincides with and is equal to DE;
3	Ax. 8.	∴ the . B shall coincide with the . E :

DEM. 4	D. 2, 3, H. 2.	Again, ∵ AB coincides with DE, and ∠ BAC = ∠ EDF,
5	Concl.	∴ the st. line AC shall fall on the st. line DF:
6	H. 1 & Con.	But AC being = DF, the . C shall fall on the . F,
7	D. 3, 6.	and . B falling on . E, and . C on . F, the st. line BC falls on the st. line EF;
8	Con. sup.	For if, though . B falls on . E, and . C on . F, base BC does not fall on base EF, then two st. lines will enclose a space;
9	Ax. 10.	But this is impossible;
10	Ax. 8.	∴ the base BC does coincide with and = the base EF:
11	D. 2, 5, 6, 10	And AB falling on and being equal to DE; AC to DF; and BC to EF,
12	Ax. 8.	∴ △ ABC coincides with and = △ DEF.
13	Hyp. 1, D. 10	Also, since side DE coincides with side AB, and side EF with BC,
14	Ax. 8, Concl.	the ∠ ABC shall coincide with and equal ∠ DEF.
15	Hyp. 1, D. 7.	And in a similar way ∠ ACB = ∠ DFE.
16	Recap.	*Wherefore, if two triangles have two sides, &c.* Q. E. D.

SCH.—1. This being the first Theorem in the *Elements*, it is exclusively proved by means of the Axioms.

2. The converse of the 8th Axiom is assumed; namely, that *if magnitudes are equal*, not merely if they are equivalent, *they will also coincide.*

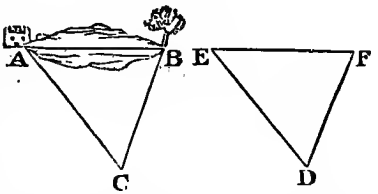
3. The equality spoken of in this Proposition, is equality of the sides and of the angles. Triangles may be *equal in area*, though the sides and angles of the one are not equal to the sides and angles of the other. When the sides and angles mutually coincide, the triangles are named *equal* triangles; when their areas only are equal, such triangles are called *equivalent* triangles.

4. Some have taken the 4th Proposition for an axiom. We perceive its truth indeed, almost without demonstration; but the number of axioms in any science should not be needlessly increased; and as this proposition can be naturally established by means of the received axioms of Geometry, it holds its proper place when classed with truths to be demonstrated. It may be more briefly enunciated, thus—"*If two triangles have each two sides and their included angle equal, the triangles are equal in every respect.*"

USE AND APP.—This Proposition contains the first of the *criteria* by which to infer the equality of triangles, and is applied to various uses ; as—

1*st.* Very frequently in all parts of Geometry *to establish the equality of triangles.*

2*nd. To ascertain an inaccessible distance,* as AB, the breadth of a lake.



With an instrument for measuring angles, as the Theodolite, p. 44, take the angle at C formed by the st. lines AC, BC, from the extremities A and B of the inaccessible distance ; and with a chain or other measure of length, find the distances CA and CB. The Representative Values of these measurements must now be taken from a *Scale of Equal Parts,* p. 55, and drawn on paper, or on any plane surface : thus, draw a st. line DF of an indefinite length, and at . D, by aid of the graduated semicircle, form an angle FDE equal to the angle BCA : from a scale of equal parts the distance from A to C is represented in proper proportion by the line DE, and the distance from B to C by DF ; consequently, on a principle established in the Sixth Book, Prop. 4, and which we now assume as a *Lemma,—that the sides about similar triangles are proportional,*—the st. line EF will represent in the due proportion the distance from A to B ; and if to the same scale we apply the st. line EF, that distance on the scale will be the *representative* measurement of the actual distance AB.

N.B.—If the ground near the lake was level enough to admit of setting out the triangle DEF in the actual measurements of CA and CB, we should have EF of the very length of AB, on the principle that two triangles having two sides and their included angle in each equal, have their third sides equal ; and if we measure one, as EF, we ascertain the other, AB ; but as it is seldom we find the actual surface of a country smooth enough for our purpose, we use the method of *Representative Values,* or of Geometrical Construction.

PROP. 5.—THEOR.

The angles at the base of an isosceles triangle are equal to each other ; and if the equal sides be produced, the angles on the other side of the base shall be equal.

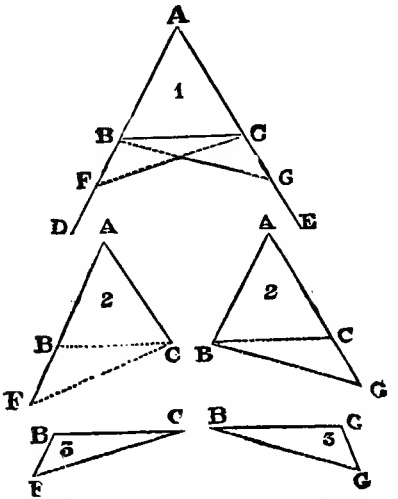
CONS.—Pst. 2. A terminated st. line may be produced to any length in a st. line.

P. 3. From the greater st. line to cut off a part equal to the less.

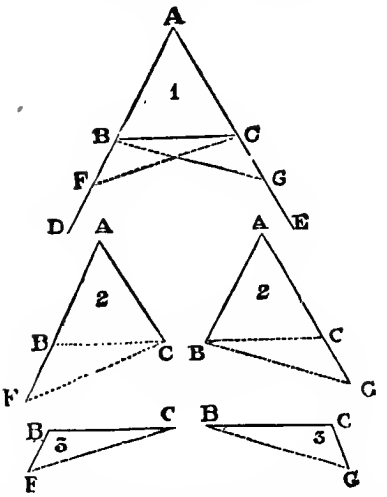
Pst. 1. A st. line may be drawn from any one point to any other point.

DEM.—P. 4. If two triangles have two sides and their included angle of one triangle equal to two sides and their included angle of another triangle, the two triangles are equal in every respect.

AX. 3. If equals are taken from equals, the remainders are equal.



EXP. 1	*by* Hyp. 1.	Let fig. ABC be an isosc. △, having the sides AB and AC equal;
2	H. 2 & Pst. 2.	and let the equal sides be produced indefinitely to .s D and E;
3	Concl. 1.	then the ∠s ABC, ACB, at the base are equal;
4	„ 2.	and the ∠s DBC, ECB, on the other side of the base are equal.
CONS. 1	*by* P. 3.	On BD take any . F, and make AG = AF;
2	Pst. 1.	join the .s F and C, G and B, by the st. lines FC and GB.
DEM. 1	*by* C. 1 & Hyp. 1.	∵ AF = AG, AC = AB, and ∠ A is common,
2	P. 4.	∴ the base FC = base GB, and △ AFC = △ AGB;
3	D 2.	also ∠ ACF = ∠ ABG, and ∠ AFC = ∠ AGB.



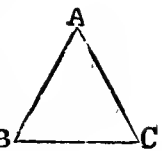
Dem. 4	C. 1 & Hyp. 1	Again ∵ the whole A F = whole A G, and part A B = part A C,
5	*Sub.* Ax. 3.	on taking away the equals A B and A C, the rem. B F = the rem. C G.
6	D. 5 & 2.	But in △s B C F, B C G, B F = C G, and F C = G B,
7	D. 3.	and ∠ B F C or A F C = ∠ C G B or A G B;
8	P. 4.	∴ △ B F C = △ C G B, ∠ F B C = ∠ G C B, and ∠ B C F = ∠ C B G,
9	D. 3 & 8.	Now, the whole ∠ A B G = whole ∠ A C F, and the part ∠ C B G = the part ∠ B C F;
10	*Sub.* & Ax. 3.	∴ on taking away ∠s C B G and B C F, the rem. ∠ A B C = the rem. ∠ A C B,
11	Remk.	and these are angles at the base.
12	D. 8.	And ∠ F B C was proved to be equal to ∠ G C B,
13	Remk.	and these are the angles below, or on the other side of the base.
14	Recap.	*Wherefore, the angles at the base. &c.*

Q. E. D.

Sch.—To assist the learner, the original figure 1 is separated into its parts, 2, 2, and 3, 3; and the equality of the triangles proved by Prop. 4.

COR.—*Every equilateral triangle is also equiangular.*

EXP. 1	*by* Hyp.	Let ABC be an equil. △ side AB = BC = AC;
2	Concl.	then shall its angles be equal, ∠ A to ∠ B to ∠ C.
DEM. 1	*by* Def. 24, P 5.	Since side AB = side AC, ∴ ∠ B = ∠ C,
2	Hyp. P. 5.	and since side CA = side CB, ∴ ∠ A = ∠ B;
3	Ax. 1.	consequently, ∠ C = ∠ A.
4	Concl.	Hence, the angles are all equal, ∠ A to B, ∠ B to C, and ∠ C to A.
5	Recap.	*Wherefore, if a △ be equilateral, &c.*



Q.E.D.

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PROP. 6.—THEOR.

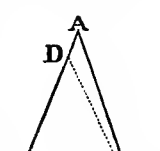
If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to the equal angles, shall be equal to one another.

CONS.—P. 3. From the greater st. line to cut off a part equal to the less.

PST. 1. A st. line may be drawn from one point to another.

DEM.—P. 4. If two triangles have two sides and the included angle equal in each, the triangles are in all respects equal.

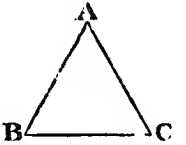
EXP. 1	*by* Hyp.	In △ ABC, supposing ∠ B = ∠ C,
2	Concl.	then the side AB = the side AC.
Sup. 1		For if AB ≠ AC, one of them is the greater;
2		let AB be greater than AC.
CONS.	*by* P. 3, & Pst. 1	In BA make BD = AC, and join DC.
DEM. 1	*by* C.	∵ DB is made = AC, and BC is common,
2	Hyp.	and ∵ ∠ DBC = ∠ ACB;
3	P. 4.	∴ base DC = base AB, and △ DBC = △ ABC.



DEM. 4	*ex abs.*	Thus the less is declared equal to the greater, which is absurd;
5	Concl.	$\therefore$ AB is not $\neq$ AC—that is, AB = AC.
6	Recap.	*Wherefore, if two angles of a triangle, &c.* Q.E.D.

COR.—*Every equiangular triangle shall also be equilateral.*

EXP. 1	*by* Hyp.	Let $\triangle$ ABC have the $\angle$s equal, A to B, B to C, and C to A;
2	Concl.	then the sides are all equal, AB to BC, BC to CA, and CA to AB.



DEM. 1	*by* H. & P. 6.	$\because \angle B = \angle C$, $\therefore$ side AC = side AB;
2	H. & P. 6.	and $\because \angle B = \angle A$, $\therefore$ side AC = side BC;
3	Ax. 1.	and $\therefore$ AB = BC.
4	Concl.	Hence, the side AB = BC, side BC = CA, and side CA = AB.
5	Recap.	*Wherefore, every equiangular triangle, &c.* Q.E.D.

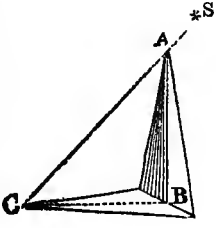
SCH.—1. This proposition is named the *converse* of the 5th. *Geometrical conversion* takes place when the hypothesis of the former proposition is made the predicate of the latter, and *vice versâ*, as in Props. 5 and 6, 18 and 19, 24 and 25, of this book.

2. Converse theorems are not universally true; for instance, the following *direct* proposition is universally true,—"*If two triangles have their three sides respectively equal, the three angles of each shall be respectively equal;*" but the *converse* is not universally true, namely,—"*If two triangles have the three angles in each respectively equal, the three sides are respectively equal.*"—*Potts' Euclid*, p. 48. To the equality of triangles it is always indispensable that *one side at least* of the one triangle should be given *equal to one side* of the other triangle.

3. In Geometry there are two modes of Demonstration: the *direct*, showing *why* a thing is so; and the *indirect*, proving that it *must* be so; the former being the usual method. *Direct Demonstration*, as in Prop. 5, is that in which we find intermediate steps which proceed regularly to prove the truth of the proposition: the *Indirect Method* is only employed, as in Prop. 6, when the predicate of it admits of an alternative, and one of them *must* be true, because they exhaust every case that can possibly exist. We prove that the alternative *cannot* be true, and infer, therefore, the predicate *must* be true. With respect to equality between magnitudes, there are two alternatives—*equal* or *unequal;* and if we prove that inequality cannot or does not exist, of necessity equality must exist.

USE AND APP.—Hieronymus, the historian, records that THALES of Miletus, who was living 546 B.C., measured *the height* of the pyramids of Egypt, *by observing the shadows* which they cast when the shadows were as long as the pyramids were high. This would be the case when the altitude of a pyramid, of any other object, and the perpendicular length of the shadow, were ual.

The height of an object and the length of its shadow are the same, when the light S, which the object AB intercepts, is at an elevation of 45°: this condition being observed, the shadow BC is equal to the height BA; because the angles BCA, BAC, being each half a rt. angle, the sides which subtend them are equal. Thus, by measuring the shadow CB, we obtain the height BA.



The height would also be obtained by making an observation with a quadrant of altitude: thus—walk away in a line perpendicular to the altitude of the object, until, at the station C, the quadrant shows A to have an elevation of 45°: the distance gone over from B to C, or BC, will equal BA, the altitude.

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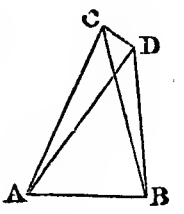
PROP. 7.—THEOR.

Upon the same base and upon the same side of it there cannot be two triangles that have their sides, which are terminated in one extremity of the base, equal to one another, and likewise those equal which are terminated in the other extremity.

CONS.—Pst. 1. A st. line may be drawn from one point to another.
Pst. 2. A st. line may be produced to any length in a st. line.

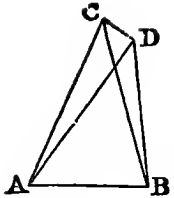
DEM.—P. 5. The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, the angles upon the other side of the base are equal.
Ax. 9. The whole is greater than its part.

EXP. 1	*by* Hyp. 1.	On base AB let there be two △s, ACB, ADB;
2	,, 2.	and let the side CA = the side DA:
3	Concl.	then it is impossible that side CB should equal side DB.
SUP.		If possible, let side CA = DA, and side CB = DB.



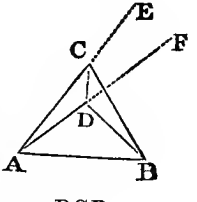
CASE I.—*Let the vertices* C *and* D *be without each other.*

CONS. 1	*by* Pst. 1.	Join .s C and D.
DEM. 1	*by* Hyp. & P. 5	∵ AC = AD in △ ACD, ∴ ∠ ACD = ∠ ADC:
2	C. & Ax. 9.	But ∠ ACD > ∠ BCD, ∴ ∠ ADC > ∠ BCD:
3	*à fort.*	and much more is ∠ BDC > ∠ BCD.
4	H. & P. 5.	Again, ∵ side BC = BD in △ BCD, ∴ ∠ BDC = ∠ BCD;
5	D. 3.	but ∠ BDC is also > ∠ BCD;
6	D. 3, 4.	∴ ∠ BDC is both > and = ∠ BCD, which is impossible.



CASE II.—*Let the vertex* D *of* △ ADB, *be within the* △ ACB.

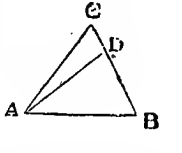
CONS. 1	*by* Pst. 1 & 2.	Join C to D, and produce AC, AD to E and F.
DEM. 1	*by* Hy. 1 & P. 5	∵ AC = AD in △ ACD, ∴ ∠ ECD = ∠ FDC.
2	C. & Ax. 9.	But ∠ ECD is greater than ∠ BCD; ∴ ∠ FDC is greater than ∠ BCD;
3	*à fort.*	and much more is ∠ BDC > ∠ BCD.
4	H. & P. 5.	Again, ∵ BD = BC in △ BDC, ∴ ∠ BDC = ∠ BCD.
5	D. 3.	But ∠ BDC is also > ∠ BCD;
6	D. 4, 3.	∴ ∠ BDC is both > and = ∠ BCD, which is impossible.



CASE III.—*When the vertex* D *is on the side* BC, *no demonstration is required.*

7	Recap.	*Therefore, upon the same base, &c.*

Q.E.D.



SCH.—The argument made use of in this proposition, is the *Dilemma*, or *double Antecedent*, in which the truth of the one is impossible, if we admit the truth of the other. The argument called the dilemma may, however, have

more than two antecedents; and Whately, p. 72, defines the true dilemma as "*a conditional Syllogism, with several antecedents in the major, and a disjunctive in the minor.*"

USE.—The only purpose for which this proposition is employed, is to prove Prop. 8.

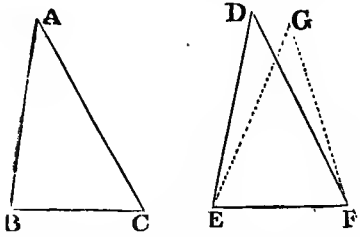
—

PROP. 8.—THEOR.—(*Important.*)

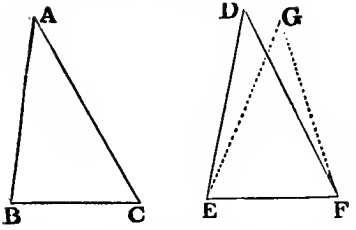
If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one, shall be equal to the angle contained by the two sides equal to them of the other.

DEM.—Pr. 7. On the same side of the same base there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity.

AX. 8. Magnitudes which coincide are equal to one another.



EXP. 1	*by* Hyp.	Let the △s ABC, DEF, have side AB = DE, side AC = DF, and also base BC = EF;
2	Concl.	then ∠ BAC shall equal ∠ EDF.
DEM. 1	*by Superp.*	Apply △ABC to △DEF, with . B on . E, and st. line BC on EF;
2	Hyp.	∵ BC = EF, ∴ . C coincides with . F.
3	D. 2, Hyp.	Wherefore, ∵ BC coincides with EF, BA and CA shall coincide with ED, FD.
4	Sup. 1.	For, suppose that base BC coincides with base EF,
	„ 2.	but sides BA and CA not with sides ED, FD, but with other st. lines EG, FG,

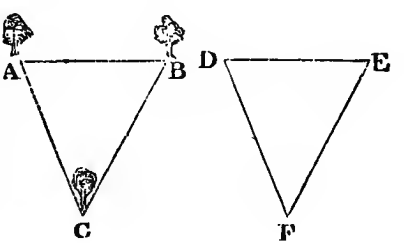


DEM. 5	Concl.	then on this sup., in △s EDF, EGF, on the same side of EF, side ED shall = EG, and also side FD = FG ;
6	P. 7.	which is impossible.
7	D. 2, 3.	∴ since base BC coincides with base EF, the sides BA, CA, coincide with sides ED, FD :
8	D. 7.	Wherefore, ∠ BAC must coincide with ∠ EDF ;
9	Ax. 8.	and ∴ ∠ BAC is = ∠ EDF.
10	Recap.	*Therefore, if two triangles have two sides,* &c. Q. E. D.

SCH.—The *Equality* established is that of the *angles*,—but the sides being equal, the triangles also must be equal. This is the *second criterion* of the equality of triangles.

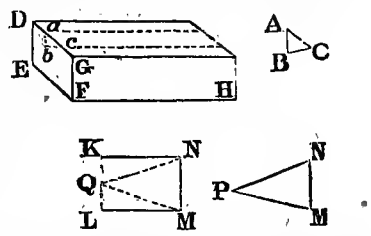
USE.—1. By the aid of this proposition, and of Prop. 22, *the angle at a given point* C, *made by st. lines from two objects, as* A *and* B, *may be determined without a theodolite.*

Measure the distances AB, BC, CA,—and from a scale of equal parts construct a triangle DEF, the sides of which, DE, EF, and FD, will be representative of the distances AB, BC, and CA : then with the semicircle find the number of degrees in angle F ; and as the triangles ACB, DEF, are similar, that number of degrees will also be the measure of angle C.



2. When the instruments for angular magnitude cannot be employed, *by reason of the inequalities of surface*, or the difficulty of placing the instruments, this proposition is useful for measuring and cutting angles in a solid body, as in a block of stone, or for bevelling, *i.e.*, for giving the desired shape to the

angular edges of timber, &c. For instance, a groove of the same triangular shape and size with the triangle ABC, is to be cut in a block of marble DEFGH. At the point in the edge DG of the block, where the groove is to commence, set off a line ac equal to AC; and on the plane surface DEFG, with ac for one side, construct a triangle abc, with sides equal to the sides in triangle ABC: abc will be the end of the groove, and if the guidance of abc be followed, the whole groove when finished will be of the same angular magnitude with ABC.



On the same principles, a beam of timber, the end of which is represented by fig. KLMN, may be bevelled so that the bevelled edge shall be of the same angle with a given angle NPM; for, on the end of the beam draw a triangle, the sides of which shall equal those of the triangle NMP; then, by Prop. 8, the bevelled edge NQM is equal to the given angle NPM.

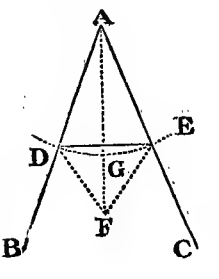
PROP. 9.—PROB.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

SOL.—P. 3. From the greater st. line to cut off a part equal to the less.
P. 1. On a st. line to draw an equil. triangle.
Pst. 1. A line may be drawn from one point to another.

DEM.—P. 8. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one, shall be equal to the angle contained by the two sides equal to them of the other.

EXP. 1	*Datum.*	Let the given ∠ be BAC;
2	*Quæs.*	it is required to bisect it.
CONS. 1	*by Assum.*	Take any point D in AB;
2	P. 3 & Pst. 1.	on AC make AE = AD, and join DE,
3	P. 1 & Pst. 1.	and on side DE, remote from ∠ A, construct an equil. △, DFE and join AF;
4	Sol.	then ∠ BAF shall = ∠ CAF, the ∠ BAC being bisected by st. line AF.



DEM. 1	*by* C. 2 & P. 1	∵ in △s DAF and EAF, AD = AE, DF = EF, and AF common,
2	P. 8.	∴ ∠ DAF is = ∠ EAF.
3	Recap.	*Wherefore, the* ∠ BAC *is bisected by the st. line* AF. Q.E.F.

SCH.—1. The bisection of the arc which measures an angle is also effected by the bisection of the angle. The arc DGF, on fig. to Prop. 9, is the measure of the ∠ BAC, or DAE; the ∠ DAG is one half, and EAG the other half, of ∠ DAE; and halves of the same being equal, the arc DG is equal to the arc GE.

2. An isosceles triangle would serve equally well for the solution and demonstration.

3. By *successive bisections* an angle may be divided into any number of equal parts, indicated *by a power of two*, as into four, eight, sixteen, thirty-two, &c., equal parts.

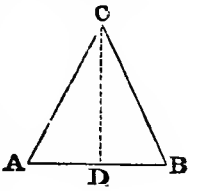
4. Hitherto no method has been discovered of geometrically *trisecting an angle*, so that the division of the quadrant of 90° into single degrees is in part effected mechanically: by simple bisection, we divide 90° into two 45°; by setting off a semi-diameter from one extremity of the quadrant, we cut off an arc of 60°; 60° bisected gives 30°; and 30° bisected gives 15°: but for the division of the 15° we require to have the means of trisecting an angle, which means Geometry does not supply. But having mechanically divided an arc of 15° into three 5°, and from one arc of 5° set off an arc of 3°, the simple bisection of the remaining arc of 2° gives an arc of 1°—a unit in the measure of the circumference. Of course this is only a practical, not a theoretical proof.

If, however, instead of taking an arbitrary quantity, 360, as the measure of the equal parts in the circumference of a circle, those who first made such division had followed the strictly geometrical process of this 9th Proposition, they would have arrived at a unit for the degrees in a given circumference with as much absolute certainty as they do now at the unit for a scale of equal parts. By making use of the powers of 2, and by their aid dividing the circle, the unit of the division is *demonstrably* accurate. Suppose the number of equal parts into which the circle had been divided, had been represented by the 9th power of 2, or 512, the bisection would have given 256 for the semicircle; 128 for the quadrant; and 64 for the octant: and 64 by successive bisection, would have given 32, 16, 8, 4, 2, and 1 equal parts. Thus, every step in the division would have been strictly in accordance with geometrical verities. Again—the unit of such degrees represented by 64, equal parts, would in the same way have been divisible into 32, 16, 8, 4, 2, and 1 minutes, and so on, to whatever extent of minuteness we might wish to carry our bisections.

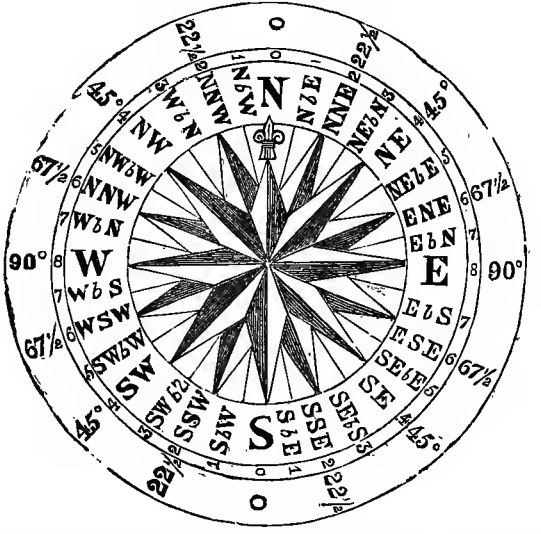
Probably no fact in geometrical measurements more clearly shows the *unscientific nature of the early geometry*, than the division of the circle into 360 equal parts. It is now, however, too late to attempt an alteration on purely geometrical grounds; and, fortunately, there is no real inconvenience or inaccuracy in the received method; for an arc of the 360th part of a circle is in practice as readily obtained as the arc of the 512th part of the same circle would be. We have only to bear in mind that Plane Geometry does not supply the means for any division of a circle, except by the method of bisections, *i.e.*, by using in regular series the powers of 2.

USE AND APPLICATION.—1. Practically the angle BAC, in the figure to Prop. 9, p. 67, would be bisected by drawing the arc DGE, and with any radius from D and E drawing arcs intersecting in . F ; AF is the bisecting line.

2. By Prop. 9, we show that *the angles at the base of an isosceles triangle are equal ;* for bisecting ∠ ACB by CD, CA is equal to CB, CD common, and ∠ ACD equal to ∠ BCD ; therefore, by P. 4, ∠ CAD equals ∠ CBD:



3. Also that the line *which bisects the vertical angle* of an isosceles triangle, *bisects the base perpendicularly ;* for AC equalling BC, DC being common, and ∠ ACD by construction equalling ∠ BCD, by Prop. 4, AD equals DB, and △ ADC equals △ BDC, and ∠ ADC equals ∠ BDC ; consequently, by Def. 10, st. line DC is perpendicular to AB.



4. The *Mariner's Compass* is divided into its 32 parts or points by Props. 9 and 10. The bisection of the diameter by another diameter at rt. angles, gives the cardinal points N., E., S., W. ; the quadrants, being bisected, give the points N.E., S.E., S.W., N.W. ; and these bisections are continued until the number of equal divisions of the circle amounts to 32—the arc at each pair of points enclosing an angle of 11° 15′. This instrument is used for taking the bearings or directions of places from some central station of observation ; and the correctness of the method depends on the physical law that a magnetised

needle or index, working freely on a pivot, does, allowing for deviation, from the same place, point in the same direction. To be quite accurate in making the observation, it is requisite to know the amount of deviation from the true North, at any given place. For Great Britain, the deviation amounts to about 24° west of north; so that when the magnetised steel index, or needle, points 24° west, the pointer on the compass card marked N. indicates the true North.

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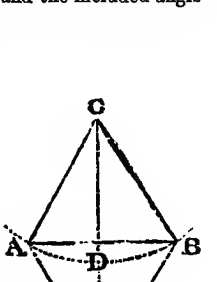
PROP. 10.—PROB.

To bisect a given finite straight line.

SOL.—P. 1. On a given st. line to construct an equilateral triangle.
P. 9. To bisect a given rectilineal angle.

DEM.—P. 4. Two triangles are equal in every respect when two sides and the included angle of one are equal to two sides and the included angle of the other triangle.

EXP. 1	*Datum.*	Given the st. line AB;
2	*Quæs.*	it is required to bisect it.
CONS. 1	*by* P. 1	On AB make an equil. △ ABC;
2	P. 9	let st. line CD make ∠ ACD = ∠ BCD;
3	SOL.	then AD = BD, *i.e*, AB is bisected in D.
DEM. 1	*by* C. 1 & 2.	∵ in △s ACD, BCD, AC = BC, CD is common, and ∠ ACD = ∠ BCD;
2	P. 4.	∴ base AD = base DB.
3	Recap.	*Wherefore*, AB *is bisected in* D.



Q.E.F.

SCH.—By successive bisections, a line may thus be divided into any number of equal parts indicated by a power of 2; as 4, 8, 16, 32, 64, &c.

USE.—1. In practice, the st. line AB would be divided into equal parts by drawing with equal radii arcs intersecting in . C and . E; the st. line CE will bisect the st. line AB.

2. Also, if the length of the line AB be ascertained by means of a scale of equal parts, the division into any required number of equal parts, as 2,

3, 4, &c., will be effected by dividing the numerical value by 2, 3, 4, &c., as the case may be.

3. But for a ready and infallible method of bisecting a line, Prop. 10 cannot be dispensed with.

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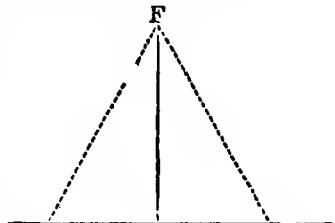
PROP. 11.—PROB.

To draw a st. line at right angles to a given st. line from a given point in the same.

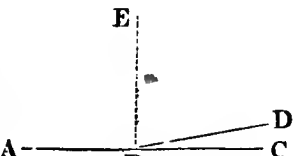
SOL.—P. 3. From the greater of two st. lines to cut off a part equal to the less.
P. 1. On a given st. line to construct an equilateral triangle.
Pst. 1. Two points may be joined by a line.

DEM.—P. 8. In two triangles, if two sides and the base of one triangle be equal to the two sides and the base of another triangle, the angle between the two equal sides of the one is equal to the angle between the two equal sides of the other.
Def. 10. When a st. line on another st. line makes the adjacent angles equal, each of the angles is a right angle.
AX. 1. Things equal to the same thing are equal to one another.



EXP. 1	*Data.*	Given the st. line AB, and C a point in it;
2	*Quæs.*	to draw from C a st. line at ⊥ AB.
CONS. 1	*by* P. 3.	In AC take any D, and make CE = CD;
2	P. 1 & Pst. 1	on DE construct the equil. △ FDE, and join FC.
3	Sol.	then ∠s DCF and ECF are equal, and consequently rt. angles.
DEM. 1	*by* C. 1 & 2.	∵ DC = EC, FC common to the △s DCF, ECF, and DF = EF,
2	P. 8.	∴ ∠ DCF = ∠ ECF:
3	C. 2.	And by construction they are adjacent angles;
4	Def. 10.	∴ the ∠s DCF and ECF are rt. angles.
5	Recap.	*Wherefore, from the given point* C, &c. Q.E.F.

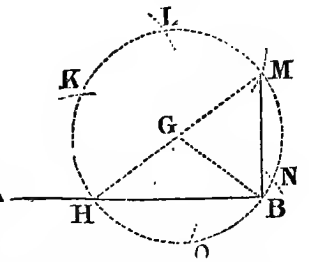
COR.—*Hence it may be shown that two st. lines cannot have a common segment.*

EXP. 1	If possible, let the segment AB be common to the st. lines ABC and ABD.	
CONS. 1	*by* P. 11.	At . B draw BE, making the ∠s with AB rt. angles.
DEM. 1	*by* Hyp.	∵ st. lines AB and BC make one st. line ABC and ∠ ABE is a rt. angle ;
2	Def. 10.	∴ ∠ ABE = ∠ EBC :
3	Hyp.	Again, ∵ st. lines AB and BD form one st. line ABD, and ∠ ABE is a rt. angle ;
4	Def. 10.	∴ ∠ ABE = ∠ EBD :
5	D. 2, 4, Ax. 1	Wherefore, ∠ EBD = ∠ EBC ; which is impossible.
6	Recap.	*Therefore, two st. lines cannot have a common segment.* Q.E.D.

SCH.—1. This corollary should be ranked among the axioms ; for it is assumed in Prop. 4, where it is taken for granted that *if certain st. lines, placed on one another, coincide for any portion of their length, they must coincide throughout.* It is also assumed in Prop. 8.

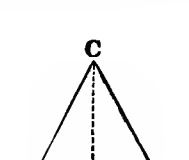
2. When the given point is at the extremity of a given st. line, the line from that extremity should be produced, and the rt. angle be then constructed :

3. Or the following method may be adopted :—Let AB be the given st. line, and . B the extremity from which the perpendicular is to be raised. Take any point G above the line AB, and with radius GB describe a circle cutting AB in . H ; join GH and produce it to M ; or from . H set the same radius GB on HK, KL, and LM ; M is the point from which if a st. line be drawn to B, that st. line MB will be perpendicular to AB. The radius equals the chord of 60° ; three times 60° equal 180° ; and 180° is the semicircle : and by P. 31, bk. iii., *the angle* HDM *in a semicircle is a rt. angle.*



USE AND APP.—1. By this proposition *the Square* is constructed, an instrument employed for ascertaining the perpendicular to an horizontal line, and for all purposes for which right angles are needed.

2. *On a given st. line* AB, *to describe an isosceles triangle of which the perpendicular height* CD, *is equal to the base* AB.



CONS.	1	*by* P. 10, 11, & 3	Bisect AB in . D, and make st. line DC perpen. and equal to AB;
	2	Pst. 1.	join AC and BC;
	3	Sol.	the figure ABC is the isosc. △ required.
DEM.	1	*by* C. 1.	∵ in △s ADC, BDC, AD=DB, DC is common, and ∠ ADC = ∠ BDC;
	2	P. 4.	∴ AC=BC.
	3	C. 1.	And the perp. DC has been made equal to AB.

Q.E.F.

PROP. 12.—PROB.

To draw a perpendicular to a given st. line of unlimited length from a given point without it.

SOL.—Pst. 3. A circle may be drawn from any centre at any distance from that centre.

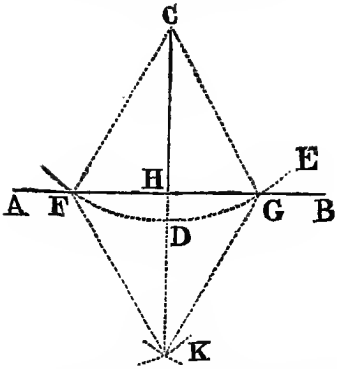
P. 10. To bisect a given st. line.

Pst. 1. A line may be drawn from one point to another.

DEM.—Def. 15. The radii of the same circle are all equal.

P. 8. If two triangles have two sides in one equal to two sides in the other, and the base equal to the base, the angles contained by the two pair of equal sides are equal.

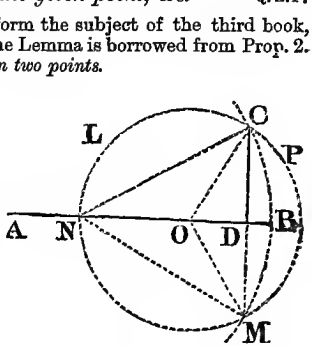
Def. 10. A st. line at rt. angles to another st. line, is perpendicular to it.



Exp. 1	*Data.*	Given the st. line AB, and the . C out of it;
2	*Quæs.*	required from . C a perpendicular to AB.
Cons. 1	*by Assum.*	Take any . D on the other side of AB;
2	Pst. 3.	from . C with rad. CD draw the arc EGF, cutting AB in . F and . G;
3	P. 10, Pst. 1.	bisect FG in H, and join C and H, C and F, and C and G;
4	Sol.	then the st. line CH is perpendicular to AB.
Dem. 1	*by* C. 3, Def 15	∵ in △s FHC, GHC, FH = HG, CF = CG, and HC common,
2	P. 8.	∴ ∠ CHF = ∠ CHG:
3	C. Remk.	Now these are adjacent angles;
4	Def. 10.	∴ the st. line CH is perpendicular to AB.
5	Recap.	*Therefore, from the given point,* &c. Q.E.F.

Sch.—1. The properties of the circle form the subject of the third book, but in the construction of the 12th Prop., the Lemma is borrowed from Prop. 2. bk. iii., *that the circle will intersect the line in two points.*

2. If the given point C is over, or nearly over, the *extremity* of the given st. line AB;—In AB take any point O, and with rad. OC describe the arc CLM; and from N another point in AB, with rad. NC, the arc CBM: join the points of intersection C and M, CD is the perpendicular required.



In the triangles NCO, NMO,—the three sides of the one are equal to the three sides of the other, and by P. 8, the angle CNO equals the angle MNO: thus in the triangles NDC, NMD, two sides and the included angle of one are equal to two sides and the included angle of the other:—therefore, by P. 4, ang. ADC equals ang. ADM, and they are adjacent angles; hence st. line CD is perpendicular to st. line AB.

USE AND APP.—1. In practice, the problem will be solved by drawing, as in the figure to P. 12, the arc FDE, and from the points F and G, with equal radii, describing arcs intersecting in . K; by joining CK, the perpendicular to AB will be drawn.

2. This problem is indispensable to all Artificers, Surveyors, and Engineers.

PROP. 13.—THEOR.

The angles which one st. line makes with another upon one side of it are either right angles, or together equal to two right angles.

CONS.—P. 11. To draw a st. line at right angles to a given st. line from a given point in the same.

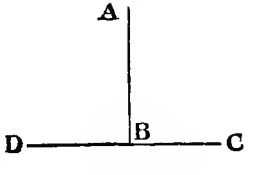
DEM.—Def. 10. When a st. line standing on another st. line makes the adjacent angles equal, each of the angles is a rt. angle.

AX. 8. Magnitudes which exactly fill the same space are equal.

AX. 2. If equals be added to equals, the sums will be equal.

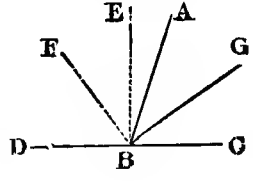
AX. 1. Magnitudes which are equal to the same, are equal to each other.

EXP. 1	*by* Hyp.	Let the st. line AB make angles with the st. line DC,
2	Concl. 1.	then the ∠s CBA, ABD, are two rt. angles,
3	„ 2.	or are together = two rt. angles.



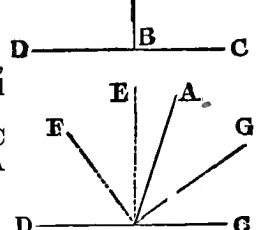
CASE I.—*Suppose that the ang.* CBA *is equal to ang.* ABD, *as in the 1st fig.;*

	Def. 10.	then each of the angles is a rt. angle.



CASE II.—*But suppose that the angle* CBA *is not equal to angle* ABD, *as in the 2nd figure below;*

CONS. 1	*by* P. 11.	At . B in DC draw BE at rt. angles to CD,
2	Def. 10.	then ∠s CBE, EBD are two rt. angles.
DEM. 1	*by* Ax. 8, *Add.*	∵ ∠CBE = ∠CBA + ∠ABE, to each equal add ∠EBD,
2	Ax. 2.	∴ ∠s CBE, EBD, = ∠s CBA, ABE, and EBD.
3	Ax. 8, *Add.*	Again, ∵ ∠ DBA = ∠s DBE & EBA, to both equals add ∠ABC;
4	Ax. 2.	∴ ∠s DBA + ABC = ∠s DBE, EBA and ABC.
5	D. 2.	But ∠s CBE, EBD = ∠s DBE, EBA, and ABC;
6	Ax. 1.	∴ ∠s CBE, EBD = ∠s DBA, ABC;
7	C.	Now ∠s CBE, EBD are two rt. angles,
8	Ax. 1.	∴ ∠s DBA, ABC together = two rt. angles.
9	Recap.	*Wherefore, the angles which one line, &c.*



Q.E.D.

SCH.—1. A rt. angle FBG is formed by bisecting the angles ABD, ABC.

2. If *one angle* be a rt. angle, *the other* is a rt. angle; if *one* be obtuse, the *other* is acute; and if *one* be acute, the *other* is obtuse.

3. *A semicircle is the measure of two right angles;* and all the angles formed by any number of lines converging to one point, on one side of another line, are together equal to two right angles.

4. The *supplement* to an angle is what it is *deficient* of *two* rt. angles; thus, ∠ ABD is the supplement of angle ABC: the *complement*, what is *wanting* to make up one rt. angle; as, ∠ ABE is the complement of ang. ABC.

USE AND APP.—This theorem is of frequent use in Trigonometry and Astronomy. When we know one of the angles which a st. line *meeting* another st. line makes, at a point in it, we in fact know the other, *for the two angles are always equal to* 180°; and if we subtract the given arc from 180° we have the other angle: thus, let ang. ABC equal 70°, ∠ ABD equals 180—70, or 110.

PROP. 14.—THEOR.

If at a point in a st. line, two other lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two lines shall be in one and the same line.

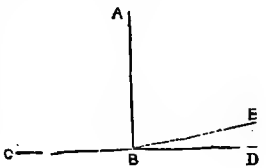
CONS.—Pst. 2. A straight line may be produced to any length in a st. line.

DEM.—P. 13. The angles made by one st. line with another on the same side of it, are equal to two rt. angles.

AX. 1. Things equal to the same thing are equal to one another.

AX. 3. If equals be taken from equals, the remainders are equal.

EXP. 1	Hyp.	At . B in st. line AB, let two st. lines BC and BD make the adjacent ∠s ABC, ABD = two rt. angles;
2	Concl.	then BC and BD will form one and the same st. line CD.



CONS. 1	*by Supp*	Should BD not be a production of the st. line CB,
2	Pst. 2.	make BE a continuation of st. line CB.
DEM. 1	*by* Hyp.	∵ AB with the st. line CBE makes the ∠s ABC, ABE,
2	P. 13.	∴ these adjacent ∠s ABC, ABE = two rt. angles;
3	Hyp.	But ∠s ABC and ABD also = two rt. angles,
4	Ax. 1.	∴ ∠s ABC and ABE = ∠s ABC and ABD;
5	*Sub.*	taking away the common ∠ ABC,
6	Ax. 3.	then the rem. ∠ ABE = the rem. ∠ ABD,
7	*ex abs.*	and the less ∠ ABE = the greater ∠ ABD, which is impossible;
8	Concl.	∴ BE is not in the same st. line with CB.
9	*Sim.*	In the same way no line except BD is a continuation of the st. line CB;
10	Concl.	∴ BD is in one and the same st. line with CB.
11	Recap.	*Wherefore, if at a point in a line, &c.* Q.E.D

SCH.—1. The words "upon the opposite sides of it" are of essential importance; for two lines may make with a third, two angles equal to two rt. angles, and yet the two lines not be in one st. line.

2. The fourteenth proposition is the converse of the thirteenth.

—

PROP. 15.—THEOR.

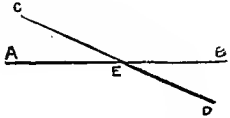
If two st. lines cut one another, the opposite or vertical angles shall be equal.

DEM.—P. 13. The angles which one st. line makes with another st. line upon one side of it, are either two rt. angles, or equal to two rt. angles.

AX. 1. Things equal to the same are equal to each other.

AX. 3. If equals be taken from equals, the remainders are equal.

EXP. 1	Hyp.	Let the st. lines AB, CD intersect in the point E;
2	Concl.	then ∠ AEC = ∠ DEB, and ∠ CEB = ∠ AED.



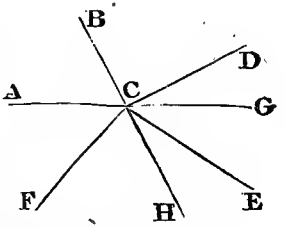
DEM. 1	*by* Hyp.	∵ st. line AE makes with CD at the . E the ∠s CEA and AED,
2	P. 13.	∴ these ∠s CEA and AED = two rt. angles.
3	Hyp.	And ∵ st. line DE makes with AB at . E the ∠s AED, DEB;
4	P. 13.	∴ these ∠s AED and DEB = two rt. angles;
5	Ax. 1.	∴ also ∠s CEA, AED = ∠s AED, DEB:
6	*Sub.*	Take away the common angle AED,
7	Ax. 3.	and rem. ∠ CEA will = rem. ∠ DEB.
8	Sim.	In like manner ang. CEB is proved equal to ang. AED.
9	Recap.	*Therefore, if two st. lines cut one another, &c.*

Q.E.D.

COR. 1.—*The angles formed by two st. lines,* AB *and* CD, *cutting each other in one point* E, *are together equal to four right angles.*

DEM. 1	*by* P. 13.	∵ ∠s AEC and CEB = two rt. angles,
2	Pr. 13.	and ∠s AED and DEB also = two rt. angles;
3	*Add.*	by adding the equals to the equals,
4	Ax. 2.	∴ the ∠s AEC, CEB, AED, and DEB = four rt. angles.

COR. 2.—*And all the angles formed by any number of st. lines,* AC, BC, DC, EC, FC, *&c., diverging from a common centre* C, *are together equal to four rt. angles.*



CONS. 1	*by* Pst. 1.	Produce any two of the given st. lines, as AC to G, BC to H.
DEM. 1	*by* Cor 1, 15	∵ st. lines AG and BH intersect in the point C, ∴ ∠s ACB, BCG, ACH, and HCG = four rt. ∠s.
2	P. 13.	But ∠s ACB, BCD, DCG = ∠s ACB and BCG;
3	P. 13.	also ∠s ACF, FCE, and ECG = ∠s ACH and HCG;
4	Concl.	∴ all the angles diverging from . C are together equal to four rt. angles.

SCH.—1. This proposition might be briefly proved by saying that *the opposite angles are equal,* because *they have the same supplement.*

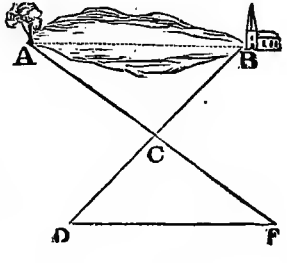
2. "This Prop. is the development of the definition of an angle. *If the st. lines at the angular point be produced, the produced lines have the same inclination to one another as the original lines have.*"—*Pott's Euclid,* p. 49.

3. The *converse* of this proposition is, "*If four lines meeting in a point make the vertical angles equal, each alternate pair of lines shall be in one and the same st. line.*"

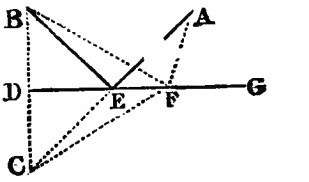
USE AND APP.—1. By an easy application of Prop. 15, we find *the breadth of a lake,* or the distance between two inaccessible objects, A and B.

From any station C, accessible both to A and B, measure the st. lines CA and CB, and produce them until CE is equal to CA, and CD to CB; join .s D and E, the distance DE will equal the distance AB.

For the st. lines CA, CB equal the st. lines CE, CD; and the angle ACB equals the angle DCE; therefore, by Prop. 4, the st. lines AB and DE are equal.



2. On the principle that *the angle of reflection equals the angle of incidence*, this proposition is useful to the billiard player to enable him with a ball at A to strike by reflection another ball at B.



Let DG be one side of the table; imagine a st. line from . B to be perpendicular to DG, and produced until DC equals BD: if now the ball A be driven in the st. line AEC, so that if it was not for the side DG it would reach . C, the ball A on striking E will be reflected from its course so as to reach the ball B; *i.e.*, the angle of reflection BED will be equal to the angle of incidence AEG.

In triangles BDE, CDE, the side BD is equal to side DC, DE common, and the included angles at . D equal: therefore, by Prop. 4, the ang. BED is equal to the ang. CED; but by Prop. 15, ang. CED is equal to ang. AEG: consequently, the ang. BED equals the ang. AEG.

3. Another use of this proposition is, *to determine the number and kind of polygons which may be joined to cover a given space.*

The circle, consisting of 360°, is *the measure for the sum of all the angles that can possibly be drawn from a point;* and no regular right-lined figures can fill up the space around a point, unless the angle contained by any two of the sides of the figure be a measure of 360°. *The only regular right-lined figures of which the angles contained by two sides are measures of a circle,* are *the equilateral triangle, the square,* and *the hexagon;* for 60°, 90°, and 120°, the respective angles of these figures, are measures of 360°. The angle formed by two conterminous sides of the pentagon contains 108°; of the heptagon, 128$\frac{4}{7}$°; of the octagon, 135°, &c.; but none of these numbers are measures of 360°.

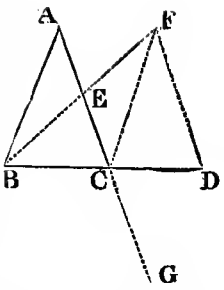
Prop. 16.—Theor.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Cons.—P. 10. To bisect a given st. line.
P. 3. From the greater st. line to take a part equal to the less.
Pst. 1. A st. line may be drawn from one point to another.
Pst. 2. A st. line may be produced to any length in a st. line.

Dem.—P. 15. If two st. lines cut one another, the vertical angles shall be equal.
P. 4. If two triangles have in each two sides and the included angle equal, they are equal in all respects.
Ax. 9. The whole is greater than its part.

EXP. 1	Hyp.	Let BC, a side of △ ABC, be produced to D;
2	Concl.	then the ext. ∠ ACD > ∠ CBA, or than ∠ CAB.
CONS. 1	*by* P10 & Pst1	Make CE = AE, and join BE:
2	Pst 2, P. 3, Pst. 1.	produce BE so that BE = EF, and join FC.
DEM. 1	*by* C. 1 & 2.	∵ AE = EC, and BE = EF, in △s ABE, CFE,
2	P. 15.	and ∠ AEB = ∠ CEF;
3	P. 4.	∴ △AEB = △CEF, and ∠ BAE = ∠ FCE.
4	C.	But ∠ ECD or ACD > ∠ ECF;
5	Ax. 9.	and ∴ ∠ ACD > ∠ BAC.
6	*Sim.* P. 10.	In like manner produce side AC to · G, and bisect BC,
7	D. 1—5.	and ∠ BCG, *i.e.*, ∠ ACD, will be shewn to be > ∠ ABC.
8	Recap.	*Therefore, if one side of a triangle be produced,* &c. Q.E.D.

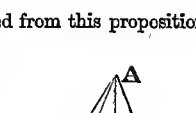


SCH.—1. We may illustrate Prop. 16 in the following way:—Let triangle ABC slide along the st. line BD until the point B covers the point C; it is obvious that vertex A, or rather F, will be at the right hand of the point A, and that the line CF must be within the ang. ACD; the ang. FCD or ABC will therefore be less than the ext. ang. ACD.

2. *Each angle of a triangle is less than the supplement of either of the other angles.*

USE AND APP.—Among the conclusions to be derived from this proposition are—

1. *Only one perpendicular* AB *can be drawn from a point* A *to a st. line* FD.



Let AB be perp. to BC, another st. line, as AC, is not perpendicular. For ext. ang. ABD by P. 16, is greater than ACB, and ABC also being a rt. angle and greater than ACB, ACB is not a rt. angle, nor is st. line AC perp. to FD.

2. If a st. line AC make the angle ACB acute, and ∠ ACF obtuse, the perpendicular AB from · A shall fall on the same side as the acute angle does; for if we suppose AE, the line on the side of the obtuse angle to be perpendicular, and AEF to be a rt. angle, then the rt. angle AEF would be greater than the obtuse angle ACE.

3. In measuring triangles, parallelograms, and trapeziums, and in reducing them to rectangular figures, these and similar conclusions are of great use.

—

PROP. 17.—THEOR.

Any two angles of a triangle are together less than two right angles.

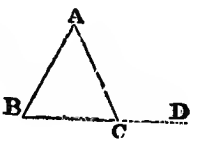
CONS.—Pst. 2. A st. line may be produced to any length in a st. line.

DEM.—P. 16. If one side of a triangle be produced, the ext. angle is greater than either of the int. opposite angles.

AX. 4. If equals be added to unequals, the wholes are unequal.

P. 13. The angles formed by one line on another, are together equal to two rt. angles.

EXP. 1	Hyp.	Let ABC be a triangle;
2	Concl.	then the ∠s A and B are < two rt. angles; also ∠s B and C, and C and A.



CONS. 1	*by* Pst. 2.	Produce any side, as BC to D.
DEM. 1	*by* Cons.	∵ ∠ ACD is the exterior angle of △ ABC,
2	P. 16.	∴ ∠ ACD > ∠ B or ∠ A int. and opposite ∠s.
3	*Add.*	To each of these unequals add the ∠ ACB.
4	Ax. 4.	Then ∠s ACD and ACB > ∠s ABC and ACB.
5	P. 13.	But ∠s ACD and ACB = two rt. angles.
6	D. 4.	∴ ∠s ABC, ACB < two rt. angles.
7	*Sim.*	In like manner ∠s BAC, ACB < two rt. angles.
8	*Sim.*	and ∠s BAC, ABC < two rt. angles.
9	Recap.	*Therefore, any two angles of a triangle,* &c.

Q.E.D.

SCH.—1. This Proposition is *explanatory of the twelfth Axiom,* and the converse of it.

2. Both the sixteenth and the seventeenth propositions will be included in the thirty-second, in which it will be proved that the *three angles of a triangle together equal two right angles.*

3. The seventeenth Prop. is useful for demonstrating some of those that follow.

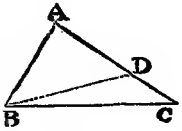
PROP. 18.—THEOR.

The greater side of every triangle is opposite to the greater angle.

CONS.—P. 3. From the greater line to take a part equal to the less.
Pst. 1. A st. line may be drawn from one point to another

DEM.—P. 5. The angles at the base of an isosceles triangle are equal.
P. 16 If one side of a triangle be produced, the ext. angle is greater than either of the int. and opposite angles.

EXP. 1	Hyp.	In $\triangle$ ABC, let side AC be greater than side AB;
2	Concl.	then $\angle$ ABC is $>$ $\angle$ ACB.
CONS. 1	*by* P. 3 & Pst. 1	Make side AD = side AB, and join BD.
DEM. 1	*by* C. & P. 5.	$\because$ AD = AB, in the $\triangle$ ABD, $\therefore$ $\angle$ ADB = $\angle$ ABD:
2	P. 16.	But ext. $\angle$ ADB $>$ int. and opp. $\angle$ DCB,
3	D. 1.	and $\angle$ ADB = $\angle$ ABD;
4	D. 2, 3.	$\therefore$ $\angle$ ABD $>$ $\angle$ DCB;
5	*à fort.*	much more is $\angle$ ABC $>$ $\angle$ ACB.
6	Recap.	*Therefore, the greater side of every triangle, &c.* Q.E.D.



SCH.—The argument on which the conclusion depends is named "*à fortiori*," *by the stronger reason*, and proves that *a given predicate belongs in a greater degree* to one subject than to another; as in the Syllogism,—Y is greater than Z, and X greater than Y; much more is X greater than Z.

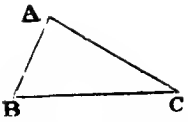
USE AND APP.—For the demonstration of other propositions, as Prop. 19.

PROP. 19.—THEOR.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

DEM.—P. 5. The angles at the base of an isosceles triangle are equal.
P. 18. The greater side of every triangle is opposite to the greater angle.

Exp. 1	Hyp.	In $\triangle$ ABC let $\angle$ ABC $>\angle$ ACB;
2	Concl.	then side AC is $>$ side AB



Sup.—*The side* AC *must be greater than, equal to, or less than* AB.

Dem. 1	*by Sup.* & P. 5.	If AC = AB, then $\angle$ ABC = $\angle$ ACB:
2	Hyp.	but $\angle$ ABC $\neq$ $\angle$ ACB;
3	D. 1 & 2.	$\therefore$ side AC $\neq$ AB.
4	*Sup.* & P. 18.	Again, if AC is $<$ AB, then $\angle$ ABC is $<$ $\angle$ ACB:
5	Hyp.	but $\angle$ ABC is $\not<$ $\angle$ ACB;
6	D. 4 & 5.	$\therefore$ side AC is $\not<$ AB.
7	D. 3 & 6.	Now, AC is neither equal to, nor less than, AB;
8	Concl.	$\therefore$ side AC is $>$ side AB.
9	Recap.	*Wherefore, the greater angle of every triangle,* &c. Q.E.D.

Sch.—1. This proposition is *the converse* of the eighteenth, and bears the same relation to it as the 6th does to the 5th Prop.

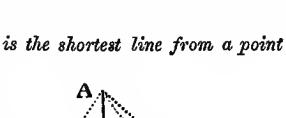
2. Propositions 5, 6, 18, and 19, may be combined into one proposition, thus,—"One *angle* of a triangle *is greater than, equal to,* or *less than* another angle, as *the side* opposed to the one *is greater than, equal to,* or *less than, the side* opposed to the other; and *vice versâ.*"

3. By aid of Props. 17, 18, and 19, we may prove that *from the same point there can be drawn but one perpendicular to a given line,* and that *this perpendicular is the shortest of all the lines from the given point to the given line.*

4. As from a given point only three lines can be drawn perpendicular to each other, *it is impossible to imagine that there are more than three species of quantity,*—a Line, a Surface, and a Solid.

Use and App.—1. *The perpendicular* AD *is the shortest line from a point* A *to a given st. line* BC.

Because ang. ADB is a rt. ang., the ang. formed by any other line from A, as $\angle$ ABC, is acute, by P. 17; and by P. 19 the side AD is less than AB: and in the same way we can prove that no other line is less than AD: therefore, *the perpendicular is the shortest line from a point to a given st. line.*

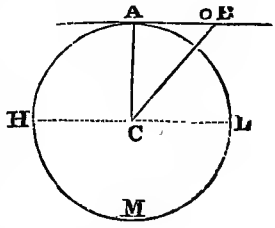


For this reason the Perpendicular is made use of in measuring: and irregular figures are reduced to those of which the angles are rt. angles.

2. *From the same point* A *only two equal st. lines,* A B *and* A C *can be drawn to a given st. line* B E. See the last figure.

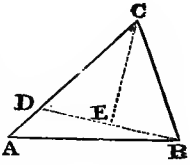
Suppose another st. line A E also equal to A C. Since A C is equal A B, by P. 5, ang. A C B is equal to ang. A B C; but in triangle A E C, the ext. ang. A C B, by P. 16, is greater than the int. ang. A E C, and ∠ A B C, being equal to ∠ A C B, is greater than ∠ A E C; therefore, by P. 19, the side A E is greater than B A, or than C A: thus it is proved that another st. line besides A B and A C cannot be drawn from . A to B E equal to those lines.

3. *All heavy bodies free to move, continually descend, or seek the point which is nearest to the earth's centre.*



Let H A L M represent the earth's circumference, H C L the rational horizon of station A, and A C a perpendicular to the centre; also let A B be a plane parallel to H L,—suppose it the channel of a canal; then water poured in at B will flow towards A, because at A, by P. 19, the distance A C is less than the distance B C. A canal thus constructed might be full at A, and almost empty at B. In the same way a sphere placed at B would roll towards A, and finally settle or come to rest at A, because A C is the shortest line to the earth's centre.

4. *By aid of the* 19*th and the* 4*th Propositions we can construct a triangle when the base* A B, *the less angle,* A, *at the base, and the difference,* A D, *of the sides, are given.*



On the st. line A D, produced indefinitely, and forming with A B the ang. A, take A D, the difference of the sides; join . s D and B, and bisect D B in . E; at . E raise a perpendicular till it meets A C in . C, and join C B; the triangle A B C is the triangle required.

By Hyp. and P. 19, B C is the least side, being opposite to the least ang. A. By Const., D E is equal to E B, E C common, and the angles at . E equal; therefore, by P. 4, the side D C is equal to the side B C; and C A is greater than C B by the given difference A D. The figure A B C, therefore, is the triangle required.

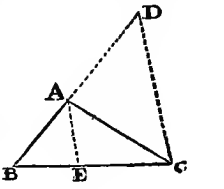
PROP 20.—THEOR.

Any two sides of a triangle are together greater than the third side.

CONS.—Pst. 2. A st. line may be lengthened out in a st. line.
P. 3. From the greater st. line to cut off a part equal to the less.
Pst. 1. A st. line may be drawn from one point to another.

DEM.—P. 5. The angles at the base of an isosceles triangle are equal.
Ax. 9. The whole is greater than its part.
P. 19. The greater angle of every triangle is subtended by the greater side.

EXP. 1	Hyp.	Let ABC be a triangle;
2	Concl. 1.	then the sides AB and AC are > BC;
	„ 2.	sides AB and BC > AC;
	„ 3.	„ BC and CA > AB.
CONS. 1	*by* Pst. 2, P. 3	Produce side BA to . D, and make AD = AC;
2	Pst. 1.	join D and C.
DEM. 1	*by* C. 1 & P. 5.	∵ AD = AC, ∠ ADC = ∠ ACD;
2	C. 1 & 2.	but ∠ BCD is > ∠ ACD;
3	Ax. 9.	∴ ∠ BCD is > ∠ ADC or BDC;
4	P. 19.	and ∴ also side BD is > side BC.
5	C. 1.	But side BD = BA and AC together;
6	D. 4 & 5.	∴ sides BA and AC together are > BC
7	*Sim.*	In like manner it may be proved that sides AB and BC are > AC,
8	„	and BC and AC > AB
9	Concl. & R.	*Therefore, any two sides of a triangle,* &c. Q.E.D.



OR,

CONS.	*by* P. 9.	Let st. line AE bisect ∠ BAC.
DEM. 1	*by* P 16.	∵ ∠ BEA is > ∠ EAC, and ∠ CEA than ∠ EAB;
2	P. 19.	∴ side BA is > BE, and side AC than CE:
3	Concl.	And BA and AC > BE and CE, which are equal to BC.
4	Recap.	*Therefore, any two sides,* &c. Q.E.D.

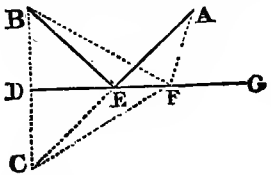
COR.—*The difference of any two sides of a triangle is less than the remaining side.*

DEM. 1	by P. 20.	The sides AC and BC are > AB;
2	*Sub.*	take away AC both from (AC + BC), and also from AB;
3	Ax. 5.	∴ side BC is > AB diminished by AC, or than the difference between AB and AC.

N.B.—In the proof of this Corollary, says *Lardner*, p. 32, "we assume something more than is expressed in the fifth Axiom. For we take for granted, that if one quantity (a) be greater than another quantity (b), and that equals be taken from both, the remainder of the former (a) will be greater than the remainder of the latter (b). This is a principle which is frequently used, though not expressed in the Axiom."

SCH.—"Let the beginner remember that the object of this proposition is not to *convince* him of the truth stated, but to show how it may be connected with and deduced from the fundamental axioms and definitions."—*Mason*, p. 56.

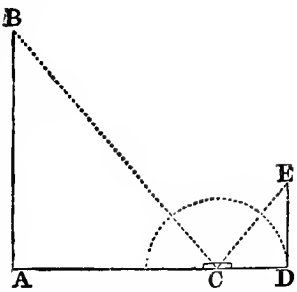
USE and APP.—1. *Of all st. lines that can be drawn from one point*, A, *to another*, E, *and reflected to a third point*, B, *those are the shortest*, AE, EB, *which make the angle of incidence*, AEG, *equal to the angle of reflection*, BED.



CONS. 1	*by* P. 12 & Pst. 2	From . B draw BD perp. to DG, and produce BD indefinitely;
2	P. 3.	make st. line DC = DB, and join EC.
3	*Assum.*, Pst. 1.	Also in st. line DG assume another point F, and join AF, BF, and CF
DEM. 1	*by* C. 1, 2.	In triangles BED, CED, side DE is common, BD equal to DC, and the angles at D equal.
2	P. 4.	∴ side BE is = CE, and ∠ BED = ∠ CED.
3	*Sim.*	In the same way we prove side BF to be = side CF.
4	Hyp. & C. & D.2	Now ∠s BED, DEC, and AEG are all equal;
5	C.	and AEC is a st. line.
6	D.5 & C. 3, P. 20	Also AFC is a △; ∴ sides AF and FC > AC:
7	D. 2, 3.	but AC = AE and EB; and FB = FC;
8	Concl.	∴ AF, FC, or AF, FB, > AE and EB.
9	Recap.	*Wherefore, of all st. lines that can be drawn*, &c. Q.E.D.

2. We may observe that natural causes act by the shortest lines; therefore, all reflections are made by the lines which cause the angle of reflection to equal the angle of incidence. Hence, *by means of a mirror placed horizontally, we may construct a triangle, the perpendicular of which shall be representative of the height of any object.*

Let A B be the height of a tower, AC an horizontal line, with a mirror at C. BCA will be the ang. of incidence of the light from . B on the mirror at · C : if now an observer at D measures the angle of reflection ECD, and the distance AC, he will have the means of constructing a triangle representative of ACB : for make the ang. ECD the angle of reflection, let C D contain the equal parts from a scale that represents the yards or feet in A C, and at D let a perpendicular D E be raised,—D E applied to the same scale of equal parts, will give the height of A B.



3. Were we to take as a lemma, prop. 4, bk. vi., that "*the sides about the equal angles of equiangular triangles are proportionals,*" we should have from the foregoing construction the proportion, and, consequently, the equation, which would determine the height A B. For triangles B A C and E D C being similar, the base of the one is to the base of the other as the perpendicular of the one to the perpendicular of the other : or,

D C : A C :: E D · B A, and the *product of the extremes equalling the product of the means.*

D C . B A = A C . E D; whence $AB = \frac{AC \cdot ED}{DC}$, or, $\frac{50 \times 6}{10} = 30$; when A C = 50, E D = 6, and D C = 10.

Prop. 21.—Theor.

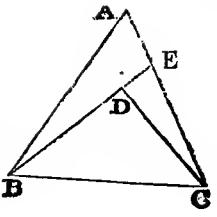
If from the ends of the side of a triangle there be drawn two st. lines to a point within the triangle, these lines shall be less than the other two sides of the triangle, but shall contain a greater angle.

Cons.—Pst. 2. A terminated st. line may be produced.

Dem.—P. 20. Any two sides of a triangle are together greater than the third side.

Ax. 4. If equals be added to unequals, the sums will be unequal.

P. 16. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.



Exp. 1	Hyp.	Given the △ ABC, and from . s B and C the st. lines BD, CD meeting within the △ in the pt. D;
2	Concl. 1.	then BD and CD shall be < BA and CA, the other two sides.
	„ 2.	but the ∠ BDC shall be > ∠ BAC.
Cons.	*by* Pst. 2.	Produce BD to meet side CA in the . E.
Dem. 1	*by* P. 20.	In △ ABE the sides AB and AE are > BE;
2	*Add.*	to each of these add the line EC;
3	Ax. 4.	then the lines AB, AE, and EC, are > BE and EC.
4	P. 20.	Again, in △ CED, the sides CE and ED are > CD;
5	*Add.*	to each of these add the line DB;
6	Ax. 4.	then sides CE, ED, and DB together are > CD and DB together.
7	D. 3.	But AB and AC are > BE and EC;
8	*à fort.*	much more are AB and AC > CD and DB.
9	P. 16.	Again, in △ CDE the ext. ∠ BDC is > the int. ∠ CED,
10	P. 16.	and in △ ABE the ext. ∠ CED is > the int. ∠ BAC;
11	*à fort.*	much more is ∠ BDC > ∠ BAC.
12	Recap.	*Therefore, if from the ends of a side, &c.*

Q.E.D.

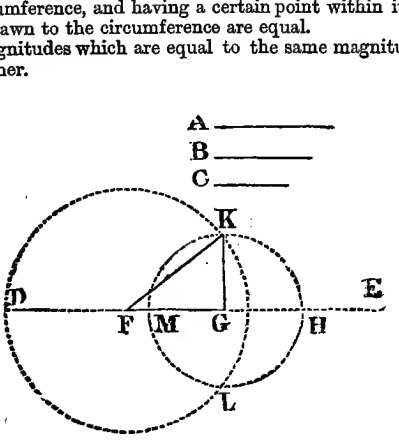
Use and Application.—In Optics this Proposition is used to prove that if from . A we could see the st. line BC, and also from D a point nearer to the line, the base BC would appear less from A than from D: it does this on the principle that quantities seen under a greater angle appear greater. For this reason the apparent diameter of the sun measures more when the earth is in *perihelion*, than when it is in *aphelion*. And thus—according to Vitruvius, who composed his work on Architecture about 15 B.C.—the tops of very high pillars should be made but little tapering, because they will, from the distance, of themselves seem less.

Prop. 22.—Prob.

To make a triangle, of which the sides shall be equal to three given st. lines, but any two whatever of these must be greater than the third.

Sol.—P. 3. From the greater st. line to cut off a part equal to the less.
Pst. 3. A circle may be drawn with any radius.
Pst. 1. A line may be drawn from point to point.

Dem.—Def. 15. A circle is a figure bounded by one continued line, called its circumference, and having a certain point within it from which all lines drawn to the circumference are equal.
Ax. 1. Magnitudes which are equal to the same magnitude, are equal to each other.



Exp. 1	*Data.*	Given the st. lines A, B, and C, of which A and B > C, A and C > B, and B and C > A;
2	*Quæs.*	to make a △ with sides = A, B, and C.
Cons. 1	*by Assum.*	Assume a st. line DE unlimited towards E;
2	P. 3.	make DF = A, FG = B, and GH = C;
3	Pst. 3.	from centre F with rad. DF draw the ⊙ DKL,
4	,, 3.	and from centre G with rad. GH the ⊙ HLK;
5	,, 1.	join the .s F, K and G, K;
6	Sol.	then the fig. KFG is a △, and side FK = side A, FG = B, and GK = C.
Dem. 1	*by* C3, Def. 15	∵ . F is the centre of ⊙ DKL, FK = FD;
2	C. 2 & Ax. 1.	but FD = st. line A, ∴ FK also = A.

DEM. 3	C.3, Def. 15	Again, ∵ G is the centre of ⊙ HKL, GH = GK;
4	C. 2 & Ax. 1.	but GH = C; ∴ also GK = C,
5	C. 2.	and FG is equal to B;
6	D. 2, 4, & 5.	∴ st. lines FK, FG, GK are respectively equal to st. lines A, B, and C.
7	Recap.	*And therefore the* △ FKG *has its three sides*, &c. Q.E.F.

SCH.—1. In this Proposition it is assumed that *the two circles will have a least one point of intersection.* See *last figure.*

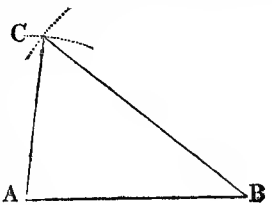
DEM. 1	*by* C. & Dat.	∵ sides FG and GK are > FK, FH > the rad. of DLK.
2		∴ the . H is outside of the ⊙ DLK.
3	C. & Dat. *Sub.*	Also, ∵ sides FK and GK are > FG, take away GK or GH;
4		∴ FM is < FK, and . M is within the ⊙ DLK.
5	D. 2 & 4.	Now, H being without, and M within, the ⊙s,
6	Concl.	therefore, the ⊙s intersect.

2. If two of the given lines were together equal to the third, *the circles would touch externally;* if the two were together less than the third, *the circles would not touch at all:* in either case no triangle could be drawn.

USE AND APP.—1. *All rectilineal figures being divisible into triangles,* this Proposition is of very extensive use in the construction of Geometrical figures, —either for making one rectilineal figure *equal* to another, or on the theory of Representative Values making one figure *like to* another: in the *first* case, the triangles into which the rectilineal figure has been divided are repeated, side for side, in another rectilineal figure of exactly the same linear dimensions: and when the construction is completed, if the one figure were placed on the other, the two would correspond, angle to angle, line to line, and point to point: in the *second* case, that of making one rectilineal figure similar to another; the sides and angles of the first must be measured, and from a scale of equal parts, lines drawn in the second, *representative* of those in the first, and angles in the second equal to those in the first;—for *equality of angles*, according to Definition 1, book vi., is essential to *similarity of figure.*

2. *Practically,* a triangle with sides equal to three given st. lines, will be drawn, by describing arcs, with radii equal to the sides, intersecting in . C, and joining CA, and CB; thus,

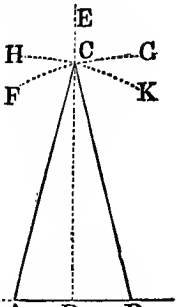
Given, three lines AB *equal to* 20, BC *to* 25, *and* CA *to* 15; *to form a triangle.*



From the scale of equal parts set off a line AB equal to 20; at A with the distance CA, 15, from the same scale, desc. an arc; and at B with the distance BC, desc. another arc: the arcs both intersect in C, and joining the points, ABC is the triangle required.

5. *On a given st. line,* AB, *to describe an isosceles triangle,* ACB, *having each of the equal sides,* AC, BC, *double of the base,* AB, *or equal to* BO.

Cons.	1	*by* P. 10, 11, Pst. 2.	Bisect AB in D, by the perp. DE, produced indefinitely to E.
	2	Pst. 3, P. 3.	From . A with rad. equal to twice AB, desc. the arc FCG;
	3	„ 3, „ 3.	and from B with the same rad. desc. the arc HCK;
	4	„ 1.	join the points CA and CB.
	5	Sol.	The fig. ACB is the isosc. △ required.
Dem.	1	*by* C. 1.	∵ side AD = side DB, DC common, and ∠ADC = ∠BDC,
	2	P. 4.	∴ the st. line AC equals the st. line BC,
	3	C. 2 & 3.	and they are each double of AB.
	4	Concl.	Wherefore, on a given line AB, &c. Q.E.F.



4. *On a given st. line,* AB, *and with a given side equal to* BO, *to describe an isosceles triangle.*

The same construction being made, and the same demonstration followed, we arrive at the conclusion—that side BC equals side CA, and that each of them is equal to the given side BO.

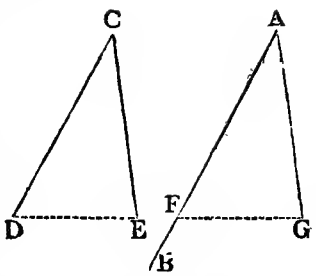
Prop. 23.—Prob.

At a given point in a given st. line to make a rectilineal angle equal to a given rectilineal angle.

Sol.—Pst. 1. A st. line may be drawn from one point to another.
P. 22. With three given st. lines, any two of which are greater than the third, to make a triangle.

Dem.—P. 8. If two triangles have the three sides of one equal to the three sides of another, each to each, the angle contained by two equal and conterminous sides of the one, shall be equal to the angle contained by any two equal and conterminous sides of the other.

EXP. 1	*Data.*	Given the . A in the st. line AB, and ∠ DCE;
2	*Quæs.*	required at . A in AB to make an ∠ = DCE.
CONS. 1	*by Assum. & Pst.* 1	In st. lines CD, DE take .s D and E, and join DE;
2	P. 22.	with side AF = CD, AG = CE, and FG = DE, make the △ AFG;
3	Sol.	then the ∠ FAG shall be = ∠ DCE.
DEM. 1	*by* C. 2.	∵ FA = DC, AG = CE, and DE = FG;
2	P. 8.	∴ the ∠ FAG = ∠ DCE,
3	Recap.	*Therefore, at* . A *in* AB, &c. Q.E.F.

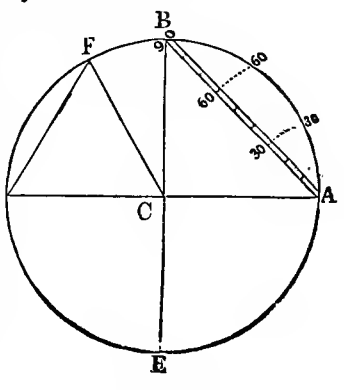


SCH.—This Proposition is an extension, or rather a generalisation, of the eleventh: by the eleventh *we draw an angle of a particular species*—a right angle; but by the twenty-third *we draw any angle whatever.*

USE AND APP.—1. This Proposition is of the widest use, in Surveying, Engineering, Perspective, and indeed in all the other parts of Practical Mathematics.

2. Next to the use of the Semicircle for measuring and making angles of a determinate magnitude, that of a *Line of Chords* is most important. Its *construction and use* we may here appropriately introduce.

Take a circle and draw a diameter DA; bisect DA in C, and at . C draw CB at rt. angles to DA, and produce BC to E;—the circle will be divided into four quadrants, each 90°. Let the arc AB of 90° be divided into arcs of 10°, 20°, 30°, &c., up to 90. Draw AB, the chord of 90°, and from . A, as a centre, set off on AB the chords of 10°, 20°, 30°, &c.: the divisions on the line AB will be a line of chords.



N.B.—*The chord of* 60° *is equal to the radius of the circle.*

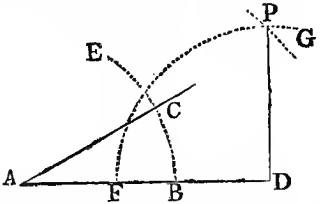
In triangle DCF, let the angle DCF measure 60°,—then, since CF and CD are equal, the angles CDF and CFD will be equal to each other,

each measuring 60°. The triangle, therefore, being equiangular, is also equilateral; and DF, the chord of 60°, equals the radius DC.

On this property depends the use of the Line of Chords in the following Problems.

PROB. I.—*To make an angle to contain a certain number of degrees, as* 30°.

Draw AD an indefinite st. line; from A with the radius equal to the chord of 60° describe an arc BE; and from the same line of chords take 30°, and set the distance from B on the arc BE: the distance BC will be the arc of 30°, and CAB the angle required.



PROB. II.—*An angle being given,* CAB, *to find the measure of it in degrees of a circle.*

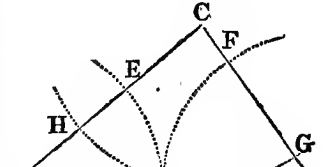
With the chord of 60°, from the line of chords, for radius, describe the arc BC; take the distance, *i.e.*, the chord, BC in the compasses, and apply the distance to the same line of chords: the number of degrees in the arc BC will be ascertained.

PROB. III.—*From the extremity* D, *of a st. line* AD, *to draw a perpendicular.*

From . D, with the chord of 60°, from the line of chords, for radius, describe the arc FG; take the chord of 90° for a radius, and from F describe another arc intersecting FG in P; draw PD,—and it is the perpendicular required.

PROB. IV.—*To construct a triangle of which the base* AB *contains* 30 *equal parts, the angle at* A 40°, *and the side* AC 25 *equal parts; and to find the angles* C *and* B, *and the other side* BC.

Take 30 from the scale of equal parts, and by P. 3, draw st. line AB; at . A, with the chord of 60°, desc. the arc DE, and from . D, with the chord of 40°, or of ang. A, cut the arc in . E; join AE,—and EAB is the required angle. Produce AE, and set on it 25, or AC; join CB,—and the triangle is completed.



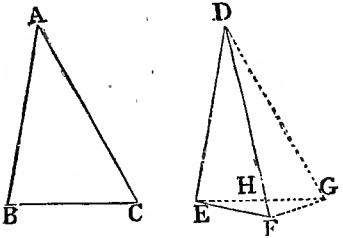
To measure the st. line CB, take the distance from C to B, and apply it to the same scale, which will give the equal parts 19: and to measure the angles,—with the chord of 60°, describe from . s C and B the arcs GH and DF; take in the compasses the distances G to H, and D to F: these distances applied to the line of chords will give the angle C 85°, and the angle B 55°.

PROP. 24.—THEOR.

If two triangles have two sides of the one equal to the two sides of the other, each to each, but the angle contained by the two sides of the one greater than the angle contained by the two sides of the other, the base of that which has the greater angle shall be greater than the base of the other.

CONS.—P. 23. To make an angle equal to a given angle.
P. 3. From the greater st. line to take a part equal to the less.
Pst. 1. Any two points may be joined by a st. line.

DEM.—P. 4. When two sides and the included angle of one triangle are equal to the two sides and the included angle of another triangle, the triangles are equal in every respect.
P. 5. The angles at the base of an isosceles triangle are equal.
Ax. 9. The whole is greater than its part.
P. 19. The greater angle of every triangle is subtended by the greater side.



EXP. 1	Hyp. 1.	In $\triangle$s ABC, DEF, let AB = DE, and AC = DF;
2	„ 2.	but $\angle$ BAC > $\angle$ EDF;
3	Concl.	then the base BC > the base EF.
CONS. 1	*by* Hyp. & P. 23	Let side DF $\not<$ DE, and make $\angle$ EDG = $\angle$ BAC:
2	P. 3 & Pst. 1.	also make DG = AC or DF, and join EG, GF.
DEM. 1	*by* Hyp. & C. 1.	$\because$ AB = DE, AC = DG, & $\angle$ BAC = $\angle$ EDG
2	P. 4.	$\therefore$ base EG is equal to base BC.
3	C. 2, & P. 5.	And $\because$ DG = DF, in $\triangle$ DFG, $\therefore$ $\angle$ DGF = $\angle$ DFG.
4	Ax. 9.	But $\angle$ DGF is > EGF, and $\angle$ DFG > $\angle$ EGF;
5	*à fort.*	much more is $\angle$ EFG > $\angle$ EGF.

DEM. 6	P. 19.	But the greater angle is opposite to the greater side ;
7	D. 5 & P. 19.	∴ the side EG is > side EF.
8	D. 2.	But EG = BC, and ∴ BC is > EF.
9	Recap.	*Therefore, if two triangles have two sides, &c.* Q.E.D.

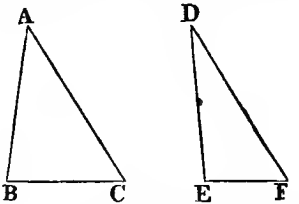
SCH.—In this Proposition it is assumed that D and F will be on different sides of EG ; or in other words, that DH is less than DF or DG.

PROP. 25.—THEOR.

If two triangles have two sides of the one equal to the two sides of the other, each to each, but the base of one greater than the base of the other, the angle contained by the sides of the one which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

DEM.—P. 4. If two triangles have two sides and the included angle in each equal, the triangles are equal in every respect.

P. 24. If two triangles have two sides equal in each, but of the included angles one greater than the other, the base of that which has the greater angle shall be greater than the base of the other.



EXP. 1	Hyp. 1.	In △s ABC, DEF, let side AB = side DE, and side AC = DF,
2	„ 2.	but the base BC > base EF ;
3	Concl.	then the ∠ BAC is > ∠ EDF.
DEM. 1	*by Sup.*	If ∠ BAC is ⋡ ∠ EDF, it is either equal or less :
2	*Sup.*	Suppose that ∠ BAC = ∠ EDF ;
3	Hyp. & P. 4.	then base BC = base EF :

DEM. 4	Hyp.	But B C is $\neq$ E F;
5	Concl.	$\therefore \angle$ B A C is $\neq \angle$ E D F.
6	*Sup.*	Again, suppose $\angle$ B A C $< \angle$ E D F;
7	H. & P. 24.	then base B C is < base E F:
8	Hyp.	But B C is $\not<$ E F;
9	Concl.	$\therefore \angle$ B A C is $\not< \angle$ E D F.
10	D. 5 & 9.	Now, $\angle$ B A C is, neither =, nor <, E D F;
11	Concl.	$\therefore \angle$ B A C is $> \angle$ E D F.
12	Recap.	*Therefore, if two triangles have two sides, &c.* Q.E.D.

SCH.—Propositions 24 and 25 have the same relation to each other as Props. 4 and 8, and the four may be combined thus:—*If two triangles have two sides of the one respectively equal to two sides of the other, the remaining side of the one will be greater or less than, or equal to, the remaining side of the other, according as the angle opposed to it in the one is greater or less than, or equal to, the angle opposed to it in the other; or vice versâ.*—LARDNER'S *Euclid*, p. 56.

USE AND APP.—The principal use of Props. 24 and 25 is to assist in the demonstration of the following propositions.

PROP. 26.—THEOR.—(*Important.*)

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side—viz., either the sides adjacent to the equal angles in each, or the sides opposite to them—then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other.

CONS.—P. 3. From the greater of two st. lines to cut off a part equal to the less.

Pst. 1. A st. line may be drawn from one point to another.

DEM.—P. 4. Two triangles are equal, when two sides and the included angle in one are equal to two sides and the included angle in the other.

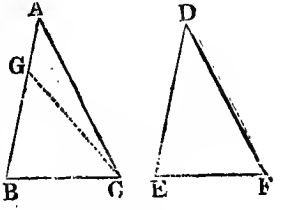
Ax. 1. Magnitudes which are equal to the same, are equal to each other.

P. 16. If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

EXP. 1	Hyp. 1.	In $\triangle$s ABC, DEF, let $\angle$ ABC = $\angle$ DEF, and $\angle$ ACB = $\angle$ DFE;
2	„ 2.	also let one side = one side;
3	Concl. 1.	then the other sides shall = the other sides.
4	„ 2.	and the third $\angle$ of the one = the third $\angle$ of the other.

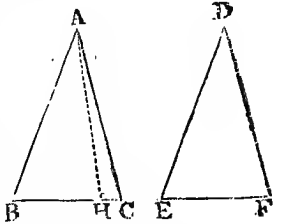
CASE I.—*Let the equal sides be adjacent to the equal angles.*

EXP. 1	Hyp.	Let side B C = side E F;
2	Concl. 1.	then A B = D E, and A C = D F,
3	„ 2.	and third ∠ BAC = third ∠ E D F.
CONS. 1	*by Sup.*	For if A B is ≠ D E, one is > the other.
2	*Sup.*	Let A B be > D E:
3	P. 3 & Pst. 1	Make B G = D E, and join G C.
DEM. 1	*by* C. 3 & Hyp	∵ in △s, G B C, D E F, B G = E D, B C = E F, and ∠ G B C = ∠ D E F;
2	P. 4.	∴ base G C = base D F, △ G B C = △ D E F, and ∠ B C G = ∠ D F E:
3	Hyp.	But ∠ D F E = ∠ B C A;
4	Ax. 1.	∴ ∠ B C G = ∠ B C A, the less = the greater, *which is impossible.*
5	Concl.	∴ A B is not ≠ D E, *i.e.*, A B does = D E;
6	D. 5 & Hyp.	Hence in △s A B C, D E F, ∵ A B = D E, B C = E F, and ∠ A B C = ∠ D E F;
7	P. 4.	∴ base A C = base D F, and ∠ B A C = ∠ E D F



CASE II.—*Let the equal sides be opposite the equal angles in each.*

EXP. 1	Hyp.	Let A B = D E, ∠ B = ∠ E, and ∠ A C B = ∠ D F E;
2	Concl. 1.	then A C shall = D F, B C = E F,
3	„ 2.	and ∠ B A C = ∠ E D F.
CONS. 1	*by Sup.*	For if B C ≠ E F one of them is the greater.
2	*Sup.*	Let B C be greater than E F;
3	P. 3, Pst. 1.	Make B H = E F, and join A H.
DEM. 1	*by* C. & Hyp	∵ in △s A B H, D E F, B H = E F, A B = D E, and ∠ A B H = ∠ D E F;

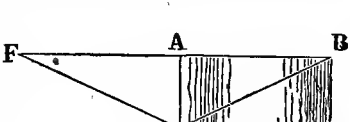


DEM. 2	P. 4.	∴ base AH = base DF, △ ABH = △ DEF, and ∠ BHA = ∠ EFD:
3	Hyp.	But ∠ EFD = ∠ BCA;
4	Ax. 1.	∴ ∠ BCA = ∠ BHA, or the int. = the ext. ang.;
5	P. 16.	which is impossible:
6	Concl.	∴ BC is not ≠ EF, *i.e.*, BC = EF.
7	D. 6 & Hyp.	Hence in △s ABC, DEF, ∵ AB = DE, BC = EF, and incl. ∠ ABC = incl. ∠ DEF;
8	P. 4.	∴ base AC = base DF, and 3rd ∠ BAC = 3rd ∠ EDF.
9	Recap.	*Wherefore, if two triangles have two angles of the one, &c.* Q.E.D.

SCH.—1. This is the last of the *three criteria* for inferring the equality of triangles; the *first* criterion contained in Prop. 4, is—*that the two sides and the included angle of each triangle, be equal;* the *second*, in Prop. 8, *that the three sides of one triangle equal the three sides of the other;* and the *third*, in Prop. 26, *that in each triangle two angles be equal and one side, either adjacent to the equal angles, or opposite to one of them.*

2. Of the *six parts* of a triangle *any three, except the three angles*, being given equal to one another, *the equality of the other parts* may be readily proved.

USE AND APP.—1. The discovery of this Proposition is ascribed to THALES, who made use of it *for measuring inaccessible distances*, as from station A to station B across a river.



At the station A set out a st. line AC, perpendicular to AB; and at C observe the ang. ACB, and by P. 23 make the ang. ACF equal to ang. ACB; the lines BA and CF produced meet in the . F; and AF will be found equal to AB.

For in triangles ACB, ACF, by construction, we have the angles at A equal,—the angles ACB and ACF equal,—and the side AC common; therefore, by P. 26, AF will equal AB; if now we measure AF, we have a measurement equal to that of AB.

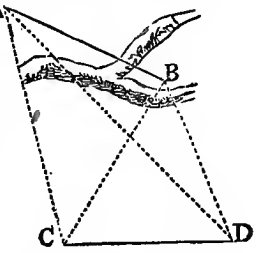
Another construction might be used, by producing st. line AC to D, and making CD equal to CA; then at . D draw DE perp. to AD, and produce BC, till it cuts DE in . E; in this case DE may be proved equal to the distance from A to B.

2. *By the Theory of Representative Values the distance of two stations* A *and* B *may be found.* Suppose AC to measure 70 feet or yards, ang. A 90°, ang. ACB 66°; at . A make a right angle, and on AC set off 70 equal parts; at . C draw an angle of 66°; the line CB will cut off B, the distance from A: let the distance from A to B be applied to the same scale, and AB will be found equal to about 150 feet or yards, according to the unit of length fixed on for AC.

3. *When neither* A *nor* B *are accessible*, a process rather different has to be employed, founded, however, on strictly geometrical principles.

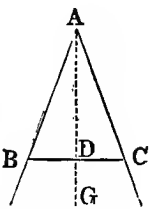
Select two stations, C and D, from both of which A and B are visible; and measure the st. line CD, suppose 150; take the angles BCD 60°, and BDA, 22½°; ADC, 45°; and ACB, 45°: it is required to ascertain the distance AB.

From a scale of equal parts make CD 150; draw the angles BCD 60°, and BDA 22½°; the st. lines CB and DB meet in B: again draw the angles ACB 45°, and ADC 45°; the st. lines DA and CA meet in A: take the distance AB in the compasses, and apply it to the same scale;—AB will be found to be about 153.



4. *Given the vertical angle* A, *and the perpendicular height* H, *of an isosceles triangle, to construct it.*

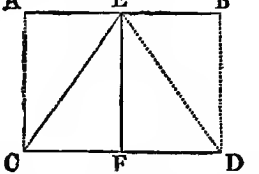
Cons. 1	*by* P. 9 & 3.	Bisect ∠ A by AG, and cut off AD = H;	
2	P. 23.	through D draw BC at ⊥ to DA:	
3	Sol.	the figure ABC is the isosc. △ required.	H
Dem. 1	*by* C. 1 & 2.	∵ ∠BAD = ∠CAD, ∠BDA = ∠CDA,	
2	C.	and AD, which is equal to H, is common;	
3	P.26,Def.25	∴ AB = AC, and ABC is an isosc. △.	
4	Recap.	*Therefore, with the vertical angle* A *given*, &c.	Q.E.F.



Lemma.

A st. line which is Perpendicular to one Parallel, is also Perpendicular to the other.

Exp. 1	Hyp.	Let AB be ‖ CD, and EF perpen. to CD;
2	Concl.	then EF shall also be perpen. to AB.
Cons. 1	*by* P. 3 or 10	Make CF = FD;
2	P. 11.	at C and D erect perpendiculars CA and DB.



CONS. 3	D. 35, Ax. 12	which by the note to the definition of parallels equal EF;
4	Pst. 1.	and draw EC, ED.
DEM. 1	*by* C. & Hyp.	In $\triangle$ CEF, DEF, the line CF = FD, FE common, and $\angle$ CFE = $\angle$ DFE
2	P. 4.	$\therefore$ side EC = side ED, $\angle$ FED = $\angle$ FEC, and $\angle$ FDE = $\angle$ FCE;
3	*Sub.*	take away the equals ECF and EDF from the equals ACF and BDF;
4	Ax. 3.	$\therefore$ the rem. $\angle$ ECA = the rem. $\angle$ EDB.
5	P. 4.	Now $\triangle$s CAE, DBE shall have $\angle$ CEA = $\angle$ DEB;
6	*Add.*	Adding the equal angles DEB, CEA to the equals DEF, CEF,
7	Ax. 2.	$\therefore$ the whole $\angle$ BEF = the whole $\angle$ AEF;
8	Const.	and these angles are adjacent:
9	Def. 10.	$\therefore$ EF is also perpendicular to AB.
10	Recap.	*Wherefore, a line which is perpendicular*, &c. Q.E.D.

PROP. 27.—THEOR.

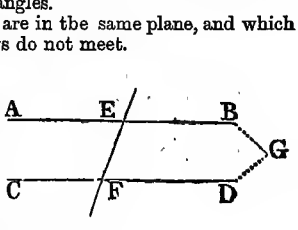
If a st. line falling upon two other st. lines makes the alternate angles equal to one another, these two st. lines shall be parallel.

CONS.—Pst. 2. A st. line may be produced indefinitely in a st. line.

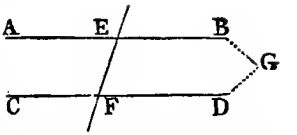
DEM.—P. 16. If one side of a triangle be produced, the ext. angle is greater than either of the interior opposite angles.

Def. 35. Parallel st. lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

EXP. 1	Hyp. 1.	Let EF fall on the two st. lines AB and CD,
2	,, 2.	and make the altr. $\angle$ AEF = the altr. $\angle$ EFD;
3	Concl.	then the st. line AB is $\parallel$ CD.



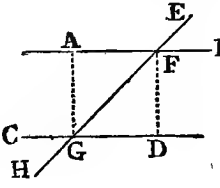
Cons. 1	*by Sup.*	If AB and CD are not ‖,
2	Def. 35.	on being produced they will meet towards A, C, or B, D;
3	Pst. 2 & *Sup.*	let them be produced, and if possible meet in G, and form a △ EFG.



Dem. 1	*by* C. 3 & P. 16	∵ fig. GEF is a △, the ext. ∠ AEF > the int. ∠ EFG;
2	Hyp.	but ∠ AEF = ∠ EFG;
3	D. 1 & 2.	∴ ∠ AEF is both > and = ∠ EFG,
4	*ad imposs.*	which is *impossible:*
5	Concl.	∴ AB, CD do not meet towards B, D.
6	*Sim.*	In the same way it may be shown, they do not meet towards A, C:
7	Def. 35.	∴ AB is parallel to GD.
8	Recap.	*Wherefore, if a st. line falling*, &c. Q.E.D.

Sch.—Since, however, there are some curved lines which are not parallels, though they never intersect, as in the Introduction, § vi., p. 30, another demonstration may he made.

Exp. 1	Hyp.	Let EH fall on AB, CD, and make the altr. ∠s AFG and FGD equal;
2	Concl.	then the lines AB and CD are parallel.
Cons. 1	*by* P. 12.	At G draw GA perpend. to AB;
2	P. 3, Pst. 1	take GD = AF, and join FD.



Dem. 1	*by* C 2	In △s AGF, DFG, side GD = side AF, GF common,
2	Hyp.	and ∠ AFG = ∠ FGD;
3	P. 4.	∴ side AG = side DF, and ∠ GDF = ∠ GAF:
4	C. 1.	But ∠ GAF is a rt. angle.
5	Def. 10.	∴ ∠ GDF is a rt. angle, and DF perpendicular to CD.
6	Hyp.	Moreover, the par. to CD is to be drawn from F;
7	D. 3.	and since the perpendiculars AG and DF are equal,
8	Ax. 12, *n.*	the parallel to CD must pass through . A:
9	Concl.	∴ st. line AB is parallel to st. line CD. Q.E.D.

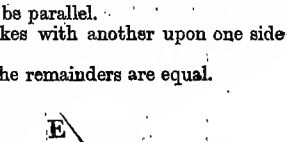
Sch.—Of the angles which two st. lines, one at each extremity of a third st. line, make with it, the *alternate angles* are those which are *on opposite sides* and *at opposite extremities* of the third line.

PROP. 28.—THEOR.

If a st. line falling upon two other st. lines makes the exterior angle equal to the interior and opposite angle upon the same side of the line; or makes the interior angles upon the same side together equal to two rt. angles; the two st. lines shall be parallel.

DEM.—P. 15. If two st. lines cut one another, the vertical angles shall be equal.
Ax. 1. Magnitudes equal to the same are equal to one another.
P. 27. If a st. line falling upon two other st. lines makes the alternate angles equal, these two st. lines shall be parallel.
P. 13. The angles which one st. line makes with another upon one side of it, are equal to two rt. angles.
Ax. 3. If equals be taken from equals, the remainders are equal.

EXP. 1	Hyp. 1.	Let the st. line EF fall on the two st. lines AB, CD;
	„ 2.	make the ext. ∠ EGB = the int. and opp. ∠ GHD;
	„ 3.	and the int. ∠s BGH + GHD = two rt. angles;
2	Concl.	then the st. line AB is ‖ the st. line CD.



DEM. 1	*by* Hyp.	For, ∵ ∠ EGB = ∠ GHD,
2	P. 15.	and ∠ EGB = ∠ AGH;
3	Ax. 1.	∴ ∠ AGH = GHD; and they are altr. ∠s;
4	P. 27.	and ∴ also st. line AB is ‖ CD.
5	Hyp.	Again, ∵ ∠s BGH + GHD = two rt. angles,
6	P. 13.	and ∠s AGH + BGH = two rt. angles;
7	Ax. 1.	∴ ∠s AGH + BGH = ∠s BGH + GHD:
8	*Sub.*	Take away the common ∠ BGH,
9	Ax. 3.	∴ the rem. ∠ AGH = the rem. ∠ GHD:
10	Cons.	But ∠s AGH and GHD are altr. angles;
11	P. 27.	∴ the st. line AB is ‖ the st. line CD.
12	Recap.	*Therefore, if a st. line falling upon the other,* &c.

Q.E.D.

SCH.—1. When one st. line cuts, or falls upon, two other st. lines,—*of the four angles* formed on each side of the incident line, *taken in succession*, each pair consists of an ext. and an int. adjacent angle; but *taken alternately*, each pair consists of an ext. and an int. opposite angle.

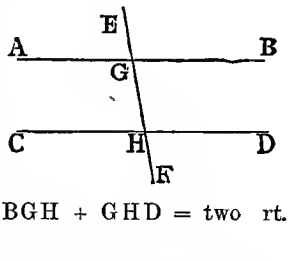
2. The principle really assumed in the twelfth Axiom is—*that two st. lines which intersect each other cannot both be parallel to the same st. line.*

Prop. 29.—Theor.

If a st. line fall upon two parallel st. lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite angle upon the same side; and likewise the two interior angles upon the same side together equal to two rt. angles.

Dem.—Ax. 4. If equals be added to unequals, the wholes are unequal.
P. 13. The angles which one st. line makes with another upon one side of it are either two rt. angles, or are together equal to two rt. angles.
Ax. 12. If a st. line meets two st. lines, so as to make the two int. angles on the same side of it together less than two rt. angles, these st. lines being continually produced, shall at length meet upon that side on which are the angles less than two rt. angles.
Def. 35. Parallel st. lines are such as are in the same plane, and which being produced ever so far both ways do not meet.
P. 15. If two st. lines cut one another, the vertical angles are equal.
Ax. 1. Magnitudes equal to the same, are equal to each other.

Exp. 1	Hyp.	Let the st. line EF fall on the ‖s AB, CD;
2	Concl. 1.	then the ∠ AGH = the altr. ∠ GHD;
	„ 2.	the ext. ∠ EGB = the int. opp. ∠ GHD;
	„ 3.	and the int. ∠s BGH + GHD = two rt. angles.



Dem. 1	*Sup.* 1.	If ∠ AGH is ≠ altr. ∠ GHD, one is > the other:
2	„ 2.	Let ∠ AGH be > ∠ GHD;
3	*Add.*	and to each angle let the ∠ BGH be added;
4	Ax. 4.	then ∠s AGH, BGH together are > ∠s BGH, GHD:
5	P. 13.	But ∠s AGH, BGH together are = two rt. angles;
6	Ax. 1.	∴ ∠s BGH, GHD together are < two rt. angles.
7	Ax. 12.	Now on this condition st. lines AB and CD, being produced, will meet.

DEM. 8	Hyp.Def.35	but st. line AB is parallel to st. line CD;
9	Concl. 1.	$\therefore$ $\angle$ AGH is not $\neq$, *i.e.*, is =, $\angle$ GHD:
10	P.15 & Ax. 1	But $\angle$ AGH = EGB, $\therefore$ $\angle$ EGB = $\angle$ GHD;
11	*Add.*	to each angle add $\angle$ BGH:
12	Ax. 2.	then $\angle$s EGB + BGH = $\angle$s GHD + BGH:
13	P. 13.	But $\angle$s EGB + BGH = two rt. angles;
14	Ax. 1.	$\therefore$ $\angle$s GHD + BGH = two rt. angles.
15	Recap.	*Wherefore, if a st. line fall on two parallel st. lines*, &c. Q.E.D.

SCH.—1. This Proposition, depending on the 12th Axiom, is the converse of Props. 27 and 28.

2. It has been objected to the twelfth axiom that it is not self-evident, and that it forms the converse to Prop. 27: now, both the assumed axiom and its converse should be so obvious as not to require demonstration.

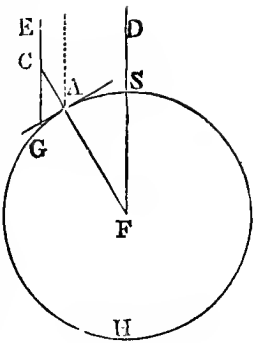
3. "The *Twelfth Axiom* may be expressed in any of the following ways:—*Two diverging rt. lines cannot be both parallel to the same rt. line:* or, *If a rt. line intersect one of two parallel rt. lines, it must also intersect the other:* or, *Only one rt. line can be drawn through a given point parallel to a given rt. line.*" —LARDNER, p. 316.

4. The *least objectionable* definition of parallel st. lines is—"*Parallel lines are such as lie in the same plane, and which neither recede from nor approach to each other.*"—POTTS' *Euclid*, p. 50.

USE AND APP.—ERATOSTHENES, who died B.C. 196, at the age of 80, applied the last three Propositions *to the measuring of the circumference of the earth*, and followed the method still employed. (*See Dictionary of Greek and Roman Biography*, vol. ii., p. 44–47.) The principle with which he set out was, *that two rays of light proceeding from the centre of the sun to two points on the earth, are physically parallel.* He learned that "at Syene, in Upper Egypt, deep wells were enlightened to the bottom on the day of the summer solstice, and that vertical objects cast no shadows. He concluded, therefore, that Syene was on the tropic, and its latitude equal to the obliquity of the ecliptic, which he had determined to be 23° 51′ 20″: he presumed that it was in the same longitude as Alexandria, in which he was *out* about 3°, which is not enough to produce what would at that time have been a sensible error. By observations made at Alexandria, he determined the zenith of that place te be distant by the fiftieth part of the circumference from the solstice; which was equivalent to saying that the arc of the meridian between the two places is 7° 12′." The distance from Alexandria to Syene, Eratosthenes gives as 5000 stadia. From these data it is a very simple operation to deduce the earth's circumference; for *if an arc of* 7° 12′ *measures* 5000 *stadia*, the question to be solved is—*How many stadia are there in* 360°?

For the demonstration of the process, we suppose the circumference GSH, on the next figure, to be *representative* of the earth's circumference; Alexandria to be at the point A, and Syene at the point S; at A the style AC is erected

perpendicular to the horizon: DS and EC are rays of light from the sun when in the Solstice, the ray DS perpendicular to Syene, and continued towards F, the earth's centre. Now at A, take the angle ACG, made by AC the perpendicular to the horizon, and by the ray EG; then, because the rays DF and EG are parallel, and the line CF meets them, the alternate angles ACG and SFA are equal. Thus, if we measure the one, we at the same time ascertain the other. The arc which measures angle SFA is SA; and if the actual distance in miles from S to A be measured, we have the elements for finding the earth's circumference; *for as the degrees in the arc* AS *are to the distance measured from* A *to* S, *so are* 360° *to the measure in units of length of the whole circumference.*



Prop. 30.—Theor.

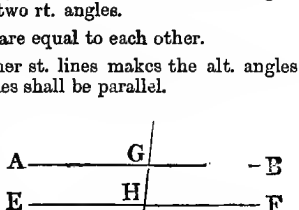
Straight lines which are parallel to the same st. line are parallel to each other.

Dem.—P. 29. If a st. line fall upon two parallel st. lines, it makes the alternate angles equal to one another; and the exterior angle equal to the int. and opp. angle upon the same side; and likewise the two int. angles upon the same side together equal to two rt. angles.

Ax. 1. Things equal to the same thing, are equal to each other.

P. 27. If a st. line falling upon two other st. lines makes the alt. angles equal to one another, these two st. lines shall be parallel.

Exp. 1	Hyp.	Let AB and CD be each ‖ to EF;
2	Concl.	then AB shall be ‖ to CD.
Cons. 1	*by* Sup.	Let the line GHK cut the lines AB, EF, and CD.



Dem. 1	*by* C. & Hyp.	∵ the line GHK cuts the ‖s AB, EF,
2	P. 29.	∴ the ∠ AGH = ∠ GHF;
3	C. & Hyp.	And, ∵ GK also cuts the ‖s EF, CD,
4	P. 29.	∴ the ∠ GHF = ∠ GKD:

DEM. 5	D. 2.	But $\angle AGH$ or $AGK = \angle GHF$.
6	Ax. 1.	$\therefore \angle AGK = \angle GKD$;
7	P. 27.	and $\therefore$ AB is $\parallel$ to CD.
8	Recap.	*Wherefore, straight lines which are parallel,* &c. Q.E.D.

SCH.—1. The corollary to this proposition, *that two lines parallel to the same line cannot pass through the same point,* is equivalent to the twelfth axiom.

2. The first axiom and this proposition are similar.

PROP. 31.—PROB.

To draw a st. line through a given point, parallel to a given st. line.

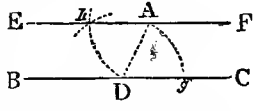
SOL.—Pst. 1. A st. line may be drawn from one point to another.

P. 23. At a given point in a st. line to make an angle equal to a given angle.

Pst. 2. A terminated st. line may be produced to any length in a st. line.

DEM.—P. 27. If a st. line falling upon two other st. lines make the alternate angles equal, these two st. lines are parallel.

EXP. 1	*Data.*	Let A be the given point, and BC the given line;
2	*Quæs.*	to draw through . A a line $\parallel$ to BC.



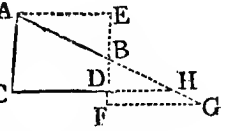
CONS. 1	*by Sup.* & Pst. 1	In BC take any point D, and join A, D;
2	P. 23.	At . A in st. line AD make the $\angle DAE = \angle ADC$;
3	Pst. 2.	produce st. line EA to F;
4	Sol.	then EF, through . A, is $\parallel$ to BC.
DEM. 1	*by* C. 1, 2.	$\because$ st. line AD meets st. lines EF and BC, and makes $\angle EAD$ = alt. $\angle ADC$,
2	P. 27.	$\therefore$ EF is $\parallel$ BC.
3	Recap.	*Wherefore, through a given point,* &c. Q.E.F.

SCH.—1. Practically, a parallel to BC through the point A, may be drawn with a *parallel ruler,* or with *the triangular square.* If the compasses are used, the way is (*see the last figure*)—join A and any point, as D, in BC: from . D with rad. DA describe the arc A*g*, and from A with AD the arc D*h*; then,

with the distance from g to A in the compasses, from D cut the arc Dh in h; join the Ah and produce it;—EF is the parallel required.

2. The *Twelfth Axiom*, that if a st. line falling on two other st. lines make the interior angles on the same side less than two rt. angles, those two st. lines on being produced shall intersect, *may very readily be demonstrated by the principles now established.*

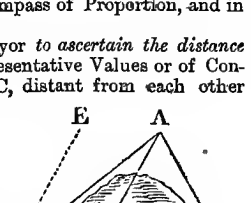
Exp. 1	Hyp. 1.	Let AC falling on AB, CD, make the int. ∠s ACD, CAB < two rt. angles;
2	„ 2.	and let ∠s ACD, CAE = two rt. angles:
3	Concl. 1.	then AB and CD being produced shall intersect
4	„ 2.	and st. lines AE and CD shall be parallels.



Cons. 1	*by Assum.* & P. 31	In AB take any point B, and through B draw EF ‖ to AC;
2	P. 3.	Along EB produced set EB, as many times as will make the distance EB repeated fall below CD; in the present instance EF equalling twice EB, or EB = BF:
3	P. 31 and 3.	From F draw FG ‖ CD or CH, and also ‖ AE;
4	Concl.	the line ABG is but one st. line.
5	Def. 35, C.	And ∵ CD cannot cut its ‖ FG, and the line AB concurs with FG;
6	Concl.	∴ CD shall cut AG between B and G.
Dem. 1	*by* C. 3 & 2, P. 29.	In △s AEB, BFG, the side AE = FG, BE = BF, and ∠ AEB = ∠ BFG:
2	P. 4.	therefore, the ∠ EBA = ∠ FBG:
3	Cor. P. 15.	consequently, AB and BG make one st. line, and CD being produced cuts AG in the point H.

Use and App.—1. The drawing of parallel lines is required in Perspective, Navigation, the Construction of the Sector, or Compass of Proportion, and in other branches of Practical Mathematics.

2. The use of *parallel lines* enables the Surveyor *to ascertain the distance of an inaccessible object*, by the method of Representative Values or of Construction: thus, there are three objects, A, B, C, distant from each other AC 6 miles, AB 8, and BC 9·4 miles; at D, a station on the line BC, the angle CDA is found to be 60°: the distance from D to A is required.



From a scale of equal parts construct the triangle ABC; and at B make the angle CBE equal to 60°: AD, drawn parallel to BE, is the representative value of the distance from D to A. Taking DA in the compasses, and applying the distance to the scale, the measurement of DA will be found equal to 6 miles.

PROP. 32.—THEOR.—*(Very Important.)*

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.

CON.—P. 31. To draw a st. line, through a given point, parallel to a given st. line.

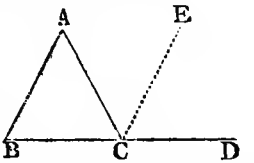
DEM.—P. 29. A st. line falling on two parallel st. lines makes the alternate angles equal, and the exterior angle equal to the int. and opp. angle upon the same side; and likewise the two interior angles on the same side equal to two rt. angles.

Ax. 2. If equals be added to equals, the wholes are equal.

P. 13. The angles which one st. line makes with another upon one side of it, are either two right angles, or together are equal to two rt. angles.

Ax. 1. Magnitudes equal to the same, are equal to each other.

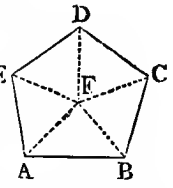
EXP. 1	Hyp.	Let the side BC of the $\triangle$ ABC be produced to D;
2	Concl. 1.	then the ext. $\angle$ ACD = the int. and opp. $\angle$s CAB, ABC;
	„ 2.	and the three int. $\angle$s ABC, ACB, and CAB together = two rt. angles.
CONS. 1	*by* P. 31.	Draw from . C a line CE $\parallel$ BA.
DEM. 1	*by* C.&P. 29	$\because$ AB is $\parallel$ to CE, and AC meets them, the $\angle$ BAC = alt. $\angle$ ACE:
2	C.	Again, $\because$ AB is $\parallel$ CE, and BD falls on them,
3	P. 29.	$\therefore$ the ext. $\angle$ ECD = the int. and opp. $\angle$ ABC:
4	D. 1.	But $\angle$ ACE = $\angle$ BAC:
5	Ax. 2.	$\therefore$ ext. $\angle$ ACD = the int. and opp. $\angle$s BAC and ABC:
6	*Add.*	To each of these equals add the $\angle$ ACB;
7	Ax. 2.	then $\angle$s ACD + ACB = the 3 $\angle$s BAC, ABC, and ACB:
8	P. 13.	But $\angle$s ACD + ACB = two rt. angles.
9	Ax. 1.	$\therefore$ $\angle$s BAC, ABC, ACB together = two rt. angles.
10	Recap.	*Wherefore, if a side of a triangle be produced,* &c. Q.E.D.



COR. I.—*All the interior angles of any rectilineal figure, together with four rt. angles, are equal to twice as many rt. angles as the figure has sides.*

DEM.—P. 15, Cor. 2. All the angles made by any number of lines meeting at a point, are together equal to two rt. angles.

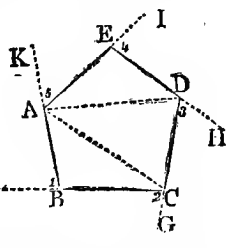
EXP. 1	Hyp. 1.	Let ABCDE be a rectil. figure, and F a point within it;
	„ 2.	and let lines be drawn from F to the ang. points A, B, C, D, E;
2	Concl.	then all the int. ∠s together with four rt. angles = twice as many rt. angles as the figure has sides.



DEM. 1	*by* Hyp.	∵ the fig. ABCDE is divided into as many triangles as there are sides,
2	P. 32.	and ∵ the ∠s in each △ = two rt. angles;
3	D. 1 & 2.	∴ all the angles of the triangles = two right angles × the number of sides:
4	Hyp.	But the same ∠s equal the angles of the fig. + the angles at F;
5	P.15,Cor.2	and all the ∠s at a point, as F, = four rt. angles;
6	Concl.	∴ the ∠s of the fig. + four rt. angles = two rt. angles × the number of sides. Q.E.D.

COR. II.—*All the exterior angles of any rectilineal figure, as* ABCDE, *are together equal to four rt. angles.*

CONS. 1	Pst. 2.	Produce all the sides of the fig. ABCDE:
DEM. 1	*by* P. 13.	∵ each int. ∠ + the adj. ext. ∠ = two rt. angles;
2	P.32,Cor.1	∴ all the int. ∠s + all the adj. ext. ∠s = two rt. ∠s × the number of sides;
3	Ax. 1.	and ∴ all the int. ∠s + all the ext. ∠s = all the int. ∠s + four rt. angles.
4	*Sub.*	from these equals take away all the int. ∠s.
5	Ax. 2.	∴ the ext. ∠s = four rt. angles. Q.E.D.



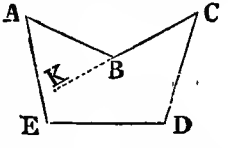
COR. III.—*If two triangles*, ABC, ADE, *have two angles of the one*, ABC, ACB, *respectively equal to two angles of the other*, AED, ADE; *then the third angle of the one*, BAC, *shall be equal to the third angle of the other*, DAE. (See the last figure.)

DM. 1	*by* Hyp.	The ∠s ABC + ACB = ∠s AED + ADE;
2	P. 32.	and the ∠s ABC + ACB + BAC = two rt. angles;
3	P. 32.	Also the ∠s AED + ADE + DAE = two rt. angles:
4	Sub.	take away the equals ABC + ACB and AED + ADE;
5	Ax. 3.	∴ the rem. ∠ BAC = the rem. ∠ DAE.

Q.E.D.

SCH.—1. The Corollaries to be deduced from Prop. 32, are very numerous: Lardner's *Euclid* gives twenty-four.

2. The first Cor. is of universal extent, applying to all rectilineal figures; the second is applicable only to figures that are called *convex*, in which each int. angle is less than two rt. angles. Some figures, as ABCDE, being *concave*, at B, have *re-entrant angles*, which are greater than two rt. angles: thus the int. angle at B is greater than two rt. angles by the angle ABK.

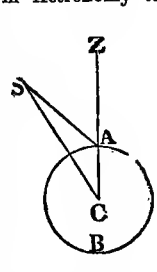


3. Without the production of a side, the three angles may be proved to be equal to two rt. angles, by drawing through any angular point a parallel to the opposite side.

4. In an equilateral triangle, each angle measures two-thirds of a rt. angle, or 60°; and in an isosceles triangle, if one angle is rt. angled, the other two are each 45°, or half a rt. angle.

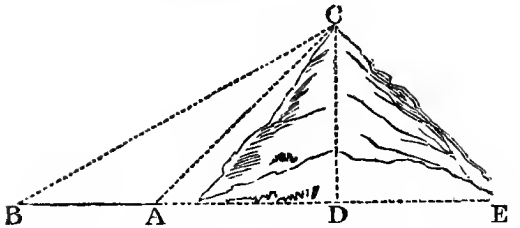
USE AND APP.—1. This Theorem may he employed in Astronomy *to determine the Parallax of a heavenly body.*

Let C represent the earth's centre, A a point on the earth's circumference AB, and Z the zenith of the station A; S is a star or any heavenly body not in the zenith.



By observation take the angle ZAS, *the zenith distance* at the earth's surface A: if the star S were viewed from C the centre, the angle ZCS would be *its zenith distance;* and the angle ZCS is less than angle ZAS. For, by P. 32, the ext. angle ZAS is equal to the angles C + S: thus, the angle S is equal to the excess of the angle ZAS above the angle ZCS. If now, from an Astronomical Table, I learn what is the zenith distance of S as viewed from C, the earth's centre, *the difference* betweeu angle ZAS and angle ZCS, or *the angle* ASC, *will be the parallax.*

2. By employing the same Theorem we may construct a figure which will give the representative value of the *perpendicular height of a mountain*, as CD, in the next figure.

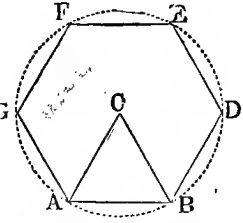


At station A with a theodolite, or other suitable instrument, ascertain the angle CAD—suppose 45°; measure AB, 300 feet, and at station B measure the angle CBD, 30°; and from these data construct the figure required.

From B draw a st. line of indefinite length BE, and take on it 300 equal parts to A: at . B draw an angle CBD of 30°; and at . A an angle CAD of 45°: the st. lines BC and AC intersect in the point C; and a perpendicular from C to BE, namely, CD, will represent the perpendicular height of the mountain. Take CD in the compasses, and apply the distance to the same scale; it will be found that CD measures 400, feet or yards, according to the unit of length in AB.

3. On the principle that all the interior angles of any rectilineal figure are equal to twice as many rt. angles as the figure has sides, diminished by four rt. angles for the amount of the angles at the centre,—*we are enabled to construct any regular right-lined figure.* The angular magnitude of each figure is equal to two rt. angles multiplied by the number of sides, *minus* four rt. angles; and the remainder divided by the number of sides or angles, gives each single angle of the regular figure. Let S equal the number of sides; 180° the measure of two rt. angles; and 360° the measure of four rt. angles; and, taking the figure below, let A represent the angle at each pair of sides, and C the angle at the centre by radii to the extremities of each side.

Then $A = \frac{(S \times 180°) - 360}{S}$; and $C = \frac{360}{S}$.



To construct a regular Polygon: Ascertain by the foregoing formula the angle at C, and the angle at A. 1st. *When the side* AB *is not given*—Draw any radius, as CA, and at . C, with another radius of the same length as the first, make the angle C as ascertained by the formula; then describe with rad. CA or CB the circle, and the distance AB will divide the circumference into as many parts as there are units in S: join the points, and the polygon is formed. 2nd. *When the side* AB *is given*—At . s A and B make angles equal to angle A, as ascertained by the formula; and bisect those angles by the st. lines AC and BC meeting in the point C; triangle CAB having the angles CAB and CBA equal, the sides CA and CB are also equal: if now, with CA or CB as radius, a circle be described, AB will cut off as many arcs from the circumference as there are units in S;—draw the chords to the respective arcs, and the polygon will be completed.

PROP. 33.—THEOR.

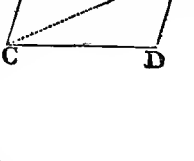
The st. lines which join the extremities of two equal and parallel st. lines towards the same parts, are also equal and parallel.

CONS.—Pst. 1. A st. line may be drawn from one point to another.

DEM.—P. 29. A st. line falling on two parallel st. lines makes the alternate angles equal.

P. 4. Two triangles having in each two sides and their included angle respectively equal, are equal in every other respect.

P. 27. If a st. line falling on two st. lines make the alternate angles equal, the two st. lines are parallel.

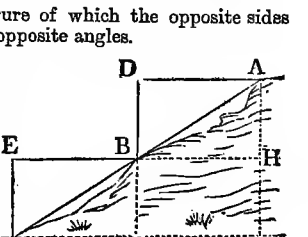


EXP. 1	Hyp. 1.	Let AB be = and ‖ to CD;
	„ 2.	and let the extremities towards the same parts be joined, A to C, and B to D;
2	Concl.	then the st. lines AC and BD are = and ‖.
CONS. 1	*by* Pst. 1.	Join B and C.
DEM. 1	*by* H.&P.29.	∵ BC meets the ‖s AB, CD, ∴ the ∠ ABC = the alt. ∠ BCD:
2	H. & D. 1.	Hence, in the two △s ABC, DCB, ∵ AB = CD, BC is common, and ∠ ABC = ∠ BCD,
3	P. 4.	∴ △ ACB = △ CBD, and base AC = base BD, and ∠ ACB = ∠ CBD.
4	D. 3.	And ∵ st. line BC with st. lines AC, BD, makes ∠ ACB = ∠ CBD,
5	P. 27.	∴ the st. line AC is ‖ to the st. line BD;
6	D. 3.	And AC has been shown equal to BD.
7	Recap.	*Wherefore, the st. lines which join the extremities, &c.* Q.E.D.

SCH.—A *Parallelogram* is a four-sided figure of which the opposite sides are equal and parallel, and the diagonals join opposite angles.

USE AND APP.—The principle contained in this Theorem enables us to *ascertain the perpendicular height of a mountain* AG, *as well as the distance from the base to the foot of the perpendicular*, CG.

Take a large right-angled triangle, ABD, and putting one end at A, let the side DB, by means of a plummet, be brought to the perpendicular; DB shows the altitude AH, and DA the horizontal distance BH: repeat



the process as often as the case requires;—then DA+EB, &c., equals the horizontal distance CG; and DB+EC, &c., the altitude AG.

PROP. 34.—THEOR.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them; that is, divides them into two equal parts.

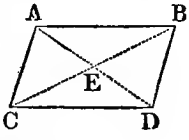
DEM.—P. 29. A st. line falling on two parallel st. lines makes the alternate angles equal.

P. 26. If two triangles have two angles and a side of one equal to two angles and a side of the other, the other sides and angle are respectively equal.

AX. 2. If equals be added to equals, the wholes are equal.

P. 4. If two triangles have two sides and the included angle equal in each, they are altogether equal.

EXP. 1	Hyp.	Let ABCD be a $\square$, and BC its diameter;
2	Concl. 1.	then $\angle A = \angle D$, and $\angle B = \angle C$;
	,, 2.	also $\triangle ABC = \triangle BCD$.



DEM. 1	*by* H.&P.29.	$\because$ BC meets the $\parallel$s AB and CD, $\therefore$ the $\angle ABC$ = the alt. $\angle BCD$;
2	H. & P. 29.	and $\because$ BC meets the $\parallel$s AC and BD, $\therefore$ the $\angle ACB$ = the alt. $\angle CBD$:
3	D. 1 & 2.	Hence, $\because$ $\angle$s ABC, ACB = $\angle$s BCD and CBD, and their adjacent side BC is common to the $\triangle$s ABC, BCD;
4	P. 26.	$\therefore \angle BAC = \angle BDC$, side AB = side CD, and side AC = side BD.
5	D. 1 & 2.	And $\because \angle ABC = \angle BCD$, and $\angle CBD = \angle ACB$,
6	Ax. 2.	$\therefore$ the whole $\angle ABD$ = the whole $\angle ACD$;
7	D. 4.	and $\angle BAC = \angle BDC$;
8	Concl.	$\therefore$ the opp. sides and angles of $\square$s are equal.
9	H. & D. 1.	Also, $\because$ AB = CD, BC is common, and $\angle ABC = \angle BCD$;
10	P. 4.	$\therefore$ the $\triangle ABC$ = the $\triangle BCD$. and diam. BC divides the $\square$ into two equal parts.
11	Recap.	*Wherefore, the opposite sides and angles of parallelograms,* &c. Q.E.D.

SCH.—1. In the last figure, the two diagonals, AD and BC, bisect each other.

2. The converse to Prop. 34 is—*If the opposite sides or opposite angles of a quadrilateral figure be equal, the opposite sides shall be parallel; i.e.*, the figure shall be a parallelogram.

3. Both the diagonals, AD and BC, being drawn, it may, with a few exceptions which require subsequent propositions, be proved, that a quadrilateral figure which has any *two* of the following properties, will also have the others :—

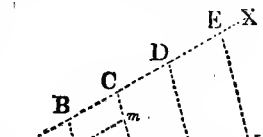
1°. The parallelism of AB and CD;
2°. The parallelism of AC and BD;
3°. The equality of AB and CD;
4°. The equality of AC and BD;
5°. The equality of the angles A and D;
6°. The equality of the angles B and C;
7°. The bisection of AD by BC;
8°. The bisection of BC by AD;
9°. The bisection of the area by AD;
10°. The bisection of the area by BC.

These ten data, being combined in pairs, will give 45 distinct pairs; with each of these pairs it may be required to establish any of the eight other properties, and thus 360 questions, respecting such quadrilaterals, may be raised. These questions will furnish to the student useful geometrical exercises. "The 9th and 10th data require the aid of subsequent propositions."—LARDNER'S *Euclid*, p. 49.

USE AND APP.—1. *The construction and accuracy of the parallel ruler* depends on Prop. 34.

2. *A finite st. line may be divided into any given number of equal parts.*

EXP. 1	*Data.*	Given a st. line AL, and the number of parts, as four;
2	*Quæs.*	to divide AL into those parts.
CONS. 1	*by* Pst. 1.	From one extremity of AL, as A, draw an indefinite st. line AX,
2	P. 3 & Pst. 1.	take AB, and make BC, CD, DE each = AB, and join EL;
3	P. 31.	through . s D, C, B draw D*d*, C*c*, B*b*, ‖s to EL;
4	Sol.	the line AL is so divided that A*b* = *bc*, *bc* = *cd*, and *cd* = *d*L.
5	P. 31.	Also through *b* draw *bm* ‖ AX.
DEM. 1	*by* C.3 & 5, P.34	∵ B*b* C*m* is a ▭, ∴ *bm* = BC or AB;
2	P. 29.	also ∠ A = ∠ *cbm*, and ∠ A*b*B = ∠ *bcm*:
3	P. 26.	therefore A*b* is equal to *bc*.
4	*Sim.*	And in the same way *dc* = *bc*, and *d*L = *cd*.

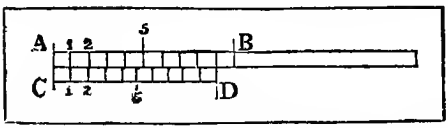


Q.E.D.

3. On the same principle the *Sliding Scale*, called from the inventors the *Vernier* or *Nonius*, is constructed. This scale is very useful, as in the

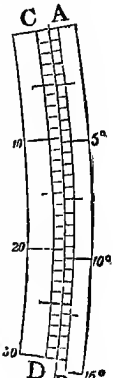
Barometer, for measuring the hundredth part of an inch ; or in the Theodolite, for measuring the minutes into which a degree is divided.

1°. *For measuring the hundredth part of an inch*, we may take, because of its clearness, *Ritchie's* description, in his *Geometry*, p. 32. "Suppose an inch AB,

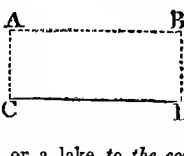


divided into 10 equal parts, each part will be $\frac{1}{10}$ of an inch. Again, suppose a st. line CD equal to 9 of these parts to be also divided into 10 equal parts, each of these will be $\frac{1}{10}$ of $\frac{9}{10}$ of an inch, which is $\frac{9}{100}$. But the length of one of the first divisions being $\frac{1}{10}$, or $\frac{10}{100}$, and that of the second $\frac{9}{100}$, one of the first divisions is $\frac{1}{100}$ of an inch longer than one of the second. If the st. line CD slide along parallel to AB till the two divisions marked 1, 1, form a continuous st. line, the sliding scale will have moved $\frac{1}{100}$ part of an inch towards B. If it slide along till the next two divisions coincide, it will have moved $\frac{2}{100}$ of an inch, &c."

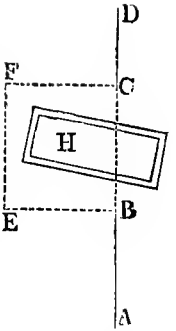
2°. *For measuring the minutes into which a degree on a circle is divided.* "If 59 degrees on the circumference or *limb* of the instrument be divided into 60 equal parts, the difference between the length of one degree and one of the latter divisions will obviously be $\frac{1}{60}$ of a degree, or one minute. This kind of vernier, on account of its great length, is seldom employed. If each degree on the arc AB of 15° be divided into two equal parts or half degrees, and if 29 of these be taken and divided into 30 equal parts, as in CD, the difference between the length of half a degree and one of the new divisions in CD, will obviously be $\frac{1}{30}$ of half a degree, or $\frac{1}{60}$ of a degree, that is, one minute. This is a very common vernier." If the limb CD move along the arc AB, so that the first division on CD and the first on AB are in one and the same st. line, the vernier will have moved $\frac{1}{2}$ of $\frac{1}{30}$, or $\frac{1}{60}$ of a degree, *i.e.*, one minute ; if the second division on CD coincide with the second on the arc AB, the vernier will have moved $\frac{2}{60}$ of a degree, or two minutes, &c.



4. *When two objects*, A *and* B, *are inaccessible from each other*, with an instrument to set out perpendiculars from them to a line CD, parallel to the line joining A and B, *the distance may be obtained ;* for, by Prop. 34, CD is equal to AB ; if therefore CD be measured, we learn the distance from A to B.

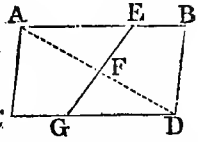


5. Also, *when there intervenes an obstacle*, as a house, or a lake, *to the continuation of a st. line* AB, beyond the obstacle H ; the direction of AB may be ascertained by the method of taking perpendiculars, and of drawing parallel st. lines.



At B raise a perpendicular BE, and at E another perpendicular EF; at F, a point beyond the obstacle, also draw a perpendicular FC; and set out FC of the same length as BE; then at C draw another perpendicular CD, and the st. line CD will be in the same direction with the st. line AB: and were H the obstacle removed, and C and B joined, ABCD would be in one and the same st. line. By construction BEFC is a rectangle, and the angles at C and B rt. angles; therefore, by Prop. 14, AB and CD will be in the same st. line with BC.

6. A field of the shape of *a parallelogram*, ABCD, *may be divided into two equal parts* by the diagonal AD; but if it has to be divided from the point E, bisect the diagonal in F, join EF, and produce EF to G; the st. line EG will divide the field into two equal portions. For in the triangles AFE, DFG, the angles EAF, AEF, are respectively equal to the angles FDG, FGD, by Prop. 29, and AF equal to FD; therefore, by Prop. 26, the triangles AFE and FDG are equal. And since the trapezium BEFD and the triangle AFE, by Prop. 30, together make up half the field ABD; the same trapezium with the triangle FGD, which is equal to the triangle AFE, will also make up half the field: therefore the st. line EG divides the field into two equal portions.



PROP. 35.—THEOR.

Parallelograms upon the same base and between the same parallels are equal, or rather equivalent, to one another.

DEM.—P. 34. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.

Ax. 6. Doubles of the same magnitude, equal each other.

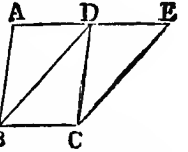
Ax. 1. Things equal to the same, are equal to one another.

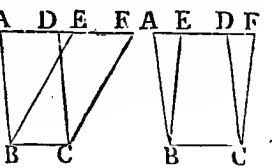
Ax. 3. If equals are taken from equals, the remainders are equal.

P. 29. A st. line falling on two parallel st. lines makes the exterior angle equal to the int. and opp. angle.

P. 24. Triangles with two sides and their included angle in each equal, are equal in all other respects.

Exp. 1	Hyp. 1.	Let the ▱s ABCD, EBCF, be on the same base BC,
	„ 2.	and between the same ‖s AF and BC;
2	Concl.	then the ▱ ABCD = the ▱ EBCF.
Case I.	*Sup.*	*Let the sides* AD, DE *opp. to* BC *be terminated in the same point* D.
Dem. 1	*by* P. 34.	Because ▱s ABCD, DBCE, are each double of △ DBC;
2	Ax. 6.	therefore ▱ ABCD = the ▱ DBCE.
Case II.	*Sup.*	*Let* sides AD *and* EF *be not terminated in one point* D.
Dem. 1	*by* H. & P. 34	∵ ABCD is a ▱, ∴ AD = BC;
2	H. & P. 34.	and EBCF being a ▱, EF = BC;
3	Ax. 1.	and ∴ AD = EF.
4	*Ad.* or *Sub.*	Add, or take away, the common part DF;
5	Ax. 2 or 3.	∴ the whole, or remainder, AE = the whole, or rem., DF.
6	P. 34.	And ∵ in △s EAB, FDC, AE = DF, and A B = DC,
7	P. 29.	and the ext. ∠ FDC = the int. ∠ EAB;
8	P. 4.	∴ base EB = base FC, and △ EAB = △ FDC.
9	*Sub.*	From trapezium ABCF take away the equal △s EAB and FDC;
10	Ax. 3.	the rem. ▱ ABCD = the rem. ▱ EBCF.
11	Recap.	*Therefore, parallelograms upon the same base,* &c. Q.E.D.



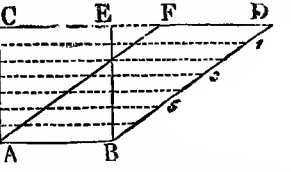


Sch.—1. The equality denoted is equality of surface or area, not equality of sides and angles.

2. Buonaventura Cavalieri, who was a pupil of Galileo, and died A.D. 1647, invented *the Method of Indivisibles*, one of the methods which preceded that of Fluxions. "He considers a st. line as composed of an infinite number of points; a surface, of an infinite number of lines; and so on." "This method,

absolutely considered, is defective and even erroneous; but the error is of the same kind as that of Leibnitz, who considered a curve as composed of an infinite number of infinitely small chords; and a surface, of infinitely small rectangles. The error in both is one which does not affect the result, for this reason—that it consists in using the simplifying effect of a certain supposition too early in the process, by which the logic of the investigation may be injured, but the result is not affected."—*Penny Cyclop.*, vi., p. 387.

The equality of parallelograms may be explained by this method of Indivisibles, which consists in the supposition that Surfaces are composed of lines, like so many threads in a piece of cloth. Now two pieces of cloth are equal, if in each there be found the same number of threads equal in length, and that the threads are as closely woven in one piece as in the other; and this will be true, though one piece be a rhomboid, and the other a square. For let A B C E represent a square piece of cloth, and A B D F a rhomboidal piece, on the same width A B, and between the same parallels A B, C D. If in the square A B C E as many lines or threads as we please be drawn parallel and equal to A B, as 1, 3, 5, and these lines or threads be produced into and across the rhomboid A B D F, there will be no more lines or threads in the one than in the other; the lines or threads are all of equal length, being equal to the same line or thread A B; and in the one the lines or threads are the same in number, or set as close together, as in the other: consequently, the two pieces of cloth, or *the square and rhomboid on the same base* A B, *and between the same parallels* A B and C D, *are equal in surface or area.*



Use and App.—In the measurement of Surfaces or Areas, the unit of surface is a rectangle; and it is therefore necessary to convert all parallelograms which are not rectangles into rectangles, in order to find their areas. The following method enables us *to convert the parallelogram* A B D F *into a rectangle* A B C E, of equal area.

Produce indefinitely the parallel D F, and at B and A, the extremities of the other parallel A B, raise, by Prop. 11, the perpendiculars B E and A C; then A B D F will be converted into a rectangle A B C E, which, by Prop. 35, is *equivalent* to A B D F.

Hence, *the area of a parallelogram is equal to the area of a rectangle having the same base and altitude; and if we multiply the linear units in the base by the linear units in the altitude, we obtain the square units, or units of surface, in the parallelogram.*

PROP. 36.—THEOR.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

CONS.—Pst. 1. A st. line may be drawn from any one point to any other point.

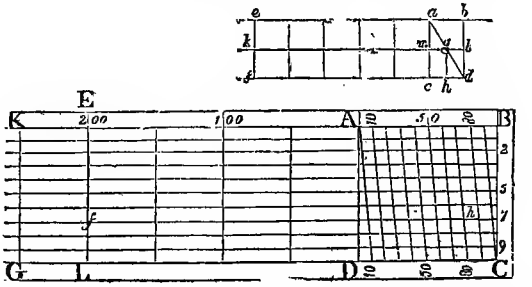
DEM.—P. 34. The opposite sides and angles of parallelograms, are equal.
Ax. 1. Magnitudes equal to the same, are equal to each other.
P. 33. Straight lines joining the extremities of two equal and parallel st. lines towards the same parts, are also themselves equal and parallel.
Def. A. A parallelogram is a four-sided figure of which the opposite sides are parallel; and the diagonal is the st. line joining two of its opposite angles.
P. 35. Parallelograms on the same base and between the same parallels, are equal.

EXP. 1.	Hyp. 1.	Let the ▱s ABCD, EFGH, be on equal bases, BC and FG,
	„ 2.	and between the same ‖s AH and BG;
2	Concl.	then ▱ ABCD = ▱ EFGH.
CONS. 1	*by* Pst. 1.	Join the points B, E, and C, H.
DEM. 1	*by* H. & P. 34	∵ BC = FG, and FG = EH;
2	Ax. 1.	∴ BC = EH.
3	H. & C.	But BC is ‖ EH, and BC and EH are joined towards the same parts by BE, CH;
4	P. 33.	∴ BE and CH are equal and parallel;
5	Def. A.	and ∴ also EBCH is a parallelogram.
6	D. 5 & Hyp.	Now the ▱s EBCH, ABCD, are on the same base BC, and between the same parallels, BC, AH;
7	P. 35.	therefore ▱ EBCH = ▱ ABCD.
8	P. 35.	Also ▱ EBCH = ▱ EFGH;
9	Ax. 1.	therefore ▱ ABCD = ▱ EFGH.
10	Recap.	*Therefore, parallelograms upon equal bases*, &c.

Q.E.D.

SCH.—The 36th Proposition may be considered as a corollary of the 35th.

USE AND APP.—1. The *Diagonal Scale* is constructed *on the principle of parallelograms on equal bases and between the same parallels being equal:* the opposite sides of a parellelogram are divided into the same number of equal parts, and the corresponding parts being joined form similar parallelograms, each one equal to the other; the diagonals to the similar parallelograms are drawn, and thus the Diagonal Scale is constructed.



The principle is thus shown—"Let *eb*, *kl*, *fd*, be three equidistant parallel lines, having other equidistant lines drawn at right angles across them. Join *ad*, then *mg* will be the half of *cd* or *ab*. For draw *gh* at right angles to *cd*, then the triangles *amg*, *ghd*, are obviously equal, and hence $mg = hd = eh$; that is, *mg* is the half of *cd*, or of its equal *ab*.

"If, instead of drawing *one line kl* between *eb*, *fd*, there be drawn *nine equidistant lines*," as between the parallels KAB, GDC, "the part *mg* in the second line would obviously be $\frac{1}{10}$ of *ab*, the part on the next line $\frac{2}{10}$, &c."—RITCHIE'S *Geom.*, p. 29.

In the Diagonal Scale KBGC thus constructed, the distance AB represents 100, and each of the divisions between A and B 10; and on the diagonal line diverging from A, the first distance from the perpendicular AD to the diagonal will be $\frac{1}{10}$ of 10, or 1; the second distance from AD to the same diagonal, $\frac{2}{10}$ of 10, or 2; the third distance, $\frac{3}{10}$ of 10, or three; and so on. Thus the spaces between E and A, are *hundreds;* between A and B, *tens;* and between B and C, *units*. If, however, the spaces between E and A are *tens*, those between A and B are *units*, and between B and C, *tenths:* indeed, the values depend on what we call the spaces between E and A. The extent from *f* on the perpendicular EL, to *h* on the seventh diagonal to the seventh parallel, may be taken for 277, 27·7, or 2·77, according as we consider EA hundreds, or tens, or units.

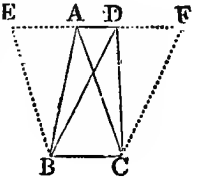
The Diagonal Scale is of very extensive use in the construction and measurement of Geometrical Figures.

PROP. 37.—THEOR.

Triangles upon the same base and between the same parallels are equal to one another.

CONS.—Pst. 2. A st. line may be produced in a st. line.
P. 31. Through a given point to draw a st. line parallel to a given st. line.
Def. A. A parallelogram is a four-sided figure of which the opposite sides are parallel, and the diagonal joins opposite angles.

DEM.—P. 35. Parallelograms on the same base and between the same parallels are equal.
P. 34. The opposite sides and angles of parallelograms are equal, and the diameter bisects them.
AX. 7. Halves of the same magnitude are equal.



EXP. 1	Hyp. 1.	Let the △s ABC, DBC, be on the same base BC,
	,, 2.	and between the same ‖s AD, BC;
2	Con. 2.	then the △ ABC = the △ DBC.
CONS. 1	*by* Pst. 2.	Produce AD both ways indefinitely.
2	P. 31.	through B draw BE ‖ CA, and through C, CF ‖ BD;
3	Def. A.	then EBCA and DBCF are parallelograms.
DEM. 1	*by* C. 3 & H.	∵ the ▱s are on the same base BC, and between the same ‖s BC, EF;
2	P. 35.	∴ the ▱ EBCA = the ▱ DBCF:
3	P. 34.	But a ▱ is bisected by its diameter, and ∴ △ ABC = half the ▱ EBCA, and △ DBC = half the ▱ DBCF;
4	AX. 7.	∴ also △ ABC = △ DBC.
5	Recap.	*Wherefore, triangles upon the same base, &c.* Q.E.D.

SCH.—On the principle that triangles are the halves of parallelograms, *the areas of triangles* are obtained; the product of the base and altitude gives the area of a parallelogram: consequently, *half the product of the base and altitude gives the area of the triangle.*

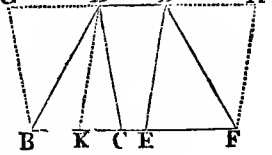
PROP. 38.—THEOR.

Triangles upon equal bases and between the same parallels are equal to one another.

CON.—Pst. 2. A st. line may be produced indefinitely.
P. 31. To draw a st. line through a given point parallel to a given st. line.
Def. A. A parallelogram is a four-sided figure of which the opposite sides are parallel, and the diagonal joins opposite angles.

DEM.—P. 36. Parallelograms upon equal bases and between the same parallels are equal.
P. 34. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.
Ax. 7. The halves of the same or of equal magnitudes are equal.

EXP. 1	Hyp. 1.	Let the △s ABC, DEF, be on equal bases BC, EF,
	„ 2.	and between the same ‖s BF, AD;
2	Concl.	then the △ ABC = the △ DEF.



CONS. 1	*by* Pst. 2.	Produce AD both ways indefinitely;
2	P. 31.	through B draw BG ‖ CA, and through F, FH ‖ ED;
3	Def. A.	then GBCA and DEFH are parallelograms.
DEM. 1	*by* H.&C.1.	∵ base BC = base EF, and BF ‖ GH;
2	P. 36.	∴ the ▱ GBCA = the ▱ DEFH:
3	P. 34.	But a parallelogram is bisected by its diagonal; ∴ △s ABC, DEF, each = half GBCA, DEFH;
4	Ax. 7.	and ∴ △ ABC = △ DEF.
5	Recap.	*Wherefore, triangles upon equal bases,* &c.

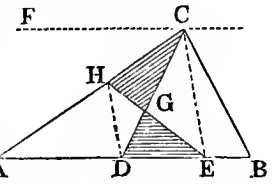
Q.E.D.

SCH.—1. The bases of the triangles are placed so as to form portions of the same st. line.

2. *The Area of a triangle may be bisected,* or *divided into any number of equal parts:* in the one case, *by bisecting the base;* in the other, *by dividing the base into equal parts, and joining the points of division and the vertex.*

USE AND APP.—1. By the last two propositions we arrive at *a practical way of dividing a triangular space, as* ABC, *into two equal parts;* for if the base BC be bisected from the vertex A by AK, then, because the two triangles ABK and ACK are on equal bases BK and CK, and between the same parallels, AD and BC, those triangles are of equal areas.

2. *When any point whatever, as* H *in the side* AC *of a triangle* ABC, *is taken, the triangle may be divided into two equal parts from that point.*



Cons.	1	*by* P. 31.	Through the vertex C draw CF ‖ AB;
	2	P. 10 & Pst. 1.	Bisect AB in D, and join CD;
	3	Pst. 1 & P. 31.	Join HD, and from C draw CE ‖ HD, and join HE;
	4	Sol.	then the △ AHE = the trapezium CHEB.
Dem.	1	*by* C. 2 & 1.	∵ AD = DB, and FC is ‖ AB;
	2	P. 38.	∴ △ ACD = △ BCD.
	3	C. 3.	Again, ∵ HD is ‖ CE, and HD the common base,
	4	P. 37.	∴ the △ CHD = the △ DHE;
	5	*Sub.*	from each of these equals take away the common part HGD;
	6	Ax. 3.	and the rem. △ CHG = the rem. △ EDG.
	7	D. 2, 6. *Sub.*	Next, from the equals ACD, BCD, take the equals CHG, EDG;
	8	Ax. 3.	and the remainder AHGD equals the remainder CGEB:
	9	*Add.*	to each of the equals AHCD, CGEB, add one of the equals EDG, CHG,
	10	Ax. 2.	and the whole AEH equals the whole CHEB.
	11	Recap.	*Thus the triangle* ABC *has been bisected from the point* H *by the st. line* HE. Q.E.F.

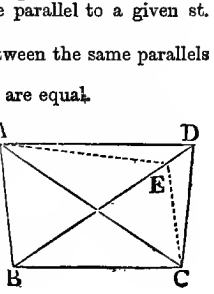
Prop. 39.—Theor.

Equal triangles upon the same base and upon the same side of it are between the same parallels.

Cons.—Pst. 1. A st. line may be drawn between two points.
P. 31. Through a given point to draw a st. line parallel to a given st. line.

Dem.—P. 37. Triangles upon the same base and between the same parallels are equal.
Ax. 1. Magnitudes equal to the same magnitude are equal.

Exp.	1	Hyp.	Let the △s ABC, DBC, be equal, and on the same base BC;
	2	Concl.	then the line joining A and D shall be ‖ to BC.



CONS. 1	*by* Pst. 1.	Join A and D, then AD is ‖ BC;
2	*Sup.*	but suppose them not to be parallel,
3	P. 31, Pst. 1.	through A draw AE ‖ BC, and join EC.
DEM. 1	*by* Hyp. & C. 3	∵ the △s ABC, EBC, are on the same base BC, and between the same parallels AE, BC;
2	P. 37.	∴ the △ ABC = △ the EBC:
3	Hyp.	But the △ ABC also = the △ DBC;
4	Ax. 1.	∴ the △ DBC = the △ EBC,
5	*ex abs.*	or, the greater magnitude is equal to the less;
6	Concl.	∴ AE is not parallel to BC.
7	*Sim.*	In the same way it may be shown that no other st. line except AD is par. to BC;
8	Concl.	∴ AD is parallel to BC.
9	Recap.	*Therefore, equal triangles upon the same base,* &c. Q.E.D.

SCH.—1. "*If the vertices of all the equal triangles which can be described upon the same base, or upon equal bases,* as in Prop. 40, *be joined, the line thus formed will be a straight line,* and is called the *locus of the vertices* of equal triangles upon the same base, or upon equal bases."

2. "A *locus* in Plane Geometry is a straight line or a plane curve, every point of which, and none else, satisfies a certain condition."—POTTS' *Euclid*, p. 51.

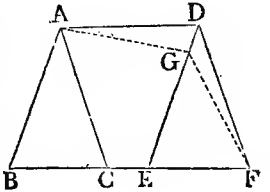
PROP. 40.—THEOR.

Equal triangles upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

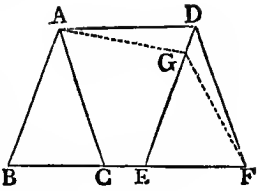
CONS.—Pst. 1. A st. line may be drawn between two points.
P. 31. Through a given point to draw a st. line parallel to a given st. line.

DEM.—P. 38. Triangles upon equal bases and between the same parallels are equal.
Ax. 1. Equals to the same are equal.

EXP. 1	Hyp. 1.	Let the equal △s ABC, DEF, be on equal bases BC, EF;
	,, 2.	and let BC and EF be in the same st. line and towards the same parts;



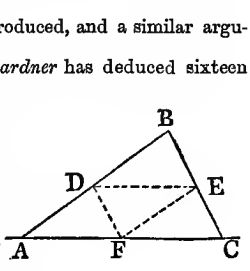
Exp. 2	Concl.	then AD is parallel to BF.
Cons. 1	*by* Pst. 1.	Join A, D; and AD is parallel to BF.
2	*Sup.*	But suppose AD not parallel to BF,
3	P. 31, Pst. 1.	through A draw AG ‖ BF, and join G, F;
Dem. 1	*by* H. & C. 3.	∵ the △s ABC, GEF, are on equal bases, BC, EF, and BF ‖ AG;
2	P. 38.	∴ the △ ABC = the △ GEF:
3	Hyp.	But △ ABC also = △ DEF;
4	Ax. 1.	∴ △ DEF = △ GEF,
5	*ex abs.*	or, the greater magnitude is equal to the less, which is absurd:
6	Concl.	∴ A G is not parallel to BF.
7	*Sim.*	In like manner no line except AD is parallel to BF;
8	Concl.	therefore AD is parallel to BF.
9	Recap.	*Wherefore, equal triangles upon the same bases,* &c. Q.E.D.



Sch.—1. The point G might be taken in ED produced, and a similar argument pursued.

2. From this and the preceding propositions, *Lardner* has deduced sixteen corollaries. Of these the principal are—

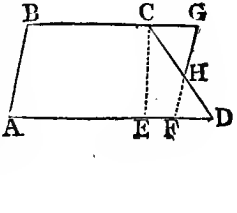
1°. A parallel to the base of a triangle through the point of bisection of one side, will bisect the other side.



2°. The lines which join the middle points D, E, F, of the three sides of a triangle divide it into four triangles which are equal in every respect.

3°. The lines joining the points of bisection of each pair of sides, is equal to half the third side.

4°. A *trapezium is equal to a parallelogram of the same attitude, and whose base is half the sum of the parallel sides.*



Let CD be bisected in H, and through H a parallel, GHF, to AB be drawn.

Since CG and FD are parallel, the angles GCH and G are respectively equal to ∠s D, and HFD (29) and CH is equal to HD; therefore (26) CG is equal to FD, and the triangle CHG

to the triangle DHF. Therefore, AF and BG are together equal to AD and BC, and the parallelogram AG to the trapezium AC; and since AF and BG are equal, AF is half the sum of AD and BC.

Thus the *Area of a trapezium* will be found *by taking half the sum of the parallel sides and multiplying that half sum by the altitude* CE.

3. "The *area of a square is found numerically* by multiplying the number of equal parts in the side of the square by itself. Thus a square whose side is twelve inches, contains in its area 144 square inches. Hence, in arithmetic, when a number is multiplied by itself, the product is called its square. Thus 9, 16, 25, &c., are the *squares* of 3, 4, 5, &c.; and 3, 4, 5, &c., are called the *square roots* of the numbers 9, 16, 25, &c. Thus *square* and *square root* are correlative terms."—LARDNER'S *Euclid*, p. 54.

PROP. 41.—THEOR.—*(Important.)*

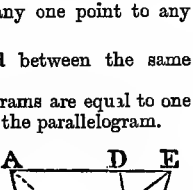
If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.

CONS.—Pst. 1. A straight line may be drawn from any one point to any other point.

DEM.—P. 37. Triangles upon the same base and between the same parallels, are equal to one another.

P. 34. The opposite sides and angles of parallelograms are equal to one another; and the diameter, or diagonal, bisects the parallelogram.

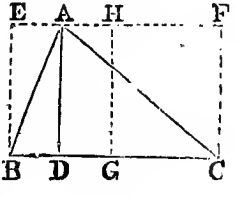
EXP. 1	Hyp. 1.	Let the ▱ ABCD, and the △ EBC, both be on BC,
	„ 2.	and between the same ‖s BC, AE;
2	Concl.	then the ▱ ABCD shall be double of the △ EBC.



CONS. 1	*by* Pst. 1.	Join the points, A, C, by the st. line AC.
DEM. 1	*by* Hyp.	∵ the △s ABC, EBC are on the same base BC, and BC ‖ AE;
2	P. 37.	∴ the △ ABC = the △ EBC:
3	P. 34.	But the ▱s are bisected by their diagonals;
4	Concl.	∴ the ▱ ABCD = twice the △ ABC;
5	D. 2.	and ▱ ABCD equals twice the △ EBC.
6	Recap.	*Wherefore, if a parallelogram and a triangle,* &c. Q.E.D.

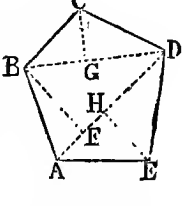
SCH.—The converse of Prop. 41 is,—*If a parallelogram is double of a triangle, and they have the same base, or equal bases upon the same straight line and towards the same parts—they shall be between the same parallels.*

USE AND APP.—The general method for finding the area of a triangle, or of any figure that may be resolved into triangles, is founded on this proposition. *The area of a parallelogram* is measured by the product of the base and altitude: and a triangle on the same, or on an equal base, being half the parallelogram, *the area of a triangle* is equal to the product of the altitude and half the base, or what is the same thing, equal to half the product of the base and altitude. *The area of polygons* is the sum of the areas of the triangles into which the polygons may be divided.



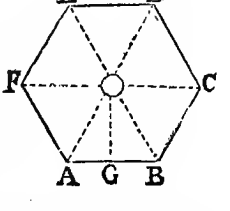
1. *For the area of the triangle*, ABC:

Take AD perpendicular to BC; bisect BC by GH, also perpendicular to BC;
and complete the ▱s ADBE, ADCF.
The area of the triangle ABC equals AD × BG, or AD × ½ BC, or ½ (AD × BC).



2. *For the area of any figure made up of straight lines, as* ABCDE:

Since the whole figure ABCDE is made up of the triangles ABD, BCD, and DEA;
The area of ABCDE equals the sum of the areas of the triangles ABD, BCD, and DEA:
That is, the area of ABCDE equals (CG × ½ BD) + (BF × ½ AD) + (EH × ½ AD).



3. *For the area of a regular polygon, as* ABCDEF:

From the centre O divide the polygon into triangles, and drop the perpendicular OG to AB;
Then the area of triangle AOB equals OG × ½ AB;
And the area of ABCDEF equals (OG × ½ AB, or AG) × the number of triangles in the figure; or equals the perp. OG × ½ the perimeter.

4. Dividing a circle by an infinite number of triangles having their common vertex in the centre, *the area of a circle equals* the product of the radius and of the semi-circumference. This method depends on the principle that the circle is the limit of the polygons inscribed in it; the circumference is the limit of their perimeters; and the radius the limit of their perpendiculars: and as the area of any polygon equals the perpendicular

multiplied by half the perimeter, the area of the circle, therefore, is equal to the radius $\times \frac{1}{2}$ the circumference.

It was ARCHIMEDES, of Syracuse, the most famous of ancient mathematicians (born B.C. 287), who first demonstrated this principle, and established the ratio between the diameter and the circumference of a circle. *The circumference of a circle* he showed "*to be greater than three times its diameter, by a quantity greater than* $\frac{10}{71}$ *of the diameter, but less than* $\frac{1}{7}$ *of the same.*" The proportion now employed is as 1 to 3·14159265, &c., or for general purposes, as 1 to 3·1416; that is, when the diameter measures 1 foot, or yard, or mile, the circumference will measure 3·1416 feet, or yards, or miles.

If we take D to represent the diameter, C the circumference, A the area of a circle, and $p = 3{\cdot}1416$; by having any two of these given, we can ascertain the others: thus,

$$1^{o}.\ D = \frac{C}{p} = \frac{4A}{C} = 2\sqrt{\frac{A}{p}}. \qquad 3^{o}.\ A = \frac{pD^2}{4} = \frac{C^2}{4p} = \frac{DC}{4}.$$

$$2^{o}.\ C = pD = \frac{4A}{D} = 2\sqrt{pA}. \qquad 4^{o}.\ p = \frac{C}{D} = \frac{4A}{D^2} = \frac{C^2}{4A}.$$

PROP. 42.—PROB.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

SOL.—P. 10. To bisect a finite st. line.

Pst. 1. A st. line may be drawn from one point to another.

P. 23. At a point in a st. line to make an angle equal to a given angle.

P. 31. To draw through a given point a parallel to a given line.

Def. A. A parallelogram is a four-sided figure of which the opposite sides are parallel.

DEM.—P. 38. Triangles upon equal bases and between the same parallels, are equal to one another.

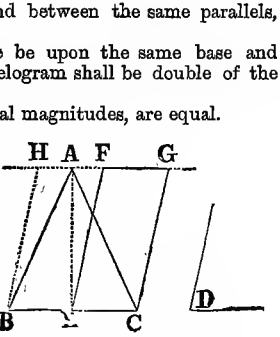
P. 41. If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.

Ax. 6. The doubles of the same or of equal magnitudes, are equal.

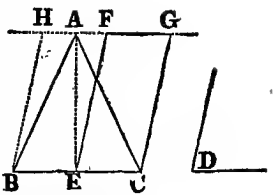
EXP. 1 | *Data.* | Given the △ ABC and the ∠ D;

Quæs. 1. | to desc. a ▱ = the △ ABC,

„ 2. | and having an angle = the given ∠ D.



Cons. 1	*by* P. 10 P st. 1	Bisect BC in E, and join AE;
2	P. 23.	at. E in EC make the ∠ CEF = the ∠ D;
3	P. 31.	and at . A draw AFG ‖ BC, and at. C, CG ‖ EF;
4	Def. A.	then EFGC is a parallelogram,
5	Sol.	and equal to the triangle ABC, and having an angle = ∠ D.



Dem. 1	*by* C. 1 & 3.	∵ base BE = base EC, and BC ‖ AG;
2	P. 38.	∴ △ ABE = △ AEC, and △ ABC = twice △ AEC:
3	C.	But the ▱ FECG and the △ AEC are both on EC, and EC is ‖ AG;
4	P. 41.	∴ the ▱ FECG = twice the △ AEC;
5	Ax. 5.	and ∴ the ▱ FECG = the △ ABC.
6	C. 2.	The ∠ CEF of ▱ FECG = the ∠ D.
7	Recap.	*Wherefore, there has been described a parallelogram, &c.* Q.E.F.

Sch.—*To describe a triangle equal to a given parallelogram* FECG, *and having an angle equal to a given angle* D. As in the last figure, produce the parallels GF and CE, and make EB equal to EC; at B construct an angle equal to the given angle D, and produce BA to meet the parallel GA in A, and join AC; then the triangle ABC, thus constructed, will, on drawing BH parallel to FE, be found by Prop. 41 equal to the parallelogram FECG.

—

Prop. 43.—Theor.

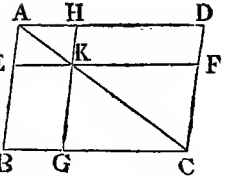
The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Dem.—P. 34. The opposite sides and angles of parallelograms are equal and the diameter bisects them.

Ax. 2. If equals be added to equals, the wholes are equal.

Ax. 3. If equals be taken from equals, the remainders are equal.

EXP. 1	Hyp. 1.	Let ABCD be a ▱, and AC its diameter,
	„ 2.	EH and GF ▱s about AC,
	„ 3.	and BK and KD the ▱s complements of the figure;
2	Concl.	then the complement BK = the complement KD.



DEM. 1.	*by* Hyp. 1.	∵ ABCD is a ▱, and AC its diameter;
2	P. 34.	∴ the △ ABC = the △ ADC:
3	Hyp. 2.	And ∵ AEKH is a ▱, and AK its diameter,
4	P. 34.	∴ the △ AEK = the △ AHK:
5	P. 34.	Also the △ KGC = the △ KFC.
6	D. 4 & 5.	Then, ∵ △ AEK = △ AHK, and △ KGC = △ KFC;
7	Ax. 2.	the △s AEK and KGC = the △s AHK and KFC:
8	D. 2.	But the whole △ ABC = the whole △ ADC;
9	Ax. 3.	therefore the rem. complement BK = the rem. complement KD.
10	Recap.	*Wherefore, the complements of the parallelograms, &c.* Q.E.D.

SCH.—The parallelograms about the diagonal, and also their complements, have each an angle in common with the whole parallelogram, and therefore are equiangular with it.

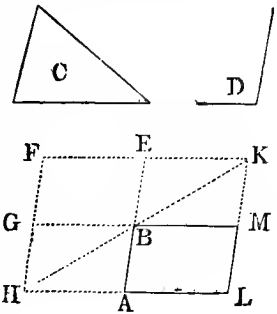
USE AND APP.—If any parallelogram, as KD in the last figure, be given, *another parallelogram*, KB, may be found, *equal to it*, and *having one side*, EK, *equal to a given line*. For, produce FK, and from K set off a distance equal to the given line EK; produce the sides DH, DF, and HK indefinitely; and through E draw AB parallel to HG or DC; draw the diagonal AK, and produce it until it cuts DF produced in C; through C draw a parallel to KF or HD, and it will cut HK and AE produced in the points G and B:—then the parallelogram BK will be equal to the given parallelogram KD.

PROP. 44.—PROB.

To a given st. line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.

SOL.—P. 42. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

P. 31. To draw a st. line, through a point, parallel to a given st. line.
Pst. 1. A st. line may be drawn between any two points.
Pst. 2. A terminated st. line may be produced to any length in a st. line.
DEM.—P. 29. If a st. line fall upon two parallel st. lines, it makes the alternate angles equal, and the ext. ang. equal to the int. and opposite angles, and likewise the two interior angles upon the same side equal to two rt. angles.
Ax. 9. The whole is greater than its part.
Ax. 12. A st. line meeting two st. lines, so as to make the two int. angles on the same side less than two rt. angles, these two st. lines, being produced, shall meet on the side on which the angles are less than two rt. angles.
P. 43. The complements of the parallelograms which are about the diameter of any parallelogram, are equal.
Ax. 1. Magnitudes equal to the same, are equal to one another.
P. 15. If two st. lines cut one another, the vertical angles are equal.



EXP. 1	*Data.*	Given the st. line AB, △ C, and ∠ D;
2	*Quæs.*	to apply to AB a ▱ = the △ C, and having an ang. = the ∠ D;
CONS. 1	*by* P. 42.	Make a ▱ BEFG = the △ C, and having an ang. EBG = ∠ D;
2	Pos.	place the ▱ so that the side BE form one st. line with AB;
3	Pst. 2.	produce FG to . H;
4	P. 31, Pst. 1.	and through . A draw AH ‖ BG or EF, and join HB:
5	Pst. 2.	Next produce HB and FE to meet in the point K;
6	P. 31, Pst. 2.	through K draw KL ‖ AE; and produce GB and HA to M and L;
7	Sol.	then the ▱ BL = the △ C, and has its. ang. ABM = ∠ D.
DEM. 1	*by* C. 3 & 4.	∵ st. line HF falls on the parallels AH and EF,
2	P. 29.	∴ the ∠s AHF and HFE together = two rt. angles;
3	Ax. 9.	and ∠s BHF and HFE together < two rt. ∠s;

DEM. 4	Ax. 12.	but when int. angles on the same side are $<$ two rt. angles, their sides meet, if produced ;
5	Concl.	$\therefore$ HB and FE, being produced, will meet:
6	Pst. 2.	Let them be produced and meet in the point K ;
7	P. 31.	through K draw KL $\parallel$ EA or FH ;
8	Pst. 2.	and produce HA to . L, and GB to . M ;
9	Concl.	then FKLH is a $\square$, HK the diameter, AG and ME $\square$s about HK, and LB and FB the complements ;
10	P. 43.	and $\therefore$ compl. LB = compl. BF :
11	C.1 & Ax. 1.	But $\square$ BF = $\triangle$ C, $\therefore$ $\square$ LB = $\triangle$ C.
12	P.15 & C.1.	And $\because$ $\angle$ GBE = $\angle$ ABM, and $\angle$ GBE = $\angle$ D,
13	Ax. 1.	$\therefore$ also $\angle$ ABM = $\angle$ D.
14	Recap.	*Therefore, to the st. line* AB *the parallelogram,* &c. Q.E.F.

SCH.—1. In this Problem the Solution and the Demonstration are almost unavoidably mixed together.

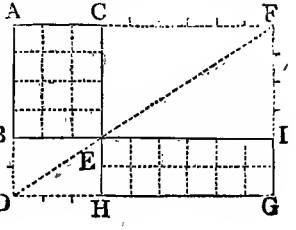
2. When a parallelogram is drawn *on a straight line*, it is said *to be applied to that line.*

USE AND APP.—This proposition contains a kind of *Geometrical Division.* The whole area of one figure, as C, or BF, in the last diagram, being given, and AB the side of another ; what is required is, so to separate the parts of the given figure, that when applied to the given line, the same number of parts shall he contained in the required figure, as existed in the given figure.

Arithmetically we can say, a given triangle contains, for example, 20 square feet—a given line, 4 lineal feet ; how can we apply a parallelogram to the 4 lineal feet, so that the area of the parallelogram shall equal that of the triangle ? We divide 20 by 4, and the quotient 5 is the lineal measurement of the other side of the parallelogram.

Geometrically we say, a given rectangle BC contains 12 square feet—a given st. line BD, or its equal EH, 2 lineal feet ; how can we apply another rectangle to BD or EH measuring 2 lineal feet, so that the area of the required rectangle may equal the area of the given rectangle ?

Produce indefinitely the parallels AC and BE, AB and CE ;
from B set off BD equal to the representative value of 2 feet, and draw the diag. DE ;
produce DE until it cuts AC produced in F, the diagonal DF will so divide or intersect AC produced, that the part CF shall equal the other side of the rectangle, of which the given side BD is 2 lineal feet.



For, complete the rectangle ADGF; EH equals BD, and EI equals CF; and the complement EG is equal to the complement EA: and if we divide AE into equal squares, and EG into equal squares, we find that in the first there are 12, and in the second 2×6, or 12 also.

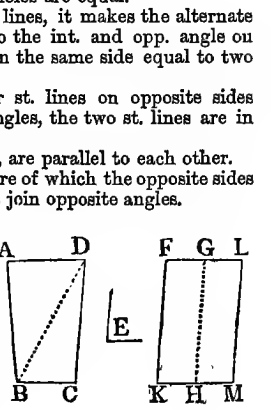
Prop. 45.—Prob.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Sol.—Pst. 1. A st. line may be drawn to join any two points.
P. 42. To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.
P. 44. To a given st. line to apply a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.

Dem.—Ax. 1. Magnitudes which are equal to the same, are equal to each other.
Ax. 2. If equals be added to equals, the wholes are equal.
P. 29. If a st. line fall on two parallel st. lines, it makes the alternate angles equal; and the ext. angle equal to the int. and opp. angle on the same side; and the two int. angles on the same side equal to two rt. angles.
P. 14. If at a point in a st. line two other st. lines on opposite sides make the adj. angles equal to two rt. angles, the two st. lines are in one and the same st. line.
P. 30. St. lines parallel to the same st. line, are parallel to each other.
Def. A. A parallelogram is a four-sided figure of which the opposite sides are equal and parallel, and the diagonals join opposite angles.

Exp. 1	*Data.*	Given a rectilin. fig. ABCD, and an $\angle$ E,
2	*Quæs.*	to describe a $\square$ = ABCD, having an angle = $\angle$ E.



Cons. 1	*by* Pst. 1.	To divide the fig. into triangles, join DB;
2	P. 42.	describe a $\square$ FH = the $\triangle$ ADB, and having an $\angle$ FKH = $\angle$ E;
3	P. 44.	to the line GH apply the $\square$ GM = the $\triangle$ DBC, and having an ang. = $\angle$ E;
4	Sol.	then the fig. FKML is the $\square$ required.

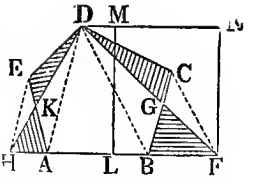
DEM. 1	*by* C. 2 & 3.	∵ ∠ E = ∠ FKH, and also = ∠ GHM;
2	Ax. 1.	∴ ∠ FKH = ∠ GHM:
3	*Add.*	To each of the equal angles add the ∠ KHG;
4	Ax. 2.	then the ∠s FKH + KHG = ∠s GHM + KHG:
5	P. 29.	But ∠s FKH and KHG = two rt. angles;
6	Ax. 1.	∴ ∠s GHM and KHG = two rt. angles:
7	D. 6.	Thus at H in st. line HG, the adj. ∠s GHM, GHK = two rt. angles;
8	P. 14.	∴ KH is in the same st. line with HM.
9	C. 9.	Again, ∵ st. line HG meets the parallels KM and FG,
10	P. 2.	∴ ∠ MHG = alt. ∠ HGF;
11	*Add.*	To each of these equal angles add ∠ HGL;
12	Ax. 2.	then ∠s MHG and HGL = ∠s HGF and HGL:
13	P. 29.	But ∠s MHG and HGL = two rt. angles;
14	Ax. 1.	∴ ∠s HGF and HGL = two rt. angles.
15	P. 14.	And ∵ FG is in the same st. line with GL;
16	C. 2 & 3.	and ∵ KF ∥ HG, and HG ∥ ML;
17	P. 30.	∴ KF is parallel to ML:
18	C. 2 & 3.	Also KM has been proved parallel to FL;
19	Def. A.	∴ fig. KFLM is a parallelogram.
20	C. 2 & 3.	And ∵ △ ABD = ▱ HF, and △ BDC = ▱ GM;
21	Ax. 2.	therefore, the whole rectil. fig. ABCD = the whole ▱ KFLM.
22	Recap.	*Wherefore, the parallelogram* KFLM *has been described,* &c. Q.E.F.

COR.—Hence a parallelogram equal to a given rectilineal figure can be applied to a given right line and in a given angle, by applying to the given right line a parallelogram equal to the first triangle.

USE AND APP.—1. By this and the preceding problem *we may measure the superficial contents of any rectilineal figure whatever,* by first reducing it to triangles, and then making a rt. angled parallelogram equal to the sum of the triangles. We may also make a rt. angled parallelogram on a given line, and *which shall be equal in area to several irregular figures.* Also, if we have several figures, *we may make another equal to their difference.*

2. By principles already established, and by the problems for the conversion of rectilineal figures into parallelograms of equal area, *we may change any right-lined figure,* as ABCDE, *first into a triangle, and then into a rectangle of equal area.*

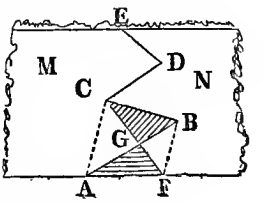
Join DA and BD to divide the given figure into triangles, and produce | AB indefinitely. Through . E draw EH parallel to DA, and through . C, CF parallel to DB : join DH and DF ;— then the triangle DHF is equal in area to the given figure ABCDE.



Next, through . D draw DN parallel to HF ; bisect HF, a side of the triangle DHF, in L ; at . L raise the perpendicular LM, and through . F draw FN a parallel to L M; the fig. LMNF is a rectangle of the same altitude as the triangle, and on half its base,—and is therefore equal in area to the triangle DHF.

3. In a similar way *a crooked boundary,* ABCDE, between two fields, M and N, *may be made straight* without changing the relative size of the fields.

Draw AC the subtend to angle B, and through . B, a st. line BF parallel to AC, and join CF ; the crooked boundary AB, BC is now converted into the single boundary CF. In a similar way FC and CD will be converted into one boundary, and this last and DE into a single boundary ; and thus the crooked boundaries AB, BC, CD, and DE, will be changed into one straight boundary without affecting the size of the fields : the shape will be altered, but the areas of fields M and N remain as at first.



PROP. 46.—PROB.

To describe a square on a given straight line.

SOL.—P. 11. To draw a st. line at right angles to a given st. line from a given point in the same.

P 3. From the greater of two st. lines to cut off a part equal to the less.

P. 31. To draw a st. line through a point parallel to a given st. line.

Def. A. A parallelogram is a four-sided figure of which the opposite sides are equal and parallel, and the diagonals join opposite angles.

DEM.—P. 34. The opposite sides and angles of parallelograms are equal.

Ax. 1. Things equal to the same, are equal to each other.

P. 29. If a line fall on two parallel st. lines, it makes the alternate angles equal to each other ; and the ext. angle equal to the int. and opposite angle on the same side ; and the two int. angles equal to two rt. angles.

Ax. 3. If equals be taken from equals, the remainders are equal.

Def. 30. Of four-sided figures a square is that which has all its sides equal, and all its angles rt. angles.

Exp. 1	*Datum.*	Given the st. line AB ;
2	*Quæs.*	it is required to draw a square on AB.
Cons. 1	*by* P. 11.	From . A draw AC at rt. angles to AB ;
2	P. 3.	make AD = AB ;
3	P. 31.	through . D draw DE ∥ AB ; and through . B, BE ∥ AD ;
4	Def. A.	then ABDE is a parallelogram,
5	Sol.	and the square required.
Dem. 1	*by* P. 34 & C. 2.	∵ AB = DE, AD = BE, and AB = AD ;
2	Ax. 1.	∴ AB = AD = DE = EB,
3	D. 1 & 2.	and ∴ the ▱ ABDE is equilateral.
4	C. 3.	Also, ∵ AD meets the parallels AB and DE,
5	P. 29.	∴ the int. ∠s BAD, ADE = two rt. angles :
6	C. 1.	but the ∠ BAD is a rt. angle ;
7	Ax. 3.	∴ ∠ ADE is also a rt. angle.
8	P. 34.	Now the opposite angles of ▱s are equal ;
9	D. 7 & 6.	∴ ∠ ABE opposite to ∠ ADE is a rt. angle, and ∠ BED opposite to ∠ BAD is a rt. angle ;
10	D. 6, 7, & 9.	∴ the figure ADEB is a rectangle ;
11	D. 3.	it is also equilateral ;
12	Def. 30.	therefore □ ABED is a square on AB.

Q.E.F.

Cor. 1.—*The squares on equal st. lines are equal ; and if the squares are equal, the st. lines are equal.*

2. *Every parallelogram having one right angle, has all its angles rt. angles.*

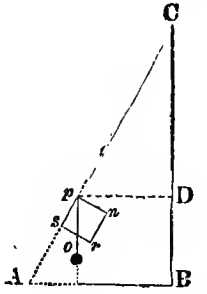
Sch.—*Given the diagonal* AB *to construct a square.*

Cons. 1	P. 11.	At . A and . B draw perpendiculars AE and BF ;
2	P. 9.	bisect the rt. angles by AC and BC meeting in C ;
3	P. 31.	through . s B and A draw BD ∥ AC, and AD ∥ CB ;
4	Sol.	then ABCD is the square required.
Dem. 1	*by* C. 2, & P. 6.	∵ the ∠s CAB and CBA are equal, side AC = side CB ;
2	P. 34.	and ∵ ACBD is a □, AD = CB, and BD = CA ;
3	Concl.	∴ AC = CB = BD = DA, and the fig. is equilateral.
4	C.	Again, the ∠s CAB, CBA being together one rt. angle,
5	P. 32.	the angle C is a rt. angle ;

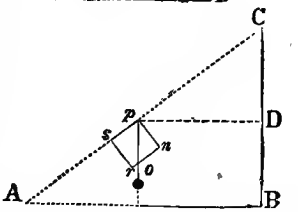
DEM. 6	P. 46, Cor. 2.	But in a ▭, as ACBD, when one angle is a rt. angle, all the angles are rt. angles ;
7	C. 3, Concl.	∴ also ACBD has its angles rt. angles ;
8	Concl.	And therefore, the figure being equilateral and rectangular, ACBD is the equare required.

USE AND APP.—The *Geometrical Square* is an instrument *by means of which*, and of the property of similar triangles that the sides about the equal angles are proportional, *the height of an inaccessible object can be ascertained*, provided a measurement to the perpendicular from the object can be made. The edges of the square are each divided into 100 equal parts, and from one corner a plummet is suspended ; when the object is seen along one edge of the instrument, the plummet cuts another edge, and forms a triangle similar to the triangle formed by lines representing the perpendicular from the object, a parallel to the horizontal line at its base, and the hypotenuse, or distance from the point of observation to the object itself.

In the adjoining figures, | AB represents the horizon ; *p*D a parallel to the horizon ; DB the height of the instrument ; CD the height of the object C above the parallel to the horizon ; *sp* the edge along which the object is to be seeu ; *sr*, *rn*, and *pn* graduated edges each of 100 parts ; and *p* the point of suspension for the plummet.



From the place of observation measure the distance *p*D, and the height of the instrument DB ; direct the edge *sp* towards the object C, and note the number of parts in *sr* or in *rn*. 1°· When the plummet cuts *sr* in o, the triangle *pso* is similar to the triangle CD*p* ; and we have the proportion $so : sp :: pD : CD$; whence $CD = \frac{sp \cdot pD}{so}$ and CB = CD + DB. 2°· When the plummet cuts *rn* in o, the triangles *onp* and CD*p* are similar ; and we have the proportion $pn : no :: pD : DC$; whence $CD = \frac{no \cdot pD}{pn}$; and CB = CD + DB.



For example, let *p*D = 60 ft. ; *so* = 50 eq. pts. ; and DB = 6 ft. ; required CB. Here 50 : 100 :: 60 : 120 = CD, and 120 + 6 = 126 ft. = CB.—*See* TATE'S *Geomctry*, pp. 49-51.

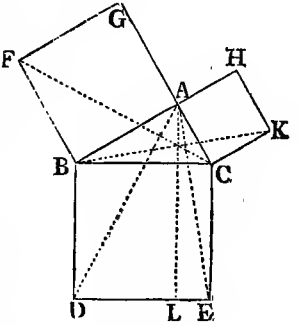
PROP. 47.—THEOR.—(*Most Important.*)

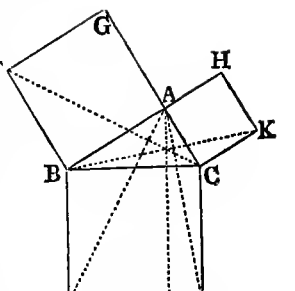
In any right-angled triangle, the square which is described upon the side subtending, or opposite to, the right angle, is equal to the squares described upon the sides containing the right angle.

CONS.—P. 46. To describe a square on a given straight line.
P. 31. Through a point to draw a st. line parallel to a given st. line.
Pst. 1. Any two points may be joined by a st. line.

DEM.—Def. 30. A square has all its sides equal, and its angles rt. angles.
P. 14. If at a point in a st. line two other st. lines on the opposite sides of it make the adjacent angles together equal to two rt. angles, these two st. lines shall be in one and the same st. line.
Ax. 1. Things equal to the same, are equal to each other.
Ax. 2. If equals be added to equals, the wholes are equal.
P. 4. If two triangles have each two sides and their included angle equal, the triangles are in every other respect equal.
P. 41. If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram is double of the triangle.
Ax. 6. Things double of the same, are equal to each other.

EXP. 1	Hyp.	Let ABC be a △ and BAC a rt. angle;
2	Concl.	then the square on BC = the squares on AB and AC.
CONS. 1	*by* P 46.	Draw on BC a square BE; on BA, a square BG; and on AC, a square HC;
2	P. 31.	through A draw AL ∥ BD or to CE;
3	Pst. 1.	join AD and FC; also AE and BK.
DEM. 1	*by* H&Def. 30	∵ ∠s BAC and BAG are each a rt. angle;
2	P. 14.	∴ the lines AC, AG on opp. sides of AB make the adj. ∠s = two rt. angles, and ∴ CA is in the same st. line with AG.
3	H. Def. 30 & P. 14.	Also AB is on the same st. line with AH.





Dem. 4	C.1, Def. 30	∵ the ∠s DBC and FBA are each a rt. angle;
5	Ax 1.	∴ the ∠ DBC = the ∠ FBA:
6	*Add.*	To each of the equals add the ∠ ABC;
7	Ax. 2.	∴ the whole ∠ DBA = the whole ∠ FBC:
8	C.1, & Def. 30	Hence, ∴ AB = FB, BD = BC, and ∠ DBA = ∠ FBC;
9	P. 4.	∴ base AD = base FC, and △ ABD = △ FBC.
10	C. 2.	Now the ▱ BL and the △ ABD are both on the same base BD, and between the same parallels BD and AL;
11	P. 41.	∴ the ▱ BL is double of the △ ABD.
12	C. 2.	Also the square GB and the △ FBC are both on the same base FB, and between the same parallels FB and GC;
13	P. 41.	∴ the ▱ or square GB, is double of the △ FBC;
14	D. 9.	But the △ ABD = the △ FBC;
15	Ax. 6.	∴ the ▱ BL = the square GB.
16	C. 3.	Also, after joining AE, BK, the ▱ CL = the square HC:
17	Ax. 2.	∴ the whole square BDEC = the two squares GB and HC;
18	C. 1.	Now the squares are BE on side BC, BG on side BA, and CH on side CA;
19	Concl.	∴ the square on side BC = the two squares on sides BA and CA.
20	Recap.	*Therefore, in any right-angled-triangle, &c.*

Q.E.D.

Cor. I.—Hence, *if the sides of a rt. angled triangle be given in numbers, its hypotenuse may be found:* for, let the squares of the sides be added together, and the square root of their sum will be the hypotenuse. Suppose AB the base, AC the perpendicular, BC the hypotenuse; the formula for BC is,

$$BC = \sqrt{AB^2 + AC^2}.$$

Example. The height of a tower is 40 feet, and the breadth of its ditch 30 feet: required the length of a ladder to reach from the further side of the ditch to the top of the tower.

$\sqrt{40^2 + 30^2}$ or $\sqrt{1600 + 900}$, or $\sqrt{2500} = 50$ ft., length of the ladder.

COR. II.—*If the hypotenuse and one side be given in numbers, the other side may be found:* for, let the square of the side be subtracted from that of the hypotenuse, and the remainder is equal to the square of the other side. The square root of this remainder will therefore be equal to the other side. Thus,

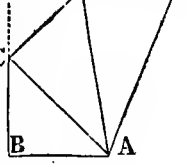
$$AB = \sqrt{BC^2 - AC^2}; \text{ and } AC = \sqrt{BC^2 - AB^2}.$$

Example. One village A is at right angles with two others B and C; the distance from B to C is 50 furlongs; from A to C 30 furlongs: required the distance from A to B.

$\sqrt{50^2 - 30^2} = \sqrt{2500 - 900} = \sqrt{1600} = 40$ furlongs from A to B.

COR. III.—*If any number of squares be given, a square equal to their sum may be found; or if one square be given, any multiple of it may be ascertained; or if two squares be given, the difference between them; or a square may be made that shall be the half, fourth, &c., of a given square.*

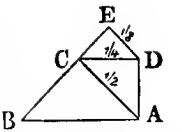
1°. Set the st. lines AB and BC, representative of the sides of the first two given squares, at rt. angles ABC,—the square on $AC = AB^2 + BC^2$; at C place CD, representative of the third square, at rt. angles to AC,—the square on $AD = AB^2 + BC^2 + CD^2$; and at D place ED, representative of the fourth square, at rt. angles to AD,—the square on $AE = AB^2 + BC^2 + CD^2 + DE^2$. And so on for any number of squares.



2°. Supposing AB to be representative of the st. line on which the given square is constructed, its multiple square will be obtained in a similar way; for in this case BC, CD, DE being each equal to AB, the square on AE is the multiple of the square on AB.

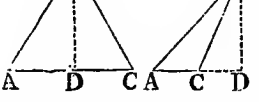
3°. Let AB be the less, and AC the greater of two st. lines; at B the extremity of the less, raise a perpendicular BG, and from A at the other extremity, with AC as radius, inflect on BG the greater st. line; the square of the intercept CB will equal the difference of the squares on AC and AB.

4°. Make the angles A and B each equal to half a right angle; C being a rt. angle, the square on AC will be one-half of the square on AB: again, at. A and. C make the angles CAD, ACD each equal to half a rt. angle, and the square on CD will be $\frac{1}{2}$ of that on AC, or $\frac{1}{4}$ of the square on AB: the process may be continued for any bisection of the rt. angles supposed to be formed at the extremities of a line, as for $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c.



Cor. IV.—*If a perpendicular* BD *be drawn from the vertex of a triangle to the base, the difference of the squares of the sides* AB *and* CB, *is equal to the difference between the squares of the segments* AD *and* DC.

For (47. I.) $AB^2 = AD^2 + BD^2$;
and $CB^2 = DC^2 + BD^2$.
Take the latter from the former,
$AB^2 - CB^2 = AD^2 - DC^2$.
The difference vanishes when
AD = DC.

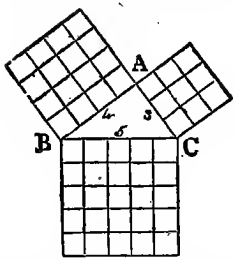


Cor. V.—*If a perpendicular be drawn from the vertex* B *to the base* AC, *or* AC *produced, the sums of the squares of the sides and alternate segments are equal.*

For $AB^2 + BC^2 = AB^2 + BD^2 + DC^2$; and $AB^2 + BC^2 = BC^2 + AD^2 + BD^2$;
therefore (Ax. 1) $AB^2 + BD^2 + DC^2 = BC^2 + AD^2 + BD^2$.
Take away the common square BD^2;
and (Ax. 3) $AB^2 + CD^2 = BC^2 + AD^2$.

Sch.—1. The 32nd and 47th Propositions are said to have been discovered by Pythagoras, born B.C. 570, and other principles of Geometry to have been brought by him from Egypt into Greece. Whatever we may think of the tale of his extravagant joy on the discovery of the 47th, certain it is that this is one of the most important propositions in all Euclid, for on it, and on the proposition in the sixth book which establishes the similitude of equiangular triangles, the whole science of Trigonometry is founded.

2. A plain and practical illustration of the 47th Proposition may be given by taking three st. lines in the proportion of 3, 4, and 5, and constructing with them a rt. angled triangle BAC: on each of the three sides draw a square, and sub-divide each square; that on AC 3, into nine smaller squares, that on AB 4, into sixteen, and that on BC 5, into twenty-five squares: the sum of the squares, 9 + 16, on AC and AB, equals the squares on BC, or 25 squares.

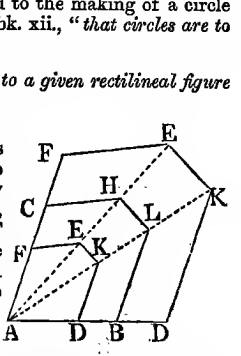


To form a rt. angle, st. lines containing 3, 4, and 5 equal parts, or any equimultiples of them, may be used: thus, measure off a st. line containing 5 feet or links, &c.; at one extremity B, with 4 feet or links, draw an arc, and at the other extremity C, with 3 feet or links, another arc: the two arcs intersect in A, and the st. lines from A to C, and from A to B, are at rt. angles to each other.

USE AND APP.—1. The 47th and its corollaries may be applied for the construction of all similar rectilineal figures, by Prop. 31, bk. vi., where it is proved "*that in right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle;*" and to the making of a circle the double or the half of another circle, by Prop. 2, bk. xii., "*that circles are to one another as the squares of their diameters.*"

1°. *To make a rectilineal figure* ADKEF, *similar to a given rectilineal figure* ABLHC.

Divide the figure into triangles by the st. lines AH, AL produced, if necessary; take AD equal to the side of the required figure, and through D draw DK parallel to BL, through K, KE parallel to LH, and through E, EF parallel to HC. Then ADKEF will be similar to ABLHC,—for, by Prop. 29, the angles D, K, E, F are equal to those at B, L, H, and C, and the triangles ADK similar to ABL, AKE to ALH, and AEF to AHC.



Now, by Prop. 19, bk. vi., "*Similar triangles are to each other as the squares of their like sides;*" and figures made up of similar triangles being simil r, all similar figures are to each other as the squares of their like sides.

For, $\frac{\text{area AEF}}{\text{area AHC}} = \frac{\text{AE}^2}{\text{AH}^2} = \frac{\text{AK}^2}{\text{AL}^2} = \frac{\text{AD}^2}{\text{AB}^2} \therefore \frac{\text{area AEF}}{\text{AD}^2} = \frac{\text{area AHC}}{\text{AB}^2}$

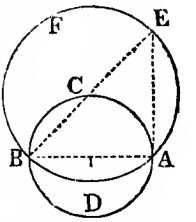
In like manner $\frac{\text{area AKE}}{\text{AD}^2} = \frac{\text{area ALH}}{\text{AB}^2}$, and $\frac{\text{area ADK}}{\text{AD}^2} = \frac{\text{area ABL}}{\text{AB}^2}$.

Adding these equals, we have,

$$\frac{\text{area ADKEF}}{AD^2} = \frac{\text{area ABLHC}}{AB^2} \therefore \frac{\text{area ADKEF}}{\text{area ABLHC}} = \frac{AD^2}{AB^2}.$$

—*See* TATE'S *Geometry*, p. 92.

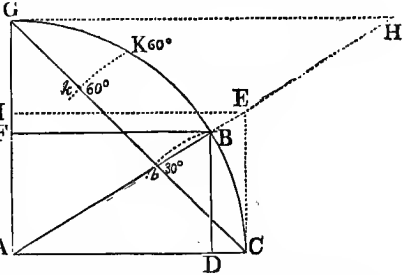
2°. *To make a circle the double or the half of another circle.*



Let AB be the diameter of ADBC; at . A raise a perpendicular AE, and at . B make the angle ABE equal to half a rt. angle; produce | BC until it cuts AE: the square on BE will be double of the square on AB, and the circle of which BE is the diameter double of the circle of which AB is the diameter.

Again, let BE be the diameter of a circle; at . s E and B make angles each equal to half a rt. angle; and the square on AB will be one-half of the square on BE; and the circle of which AB is the diameter one-half of the circle of which BE is the diameter.

2. By this 47th Proposition, the *Chords, Natural Sines, Tangents,* and *Secants of Trigonometrical Tables are constructed.*



With et. line AC as radius desc. an arc CG, and from . A raise a perpendicular AG; the arc CBKG being the measure of a rt. angle, is equal to 90°: join C and G, CG is *the line of chords;*—on which the chords of CB 30°, CK 60°, being inflected, C*b* is the chord of 30°, C*k* the chord of 60°. The *sine* of arc CB is BD, the *tangent* CE, the *secant* AE. The *co-sine* is FB, the *co-tangent* GH, the *co-sceant* AH.

Let it be supposed that the radiue AB is divided into 1,000,000 parts, and that the arc BC is 30°. Since the chord C*k* of 60° is equal to the radius AC; BD the sine of 30° shall be equal to the half of AC, or 500000, in the rt. angled triangle ADB. Now, $AB^2 = AD^2 + BD^2$; and $AB^2 - BD^2 = AD^2$ or BF^2, the sine of the complement: substituting the numbers we have $\sqrt{(1000000^2 - 500000^2)} = 866025 = FB$. Next, as the triangles ABD, AEC are equi-angular, we have the proportion AD : BD :: AC : CE; therefore the tangent of 30° $CE = \frac{BD \cdot AC}{AD}$

or, $\frac{500000 \times 1000000}{866025} = \frac{500000000000}{866025} = 577350$. Then $AC^2 + CE^2 = AE^2$,

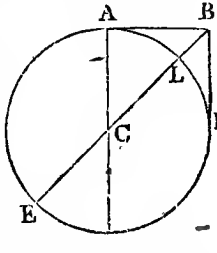
and AE is the secant of 30°;—therefore $\sqrt{(1000000^2 + 577350^2)} = 1154703$, the *Natural Secant* for an arc of 30°.

3. *Arithmetically,* when the given numbers are 3, 4, 5, or their equi-multiples, *the sum of the squares of the two less is equal to the squares of the greater,* as $(6 \times 6) + (8 \times 8) = 10 \times 10 = 100$, and when any two are given

we can find the third exactly : but with respect to all other numbers, though the sum of the squares of any two numbers always equals or coustitutes the square of a number greater than either, we cannot attain that number with perfect accuracy ; excepting in the case of *right triangular numbers*, all we can do is to approach its value by increasing the number of decimal places in the root. Thus $(5 \times 5) + (8 \times 8) = 25 + 64 = 89$; but the square root of 89 is 9·433981, &c. — *See the Introduction*, § vi., *on Incommensurable Quantities.*

Right triangular numbers may be found thus :—

Put n any odd number, then $\frac{n^2-1}{2}$ = the second number, and $\frac{n^2+1}{2}$ = the third number : thus, take 7, then $\frac{49-1}{2} = 24$, and $\frac{49+1}{2} = 25$. Now $(24 \times 24) + (7 \times 7) = 576 + 49 = 25 \times 25 = 625$; *the rt. triangular numbers being* 7, 24, and 25.



4. The height of any elevation on the earth's surface is so small when compared with the earth's diameter, that for practical purposes, as levelling, and ascertaining the height of mountains, we may consider the earth's actual diameter, and the diameter + the elevation, as the same quantity, *i.e.*, B E and L E not sensibly to differ ; nor the arc A L from the horizontal level A B. We assume L E to be 7960 miles, or that we may have an easier number, 8000 miles. If we take A B one mile, then $BL = \frac{1}{8000}$ part of a mile, or nearly 8 inches ; *i.e.*, *for every mile of survey, the surface or curvature of the earth is* 8 *inches below the horizontal level.*

5. *Heights and Distances from the curvature of the earth* are computed by Prop. 47, from the principle established in Prop. 16, bk. iii., *that the tangent* A B *is perpendicular to the radius* C A *of the arc* A L. Then if A B be required, we have $\sqrt{(LC + LB)^2 - AC^2} = AB$: if L B, the formula is $\sqrt{AB^2 + AC^2} = BC$, and $BC - LC$ or $AC = BL$.

Example 1. *Given* B L, *the height of the Peak of Teneriffe ; what will be the radius of its horizon, or the distance at which it may be seen ?*

Here $CB = CL + LB$. And $\sqrt{CB^2 - AC^2} = AB$ the horizontal radius.

Or, $4002^2 - 4000^2 = 16016004 - 16000000 = 16004$. And $\sqrt{16004} = 126$ miles $= AB$, *distance at which visible.*

Ex. 2. *A meteor* B *is seen over a distance from* A *to* D *of* 200 *miles ; required its height.*

Here $BL = BC - LC$ or AC. And $\sqrt{AC^2 + AB^2} = CB$.

Or, $\sqrt{4000^2 + 100^2} = \sqrt{16010000} = 4001{\cdot}24$ miles. And $4001{\cdot}24 - 4000 = 1{\cdot}24$ miles, *height of the meteor.*

Ex. 3. *A fountain* B *one mile from* A, *is observed from* A *to have the same apparent level : how much is* B *above* A ? i.e., *how much is* B *further from the earth's centre than* L *or* A ?

Here $BC - LC = BL$. And $\sqrt{(4000^2 + 1^2)} = 4000{\cdot}0001255 = BC$: then $4000{\cdot}001255 - 4000 = {\cdot}0001255$ of a mile = curvature 8 inches nearly.

By Prop. 36, bk. iii., the square of the tangent AB equals the rectangle of BL into BE; and as in levelling the distances are usually small, $AB^2 = BL \times EL$ nearly.

When AB is 1 mile, BL is $\frac{2}{3}$ of 1 foot, or 8 inches;
„ AB is 2 miles, BL is $\frac{2}{3}$ of 4 feet, or 32 inches;
„ AB is 3 miles, BL is $\frac{2}{3}$ of 9 feet, or 6 feet;
„ AB is 4 miles, BL is $\frac{2}{3}$ of 16 feet, or 10·6 feet.

Thus two-thirds of the square of the number of miles that the level is long, gives the height of B above A in feet, or *what the horizontal level differs from the level of the earth's curvature.*

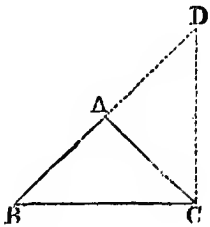
Prop. 48.—Theor.

If the squares described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by the two sides is a rt. angle.

Cons.—P. 11. To draw a st. line at rt. angles to a given st. line from a given point in it.
P. 3. From the greater st. line to cut off a part equal to the less.
Pst. 1. A st. line may be drawn from one point to another.

Dem.—Ax. 2. If equals be added to equals, the wholes are equal.
P. 47. In a rt. angled triangle the square on the side subtending the rt. angle is equal to the sum of the squares of the sides containing the rt. angle.
Ax. 1. Magnitudes equal to the same magnitude, are equal to each other.
P. 8. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them of the other.

Exp. 1	Hyp.	In triang. ABC let the square on BC = the sum of the squares on AB and AC;
2	Concl.	then the ∠ BAC is a rt. angle.
Cons. 1	*by* P. 11.	At A draw AD at right angles with AC;
2	P. 3 & Pst. 1	make AD = AB, and join DC.



DEM. 1	*by* C. 2.	∵ DA = AB, ∴ square on DA = square on AB;
2	*Add.*	Let the square on AC be added to each;
3	Ax. 2.	then the squares on DA and AC = the squares on AB and AC:
4	C. 1.	But ∠ DAC is a rt. angle;
5	P. 47.	∴ the square on DC = the squares on AD and AC.
6	Hyp. 1.	Also the square on BC = the squares on AB and AC;
7	Ax. 1.	∴ the square on DC = square on BC, and side DC = side BC:
8	C. 2 & D. 7	Thus in △s DAC, BAC, side AD = side AB, DC = BC, and AC is common;
9	P. 8.	∴ ∠ DAC = ∠ BAC:
10	C. 1.	But ∠ DAC is a rt. angle;
11	Ax. 1.	∴ ∠ BAC is a rt. angle.
12	Recap.	*Therefore, if the squares described,* &c.

Q.E.D.

SCH.—The 48th is the converse of the 47th Proposition, and may be extended thus :—*The vertical angle of a triangle is less than, equal to, or greater than, a rt. angle, as the square on the base is less than, equal to, or greater than, the sum of the squares of the sides.*

REMARKS ON BOOK I.

1. It will have been seen that the First Book is founded entirely on the Definitions, Postulates, and Axioms :—the *Definitions* fixing the meaning of the terms employed; the *Postulates* assigning the instruments that may be used; and the *Axioms* setting forth the principles on which the comparisons and arguments are conducted. In a few instances, for the illustration of certain propositions, other principles, not belonging to the first book, have been assumed; but these are to be regarded in their proper light, *not* as strict proofs, *but* as methods of explanation.

2. A few only of the properties of the circle are mentioned: those of the straight line and rectilineal angle are subservient to the proof of the properties of the triangle; and all rectilineal figures are either triangles, or may be resolved into triangles. The *First Book* therefore may in general terms be described as treating of the *Geometry of Plane Triangles.*

3. Excluding the Definitions, Postulates, and Axioms, it is not unusual to make a three-fold division of the contents of this Book. The *first* part, extending from the 1st Prop. to the 26th, unfolds the properties of triangles; the *second*, from Prop. 27 to 32, those of parallel lines; and the *third*, from Prop. 33 to 48, those of parallelograms, of course including the square.

4. The most important Propositions are, — *three*, — namely, Props. 4, 8, and 26, containing the *criteria*, or conditions of equality between triangles; *one*, Prop. 32, the equality of the exterior angle to the two interior and opposite angles, and of the three interior angles of every triangle to two right angles; *one*, Prop. 41, the proportion of the parallelogram to the triangle on the same base and between the same parallels; and *one*, Prop. 47, the relation between the hypotenuse and the sides about a right angle. These propositions at least must be thoroughly mastered, not by committing them to memory,* but by becoming so perfectly familiar with the principles contained in them, and with the connexions which exist between the arguments or reasonings employed, as never to feel at a loss for the demonstration, however diversified may be the figures constructed, nor even though no figure at all be drawn. The great aim should be to understand, and as a means to this, to follow up each proposition regularly through all its gradations, and verify it by its appropriate proofs.

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* To check the practice of committing to memory, and to induce exact familiarity with the principles and reasons of propositions, *the letters in the figures used should often be varied.*

GRADATIONS IN EUCLID.

BOOK II.

CONTAINING THE PROPERTIES OF RIGHT-ANGLED PARALLELOGRAMS, OR RECTANGLES.

In this Book, the relations will be investigated between the rectangles formed by the segments of straight lines, or of st. lines produced. When a st. line is cut or divided at any point, the segments are the portions between the point and the extremities of the line; when that point is within the extremities, the line is cut *internally;* when the point assumed is without the given line, and the line has to be lengthened, it is cut *externally*,—the production of the line in this case containing the point of section. If a st. line is cut internally, the line is the *sum* of the segments; but if cut externally, the st. line is their *difference*.

The subject of Geometry being magnitude, and not number, it is necessary, as we have said (p. 22), to discriminate between the Geometrical conception of a rectangle, and the Algebraical or Arithmetical representation of it: yet the latter, as illustrative of the Geometrical truth, will materially assist the former,—our ideas of number being more definite than our ideas of space. Accordingly, to each of the Propositions will be appended, what some have named, though loosely, the Algebraical or Arithmetical proof.

The *numerical area* of a rectangle is obtained by supposing the two sides containing the rectangle to be divided into a number of linear units of the same kind, as inches, feet, &c., and then multiplying the units in one side by the units in the other; *the product represents the Area or enclosed space.*

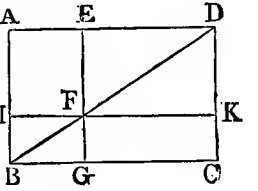
Of the two sides, one is considered as the *base*, the other as the *altitude;* and they may be represented by the letters b and a: thus the formula for the area of a rectangle will be ab; and for that of a triangle $\frac{ab}{2}$ or $\frac{1}{2}ab$; and for a square a^2 or b^2, according to the side taken,—the sides in this case being equal.

Definitions.

1. Every right-angled parallelogram, or rectangle, is said to be contained by any two of the st. lines which contain one of the right angles.

The rectangle is contained by any two conterminous sides.

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called the Gnomon. Thus the parallelogram HG, together with the complements AF, FC is the gnomon; which is more briefly expressed by the letters AGK or EHC, which are at the opposite angles of the parallelograms which make the gnomon.



Axiom.

"The leading idea, which runs through the demonstrations of the first eight propositions of book ii., is the obvious axiom, founded on the 8th Axiom, bk. i., *that the whole area of every figure, in each case, is equal to all the parts of it taken together.*"—Potts' *Euclid*, p. 68.

N.B.—The propositions, &c., required for the Construction and Demonstration will not in every instance be given. The learner is supposed to be familiar with most of them.

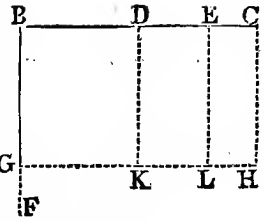
Prop. 1.—Theorem.

If there be two st. lines, one of which is divided into any number of parts, the rectangle contained by the two st. lines is equal to the rectangles contained by the undivided st. line and the several parts of the divided st. line.

Cons.—11. I. At a point in a st. line to draw a right angle.
3. I. From the greater of two st. lines to cut off a part equal to the less.
31. 1. Through a point to draw a st. line parallel to a given st. line.

Dem.—34. I. The opposite sides and angles of parallelograms are equal.
Ax. 8. Magnitudes which coincide are equal: *i.e.*, the whole area of every figure, in each case, is equal to all the parts of it taken together.

Exp. 1	Hyp.	Let A and BC be the two st. lines, BC being divided in .s D and E;
2	Concl.	then □ A . BC = □ A . BD + □ A . DE + □ A . EC.



Cons. 1	*by* 11. I.	At . B draw BF at rt. angles to BC;
2	3. I.	make BG = A:
3	31. I.	through D, E, and C draw DK, EL, and CH ‖s BG, and through G, GH ‖ BC;
4	Concl.	then □ BH = □s BK + DL + EH.
Dem. 1	*by* Def. 1 & C.	∵ □ BH is contained by the lines GB, BC, of which GB = A;
2	Concl.	∴ □ BH = □ A . BC:
3	Def. 1 & C.	Also ∵ □ BK is contained by GB, BD, of which GB = A;
4	Concl.	∵ □ BK = □ A . BD:
5	C. & 34. I.	And ∵ □ DL is contained by DK, DE, of which DK = GB = A;
6	Concl.	∴ □ DL = □ A . DE:
7	*Sim.*	In like manner □ EH = □ A . EC:
8	Ax. 8.	∴ □ A . BC = □s A . BD + A . DE + A . EC.
9	Recap.	*Wherefore, if there be two st. lines, one of which,* &c. Q.E.D.

Cor. 2 A . $\frac{1}{2}$ BC; or 3 A . $\frac{1}{3}$ BC; or 4 A . $\frac{1}{4}$ BC, &c. = A . BC.

Sch.—The propositions of this Book may be verified by Algebra and by Arithmetic; and in doing this we shall first state the Hypothesis algebraically and numerically, and then separately give, what are denominated, the *Algebraic* and *Arithmetical* Proofs.

Alg. & Arith. Hyp.—Let A = a = 6; BC = b = 10; BD + DE + EC = $m + n + p$ = 5 + 3 + 2 = 10.

Alg.	*Arith.*
$b = m + n + p$	$10 = 5 + 3 + 2$
$(\times a)$ $ab = am + an + ap$	$(\times 6)$ $6 \times 10 = (6 \times 5) + (6 \times 3) + (6 \times 2)$
	or, $60 = 30 + 18 + 12$

Use and App.—One of the methods of *Demonstrating the Rule for the Multiplication of numbers* depends on this proposition.

In the last figure, let A represent 8, and BC 54. We cut or separate the number 54 into as many parts as there are digits: for example, 50 + 4; each

part is multiplied by 8: the one part $4 \times 8 = 32$, and the other part $50 \times 8 = 400$. Now, all the partial products make up the whole product; therefore $(4 \times 8) + (50 \times 8) = 54 \times 8$; or $32 + 400 = 432$.

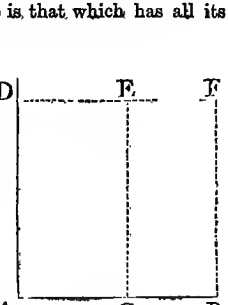
Prop. 2.—Theor.

If a st. line be divided into any two parts, the rectangles contained by the whole st. line and each of the parts, are together equal to the square of the whole st. line.

Cons.—46. I. On a given st. line to describe a square.
31. Through a given point to draw a parallel to a given st. line.

Dem.—Def. 30. I. Of four-sided figures, a square is that which has all its sides equal, and all its angles rt. angles.
Ax. 1. Magnitudes which are equal, &c.

Exp. 1	Hyp.	Let st. line AB be divided into any two parts in . C;
2	Concl.	then the □s AB . AC + AB . CB = the sq. on AB.
Cons. 1	*by* 46. I.	On AB describe the square ADFB;
2	31. I.	and through C draw CE ∥ AD or BF;
3	Concl.	then □s DC + EB = the square DB.



Dem. 1	*by* Cons.	∵ □ AF = □s AE + CF, and A is the square on AB:
2	Def. 30. I.	also ∵ □ AE is contained by AD, AC, of which AD = AB;
3	Ax. 1.	∴ □ AE = AB . AC:
4	Cons.	And ∵ □ CF is contained by BF, CB, of which BF = AB;
5	Ax. 1.	∴ □ CF = AB . CB:
6	D. 3 & 5.	Therefore □s AB . AC + AB . CB = the square on AB.
7	Recap.	*If, therefore, a st. line be divided into any two parts*, &c. Q.E.D.

Alg. & Arith. Hyp.—Let AB = *a* units = 9, and AC + CB = *m* + *n* = 5 + 4.

Alg. Then $m + n = a$ | *Arith.* Then $5 + 4 = 9$
$(\times a)$ $\therefore am + an = a \times a$, or a^2 | $(\times 9)$ $\therefore 45 + 36 = 9 \times 9 = 81$

SCH.—1. "*There is no necessity for the absolute construction of the rectangles,* to establish the relations they express."—LARDNER's *Euclid*, p. 66.

Thus, Given the line $A = *k + y$, to prove that $A^2 = Ak + Ay$.
Take the line $B = A$; then $B \cdot A = Bk + By = Ak + Ay$.

2. "The object of most of the propositions of this book is, *to determine the relations between the rectangles* under the parts of *divided lines.* We shall first confine our attention to *a finite line divided into two parts.*"

In this case there are three lines to be considered,—1st, *the whole line,* expressed by W; 2nd, *the greater part,* by P; 3rd, *the less part,* by p: then $W^2 = (W \cdot P + W \cdot p)$.

But the two parts may be considered as two independent lines, L, and l; then the whole line is their sum, S; and $S^2 = S \cdot L + S \cdot l$: and D being the difference, or $L - l$, $L^2 = L \cdot l + L \cdot D$.

USE AND APP.—*Numerical Multiplication* may also be proved by this process; for if a number be divided into its parts, the square of the number, which is the product of the number multiplied into itself, equals the sum of the products of each part into the undivided number. In the same way in *Algebraical Equations,* in which a quantity may be represented by a, and its parts by $m + n$; if both sides of the equation are multiplied by the quantity a, then $a \times a$ or $a^2 = (m \times a) + (n \times a) = ma + na$.

The wood engraver substituted *k* for *x* in the figures; hence the use of *k*.

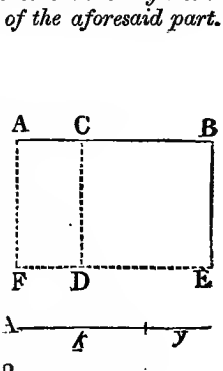
PROP. 3.—THEOR.

If a st. line be divided into any two parts, the rectangle contained by the whole line and one of the parts, is equal to the rectangle contained by the two parts together with the square of the aforesaid part.

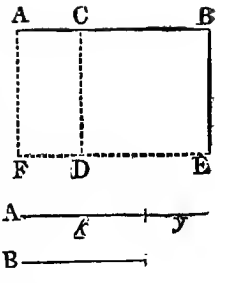
CONS.—46. I., Pst. 2, and 31. I.

DEM.—Def. 1. II., Def. 30. I., and Ax. 1.

EXP. 1	Hyp.	Let the st. line AB be cut in . C;
2	Concl.	Then $\square$ AB . BC = $\square$ AC . CB + the square on BC.
CONS. 1	*by* 46. I. Pst. 2	On st. line BC draw a square CDEB, ‖ and produce ED to F
2	31. I.	through A draw AF ; CD or BE;



CONS. 3	Concl.	then $\square$ AE, *i.e.*, AB. BC = $\square$ AD + square DB.
DEM. 1	*by* Def. 1. II. & 30. I.	$\because$ $\square$ AE is contained by AB. BE, of which BE = BC;
2	Ax. 1.	$\therefore$ $\square$ AE = AB. BC:
3	Def. 1. II. & 30. I.	Also, $\because$ $\square$ AD is contained by AC. CD, of which CD = BC;
4	Ax. 1.	$\therefore$ $\square$ AD = AC. BC:
5	Hyp.	And fig. DB is the square on CB:
6	D. 4 and 5.	$\therefore$ $\square$ AB. BC = $\square$ AC. BC + BC^2.
7	Recap.	*Thus, if a st. line be divided into any two parts, &c.* Q.E.D.



Alg. & Arith. Hyp.—Let AB = $a = 9$; BC = $m = 6$; and AC = $n = 3$.

Alg. Then $a = m + n$ $(\times m)$
$\therefore ma = m^2 + mn$

Arith. Then $9 = 6 + 3$ $(\times 6)$
$\therefore 54 = 36 + 18$

Or, Let A be a line divided into k and y, and B another line $= k$;
Then (1. II.) A . B = B . k + B . y. But (*Hyp.*) B = k.
Therefore, B . $k = k^2$; and B . $y = k$. y. Thus A . B = $k^2 + k$. y.

COR. 1. $A^2 - B^2 = (A + B) . (A - B)$; or $(k + y)^2 - k^2 = (k + y + k) . (k + y - k) = (2k + y) . y$; or $81 - 36 = 45 = 15 \times 3$.

2. $A^2 - B^2$ is greater than $(A - B)^2$ by twice B · (A − B); or $(k + y)^2 - k^2$ is greater than $(k + y - k)^2$ or y^2 by $2k . (k + y - k)$ or $2k . y$; or $81 - 36$ or 45 is greater than 9 by twice 6×3, or 36.

USE AND APP.—Multiplication of numbers may also be proved by this 3rd Prop.; for if a number, as 56, has to be multiplied by another, as 7, if the number, as 56, be separated into two parts, of which the multiplier 7 shall be one part, then on taking the square of the multiplier 7 one part, and multiplying the other part 49 by the same multiplier, the product will equal 7 times 56. Thus 56×7, or $392 = (7 \times 7) + (49 \times 7) = 49 + 343$.

PROP. 4.—THEOR.

If a st. line be divided into any two parts, the square of the whole line equals the squares of the two parts, together with twice the rectangle contained by the parts.

CONS.—46. I., Pst. 1, and 31. I.

DEM.—29. I. If a st. line fall on two parallel st. lines, it makes the alternate angles equal, and the ext. angle equal to the int. opposits angle, and the two interior angles equal to two rt. angles.

Def. 30. I. A square has its sides equal, and its angles rt. angles.

5. I. The angles at the base of an isosceles triangle are equal, and if the equal sides be produced, ths angles on the other side of the base shall be equal.

Ax. 1. Magnitudes, &c.

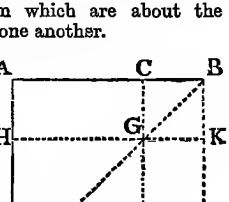
6. I. If two angles of a triangle be equal to one another, the sides also which subtend ths equal angles shall be equal to one another.

Ax. 3. If equals be taken, &c.

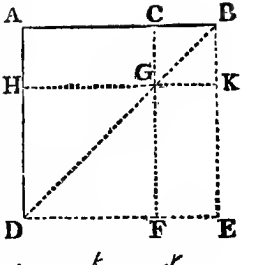
34. I. Ths opposite sides and angles of parallelograms are equal to one another, and the diagonal bisects them.

43. I. The complements of the parallelogram which are about the diameter of any parallelogram, are equal to one another.

EXP. 1	Hyp.	Let the st. line AB be divided at . C ;
2	Concl.	then the square on AB = the squares on AC and CB, together with twice the □ AC . CB.
CONS. 1	*by* 46. I. Pst. 1	On AB construct the square ADEB, and join DB ;
2	31. I.	through C draw CGF ‖ AD or BE, and through G, HK ‖ AB or DE ;
3	Concl.	then □ AE = □ HF + □ CK + 2 □ AG.
DEM. 1	*by* C. 2 & 29. I.	∵ st. line BD falls on the ‖s CF, AD, ∴ the ext. ∠ BGC = the int. ∠ ADB :
2	C. 1, D. 30. I.	But fig. ADEB being a square, side AB = side AD :
3	5. I. Ax. 1.	And ∵ ∠ ADB = ∠ ABD, and ∠ BGC = ∠ CBG :
4	6. I.	∴ side BC = side CG :
5	34. I. D. 4.	But GK = BC, CG = BK, and ∴ BC = GC = GK = BK ;
6	Ax. 1.	and ∴ the fig. CGKB is equilateral.
7	C. 2, 29. I.	Again, ∵ st. line CB meets the ‖s CG, BK, the ∠s KBC, GCB = two rt. angles :
8	D. 30, Ax. 3.	but KBC is a rt. angle, ∴ GCB is a rt. angle ;
9	34. I.	and ∴ the ∠s opposite, CGB, GKB, are rt. angles ;



Dem 10	Concl.	∴ also CGKB is rectangular :
11	D. 6 & 10.	∴ fig. CGKB is a square and on CB.
12	Sim. 34. I.	For the same reason HF is a square on HG, which = AC ;
13	D. 11 & 12.	∴ HF and CK are squares on AC and CB.
14	43. I.	And ∵ the compl. AG = the compl. GE ;
15	D. 4.	and ∵ GC = CB, and □ AG = AC . GC, or AC . CB ;
16	Ax. 1.	∴ □ GE = □ AC . CB ;
17	D. 15 & 16.	and ∴ □s AG and GE together equal twice AC . CB :
18	D. 13.	And figures HF, CK are squares on AC and CB ;
19	D. 17 & 18.	∴ figures HF + CK + AG + GE = AC^2 + BC^2 + 2 □ AC . CB.
20	Const.	But HF, CK, AG, and GE make up ADEB, the square on AB ;
21	Ax. 1.	∴ AB^2 = AC^2 + BC^2 + 2 AC . CB.
22	Recap.	*Wherefore, if a st. line be divided into any two parts, &c.* Q.E.D.



Alg. & Arith. Hyp.—Let AB = a = 12 ; AC = m = 8 ; and BC = n = 4.

Alg. Then $a = m + n$
Squaring, $a^2 = (m + n)^2$,
or $m^2 + 2mn + n^2$

Arith. Then 12 = 8 + 4
Squaring, $12^2 = (8 + 4)^2 = 144$,
or 64 + (2 × 32) + 16

Or, Let the line be made up of the parts $k + y$.

Then (2. II.) $A^2 = A . k + A . y$. But (3. II.) $A . k = k^2 + k . y$:
And $A . y = y^2 + k . y$.
Hence $A^2 = k^2 + y^2 + 2 k . y$.

Cor. 1.—*The parallelograms about the diameter of a square are also squares.*

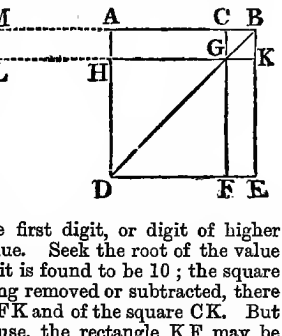
2.—*The square of a st. line is four times the square of its half;* for AC being equal to CB, $AC^2 + CB^2 = 2\,AC^2$ or $2\,CB^2$; and the rectangle AC . CB is the same as AC^2 or CB^2. Thus $AB^2 = 4\left(\frac{AB}{2}\right)^2$, *i.e.*, in the numerical example 12 × 12 = 4 (6 × 6) = 144.

3.—*Half the square of a st. line is equal to double the square of half the line;* thus $(\frac{AB}{2})^2 = 2(\frac{AB}{2})^2$ or $\frac{12 \times 12}{2} = 2(6 \times 6)$; or $\frac{144}{2} = 2 \times 36 = 72$.

4.—This proposition may also be employed for a st. line divided into any number of parts; then, *the square of the st. line will be equal to the sum of the squares of all the parts, together with double the rectangle under every distinct pair of parts.* Let $AB = k + y + z$; then $AB^2 = k^2 + y^2 + z^2 + 2(k \cdot y + y \cdot z + k \cdot z)$. Take a number $12 = 6 + 4 + 2$; then $12 \times 12 = (6 \times 6) + (4 \times 4) + (2 \times 2) + 2\{(6 \times 4) + (4 \times 2) + (6 \times 2)\}$; or $144 = 36 + 16 + 4 + 2(24 + 8 + 12)$.

USE AND APP.—1. In Algebra, the *square of a binomial,* that is, of a quantity made up of two terms, as $k+y$, is identical with the squares of the parts added to twice the product of the parts; thus $(k+y)^2 = k^2 + y^2 + 2k.y$.

2. This Proposition points out a *practical way of extracting the Square root of a number.*



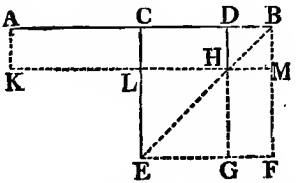
Let a number 144 be represented by the square AE, and its square root by the st. line AB. The square root of every number containing an even number of digits, consists of half that even number of digits; and the square root of every number containing an odd number of digits, consists of $\frac{n+1}{2}$ digits. We know therefore that AB the line required will be represented by a number consisting of two digits. Suppose AB divided at C, so that AC represents the first digit, or digit of higher value, and CB the second, or digit of lower value. Seek the root of the value of the first figure in 144, namely, of 100,—and it is found to be 10; the square of 10 therefore is represented by HF; HF being removed or subtracted, there remains 44 for the sum of the rectangles, AG, FK and of the square CK. But as the gnomon AKF is not convenient for use, the rectangle KF may be removed and placed as a continuation of AG, as MLHA; the whole rectangle LB therefore contains the remainder 44. As AC equals 10, twice AC, or MC = 20. We must therefore divide 44 by 20; that is, to find the divisor we double the Root already found, namely 10, and say,—how many twenties are there in 44? We find there are two for the side BK; but since 20 was not the whole side MB, but only a part MC, the new figure in the quotient, 2, must be added to the divisor 20, making 20 + 2, or 22; and 22 is contained exactly in 44. The first digit in the root is 1; the second, 2; or together, 12. Thus the Square of 144 is equal to the square of 10, to the square of 2, and to twice 20, which are the two rectangles comprehended under 2 and 10. It is for these reasons that the formula or rule given for extracting the square root of a number, requires the number to be separated into periods of two; the nearest square, and its root, to the left hand period are found,—and as twice the rectangle of the parts added to the square of the part to be found, make up the remainder,—for the division of that remainder we take twice the root, or rectangle already found, and to complete the divisor add to *that* twice the root the new term of the root: and so on, until the operation is completed.

Prop. 5.—Theor.

If a st. line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts together with the square of the line between the points of section, is equal to the square of half the line.

Con.—46. I., Pet. 1, and 31. I.

Dem.—43. I. The complements of the parallelograms about the diameter of any parallelogram, are equal to one another.
Ax. 2. If equals be added, &c.
36. I. Parallelograms upon equal bases and between the same parallels, are equal to one another.
Ax. 1. Things equal to the same, &c.
Def. 30. I. A square has its sides equal, and its angles rt. angles.
Cor. 4. II. The parallelograms about the diameter of a square are also squares.
34. I. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.



Exp. 1	Hyp. 1.	Let st. line AB be divided into two equal parts in . C,
	„ 2.	and into two unequal parts in D ;
2	Concl.	then the ▭ AD . DB + the square on CD = the square on CB.
Cons. 1	*by* 46. I. Pst. 1	On CB construct a square CEFB, and join BE ;
2	31. I.	through D draw | DHG ‖ CE or BF ;
3	„ „	through H draw | KLM ‖ CB or EF ;
4	„ „	and through A draw | AK ‖ CL or BM ;
5	Concl.	then ▭ AH + square on CD = square EB, *i.e.*, the □ on CB.

DEM. 1	*by* 43. I.	The complem. CH = the complem. HF;
2	*Add.* & Ax. 2.	adding □ DM to each, then □ CM = □ DF:
3	Hyp.	But ∵ AC = CB;
4	36. I. & Ax. I.	∴ □ AL = □ CM, and AL = □ DF;
5	*Add.* & Ax. 2.	adding to each □ CH, then □ AH = □s DF and CH.
6	Def 30. I. 4. II	But ∵ st. line DH = st. line DB, and □ AH is contained by AD, DH;
7	Concl.	∴ □ AH also = AD . DB:
8	Def. 2. II.	Also □s DF and CH make the gnomon CMG;
9	Ax. 1.	∴ gnomon CMG = the □ AD . DB:
10	*Add* Cor. 4. II	Add the square LG, that is, the square on CD;
11	Ax. 2.	then gnomon CMG + LG = □ AD . DB + CD^2:
12	Cons.	But CMG + LG make up CEFB, or the square on CB;
13	Ax. 1.	∴ □ AD . DB + CD^2 = CB^2.
14	Recap.	*Wherefore, if a st. line be divided into two equal parts,* &c. Q.E.D.

Arith. & Alg. Hyp.—Let AC = CB = a = 10, and AB = AC + CB = $2a$ = 20, or AB = AD + DB, and let CD = m = 6. Then AD = $a + m$ = 10 + 6, and DB = $a - m$ = 10 − 6.

Alg. Now $m = \frac{(a + m) - (a - m)}{2}$

and $(a + m)(a - m) = a^2 - m^2$

$(+ m^2)$ ∴ $(a + m)(a - m) + m^2 = a^2$

Arith. $6 = \frac{(10 + 6) - (10 - 6)}{2} = \frac{12}{2}$

& $(10 + 6)(10 - 6) = 100 - 36 = 64$
$(+ 36)$ ∴ $(16 \times 4) + 36 = 100$

COR.—It is manifest that *the difference of the squares of two unequal st. lines,* AC, CD, *is equal to the rectangle contained by their sum and difference; i.e.,* $AC^2 - CD^2 = (AC + CD)(AC - CD)$.

From the square EB on CB take the square DM on DB, there will remain □s EM and CH; if HF be taken from EM and placed by the side of CH, as AL, then CH and HF become AH, and □ AH is contained by AC + CD or AD, and by AC − CD or DB: hence $AC^2 - CD^2$, *i.e.*, the difference of the squares, = (AC + CD)(AC − CD), *i.e.*, the rectangle contained by their sum and difference.

Since AC = a = 10 is half the sum of AD + DB, and CD = m = 6 half their difference, the corollary may be thus expressed :—"*The rectangle contained by two st. lines* AD and DB, *is equal to the difference of the squares of half their sum* AC, *and half their difference* CD ;" *i.e.*, AD . DB = $AC^2 - CD^2$, or $(a + m)(a - m) = a^2 - m^2$.

Lardner's Corollaries to this Proposition are—1st. As the intermediate part CD diminishes, the rectangle increases, and *vice versâ*. The rectangle is a *maximum* when AB is bisected by D, its value being $\frac{(AB)^2}{2}$.—2nd. The greater the rectangle is, the less will be the sum of the squares of the parts ; the sum of the squares of the parts being at a *minimum* when the line CB is bisected in D.—3rd. Of all rectangles having the same perimeter, the square contains the greatest area.—4th. Of all rectangles equal in area, the square is contained by the least perimeter.—5th. If a perpendicular be drawn from the vertex of a triangle to the base, the rectangle under the sum and difference of the sides is equal to the rectangle under the sum and difference of the segments.—6th. The difference between the squares of the sides of a triangle, is equal to twice the rectangle under the base and the distance of the perpendicular from the middle point.—These Corollaries may be taken as the subjects of Geometrical Exercises.

Sch.—When a st. line is divided in two points *equally* and *unequally*, several linear magnitudes have to be considered ; 1st, *the whole line* AB ; 2nd, *the equal segments* AC and CB ; 3rd, *the unequal segments* AD and DB ; 4th, *the intermediate part*, that is, *the part* CD between the points of section.

The following are the principal properties connected with the equal and unequal division of a st. line :—1°. The intermediate part equals half the difference of the unequal parts ; thus, CD = ½ (AD − DB), or 6 = ½ (16 − 4). 2°. The greater segment equals half the sum added to half the difference ; thus, AD = ½ (AD + DB) + ½ (AD − DB) ; or 16 = ½ (20 + 12). 3°. The less segment equals half the sum *minus* half the difference ; thus DB = ½ (AD + DB − ½ (AD − DB), or 4 = ½ (20 − 12). 4°. The sum of two unequal lines equals twice the less added to the difference of the lines ; thus, AD + DB = 2 DB + (AD − DB) ; or 16 + 4 = (2 × 4) + (16 − 4).

Use and App.—1. By the Cor. to this Proposition *the difference between the squares* of two unequal numbers may be found *without squaring them ;* for *the product of their sum and difference will equal the difference of the squares.* Thus, if in a rt. angled triangle the hypotenuse is 100 feet, and the base 80, we may find the perpendicular without squaring the numbers ; for (100 + 80) × (100 − 80) = 3600, and $\sqrt{3600} = 60$ = the perpendicular.

2. We have also the means of finding *Quantities in Arithmetical Progression.* To be in Arithmetical Progression, *quantities must increase or diminish by a*

common difference: if there are three quantities, for instance, and the first exceeds the second by as much as the second exceeds the third, such quantities are in arithmetical progression,—the first and third being named *the extremes*, and the second *the mean*. Take, in the st. line AB, AD the greater segment = 16, DB the less = 4, and AC the half sum = 10. AC is the mean between AD and DB, the common difference being the intermediate part CD. In numbers we may take 16, 10, 4; these are in arithmetical progression, because their common difference is 6. Generally we affirm—"*The arithmetical mean is half the sum of the extremes, and the common difference is half the difference of the extremes.*"

"The fifth proposition may then be announced thus:—*The square of the arithmetical mean is equal to the rectangle under the extremes together with the square of the common difference.*"—LARDNER'S *Euclid*, p. 71.

3. This proposition is employed to establish one of the most important properties of st. lines cutting one another within a circle; as Prop. 35, bk. iii., "*If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.*"

4. By this proposition the method may be demonstrated of *finding the value of an Adfected Quadratic Equation in Algebra*, that is, of *an equation which contains the square and the first power of an unknown quantity*. In such equations the square is not completed, being defective by the square of half the co-efficient of the second term. The problem to be solved is—Given, as in the last fig., in the square CF, LG and 2 CH, required DM; *i.e.*, required the magnitude which will complete the square. The magnitude DM is the square on DB, and DB is equal to DH; and 2 DH . CD represents 2 CH. In the magnitude 2 DH . CD, 2 DH is the factor or co-efficient of CD; and if we take half of that co-efficient we have DH,—the square of which, DM, is the magnitude we are seeking to make up the square. *To solve an adfected quadratic equation*, as $x^2 + 4x = 12$, *we complete the square by adding to each side the square of half the co-efficient* 4; then $x^2 + 4x + 4 = 12 + 4 = 16$ · extracting the square root, $x + 2 = 4 \therefore x = 2$.

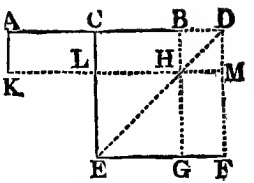
PROP. 6.—THEOR.

If a st. line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square of half the line bisected, is equal to the square of the st. line which is made up of the half and the part produced.

CONS.—46. I., Pst. 1, and 31. I.

DEM.—43. I., Ax. 2, 36, I., Ax. 1, Def. 30. I., Cor. 4. II.

Exp. 1	Hyp.	Let | AB be bisected in C and produced to D,
2	Concl.	then □ AD . DB + the square on CB = the square on CD.

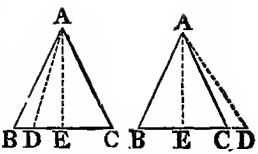


Cons. 1	*by* 46. I. Pst. 1.	On st. line CD construct the square CEFD, and join DE;
2	31. I.	through . B draw | BHG ‖ CE, or DF;
3	„ „	„ . H „ | KLM ‖ AD, or EF;
4	„ „	and „ . A „ | AK ‖ CL, or DM;
5	Concl.	then □ AM + □ LG = square CF.
Dem. 1	*by* H. & 36. I.	∵ | AC = | CB, □ AL = □ CH;
2	43. I. & Ax. 1.	but □ CH = □ HF; ∴ □ AL = □ HF;
3	*Add.* & Ax. 2.	Adding to each □ CM, then □ AM = the gnomon CMG.
4	Cons., Cor. 4. II.	But | DM = | DB; ∴ □ AM contained by AD . DM = □ AD . DB;
5	Ax. 1.	and ∴ gnomon CMG = □ AD . DB:
6	*Add.* Cor. 4. II.	Adding LG the square on CB to each,
7	Ax. 1 & 2.	then □ AD . DB and square on CB = gnomon CMG and square LG:
8	Const.	But CMG and LG make up fig. CEFD the square on CD;
9	Ax. 2.	∴ □ AD . DB + CB^2 = CD^2.
10	Recap.	*Wherefore, if a st. line be bisected and produced,* &c. Q.E.D.

Alg. & Arith. Hyp.—Let AC or CB = a = 8; AB = 2 a = 16 and BD = m = 4 AD = 2 a + m = 16 + 4; and CD = a + m = 8 + 4.

Alg. Now AD = 2 a + m (× m)
and therefore (2 a + m) m = 2 am + m^2. (Add a^2)
therefore (2 a + m) m + a^2 = a^2 + 2 am + m^2
But a^2 + 2 am + m^2 = (a + m)^2
therefore (2 a + m) m + a^2 = (a + m)^2

Arith. Now AD = 16 + 4 (× 4)
then (16 + 4) × 4 = (2 × 8 × 4) + (4 × 4) = 64 + 16. (Add 8 × 8)
and 4 (16 + 4) + 64 = 64 + 64 + 16
But 64 + 64 + 16 = (8 + 4)^2 = 144
therefore 4 (16 + 4) + 64 = (8 + 4)^2 = 144



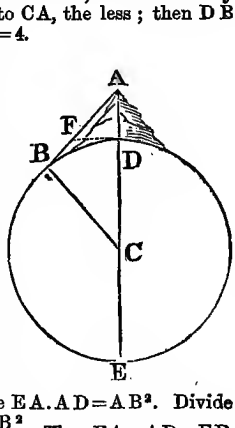
COR.—*If a st. line* A D *be drawn from the vertex* A *of an isosceles triangle to the base or its production, the difference between the squares of this line and the side of the triangle is the rectangle under the segments* B D × D C *of the base.*

For (by Cor. 4, P. 47. I.) $AD^2 \sim AC^2 = CE^2 \sim DE^2$; but (by 5 and 6. II.) $CE^2 \sim DE^2 = BD.DC$, $\therefore AD^2 \sim AC^2 = BD.DC$.

N.B.—If A D coincide with the perpendicular A E, the part D E will vanish.

SCH.—The alegebraical results of Prop. 5 and 6 differ only in appearance, "*arising from the two ways in which the difference between two unequal lines may be represented geometrically when they are in the same direction.*" In Prop. 5, the difference $DB = a - m = 10 - 6$, of the two lines $AC = a = 10$ and $CD = m = 6$, is exhibited by producing the less CD, and making CB = CA. Then DB = AC or $CB - CD = a - m = 10 - 6 = 4$. In Prop. 6, the difference DB of the two unequal lines CD and CA, is exhibited by cutting off from CD, the greater, a part CB equal to CA, the less; then $DB = CD - CB$ or $CA = (a + m) - a = (8 + 4) - 8 = 12 - 8 = 4$.

USE AND APP.—MAUROLICO, a mathematician, and abbot of Messina, who died A.D. 1575, measured the diameter of the earth by making use of this proposition.



From A the top of a mountain, the height of which A D is known, the angle C A B is measured, formed by E A and A B, a tangent to the circle in B, the limit of vision. In the rt. angled triangle A D F, the angles being obtained, the sides A F, F D will be found by Trigonometry; and F D equals F B: thus A B = A F + F B, and its square = AB^2.

Now (by Prop. 6. II.) E D being divided in C and produced to A, the rectangle $EA.AD + CD^2 = AC^2$; and (by Prop. 18. III.) A B C is a right angle: therefore $AC^2 = AB^2 + BC^2$ or CD^2, and thus the rectangle $EA.AD + CD^2 = AB^2 + CD^2$; take away the common part CD^2, and the rectangle $EA.AD = AB^2$. Divide both sides of the equation by A D, and $EA = \frac{AB^2}{AD}$. Then $EA - AD = ED$, the diameter of the earth.

Example. *A mountain is* 2½ *miles,* A D, *above the level,* F D, *of the sea; and the limit of vision, or* A B, *is* 141 *miles: required the earth's diameter,* D E.

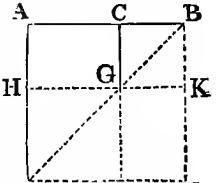
By the result just obtained, $EA = \frac{AB^2}{AD}$, and $ED = EA - AD$.

Therefore $\frac{AB^2}{AD}$ or $\frac{141 \times 141}{2.5} = \frac{19881}{2.5} = 7952.4$ miles; and $7952.4 - 2.5 = 7949.9$, *the earth's diameter.*

Prop. 7.—Theor.

If a st. line be divided into any two parts, the squares of the whole line and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with a square of the other part.

Cons.—46. I., Pst. 1 and 31. I.
Dem.—43. I., Axs. 2 and 6, Def. 30. I., Cor. 4. II., Ax. 8, and 34. I.



Exp. 1	Hyp.	Let | AB be divided into any two parts in . C;
2	Concl.	then the squares on AB and CB = twice AB . BC + AC².
Cons. 1	*by* 46. I. Pst. 1.	On AB make a square ADEB, and join DB;
2	31. I.	through . C draw | CF ‖ BE,
3	,, ,,	and ,, . G ,, | HK ‖ AB;
4	Concl.	then the squares AE + CK = twice AK + HF,
Dem. 1	*by* 43. I.	∵ the compl. AG = the compl. GE;
2	*Add.* & Ax. 1.	on adding □ CK to each, ▭ AK = ▭ CE;
3	Ax. 6.	∴ ▭ AK + ▭ CE = twice ▭ AK;
4	Const.	But AK + CE make up gnom. AKF + □ CK;
5	Ax. 1.	∴ AKF and CK together = twice AK:
6	D. 30 I., 4. II.	but | BK = | BC,
7	Ax. 6.	twice ▭ AK = twice AB . BK, and twice AB . BK = twice AB . BC;
8	Ax. 1 & 8.	∴ the gnom. AKF + □ CK = twice the rect. AB . BC;
9	*Add.* & 34. I.	Adding to both the equals HF, *i.e.*, the square on HG or AC,
10	Ax. 2.	then AKF + CK + HF = twice AB . BC and AC squared:
11	Const.	But AKF + CK + HF make up the figures ADEB and CK,
12	C. 1, 3.	and ADEB and CK are the squares on AB and CB,
13	Concl.	∴ AB² + BC² = twice AB . BC + AC².
14	Recap.	*Wherefore, if a st. line be divided into any two parts*, &c. Q.E.D.

Alg. & Arith. Hyp.—Let AB = a linear units = 16 ; AC = m = 9 ; and CB = n = 7.

Alg. Then $a = m + n$; Squaring, $a^2 = m^2 + 2mn + n^2$.
(Add n^2) $a^2 + n^2 = m^2 + 2mn + 2n^2$.
But $2mn + 2n^2 = 2(m+n)\,n = 2an$,
therefore $a^2 + n^2 = 2an + m^2$.

Arith. $16 = 9 + 7$; Squaring, $256 = 81 + 126 + 49$.
(Add 49) $256 + 49 = 81 + 126 + 98$.
But $126 + 98 = 2(9+7)\,7 = 2(16 \times 7) = 224$,
therefore, $256 + 49 = 222 + 81 = 305$.

Another form of stating the same result is,

Let AB = a = 16 ; AC = b = 9 ; and BC = $a - b$ = 16 − 9 = 7.

then $AB^2 = a^2$		And $2\,AB\ BC = 2a^2 - 2ab$	
$BC^2 = a^2 - 2ab + b^2$		$AC = \qquad b^2$	
Sum $2a^2 - 2ab + b^2$	=		$2a^2 - 2ab + b^2$

Cor. 1.—If AB and BC be considered as two independent lines, AC being their difference, "*the sum of the squares of any two st. lines is equal to twice the rectangle under them together with the square of the difference;*" *i.e.*, $AB^2 + BC^2 = 2\,AB\,.\,BC + AC^2$; or $100 + 64 = (2 \times 80) + 4$.

Cor. 2.—Hence and (by 4. II.) *the square of the sum of two st. lines, the sum of their squares, and the square of their difference, are in arithmetical progression—the common difference being twice the rectangle under the sum.*

By 4. II. $(AB + BC)^2 = AB^2 + BC^2 + 2\,AB\,.\,BC$; and by Cor. 7. II. $AB^2 + BC^2 = (AB - BC)^2 + 2\,AB\,.\,BC$; or $AC^2 + 2\,AB\,.\,BC$. Thus the common difference is $2\,AB\,.\,BC$; therefore the quantities $(AB + BC)^2$, $(AB^2 + BC^2)$, and AC^2 are in arithmetical progression ; as 324, 164, and 4,—the com. dif. being 160.

Prop. 8.—Theor.

If a st. line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts together with the square of the other part, is equal to the square of the st. line which is made up of the whole and that part.

CONS.—Pst. 2. A terminated st. line may be produced in a st. line.

3. I. From the greater st. line to cut off a part equal to the less.

46. I. On a given st. line to describe a square.

31. I. Through a given point to draw a st. line parallel to a given st. line.

DEM.—34. I. The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them.

Ax. 1. Magnitudes equal, &c.

36. I. Parallelograms upon equal bases and between the same parallels are equal.

43. I. The complements of the parallelogram which are about the diameter of any parallelogram, are equal to one another.

4. II. If a st. line be divided into any two parts, the square of the whole line equals the square of the two parts together with twice the rectangle contained by the parts.

Def. 30. I. A square is a four-sided figure having all its sides equal, and its angles rt. angles.

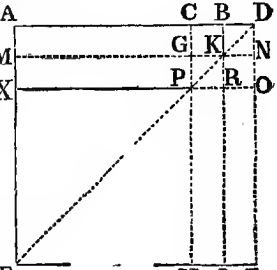
Cor. 4. II. The parallelograms about the diameter of a square are also squares.

Ax. 2. If equals be added, &c.

Ax. 8. Magnitudes which coincide are equal.

Ax. 6. Things double of the same are equal.

EXP. 1	Hyp.	Let | AB be divided in the point C;
2	Concl.	then four times AB . BC with the square on AC added = the square of (AB + BC).
CONS. 1	*by* Pst. 2 & 3. I	Produce | AB to . D, so that BD = BC;
2	46. I. Pst. 1	On st. line AD describe the square AEFD, and join ED;
3	31. I.	through . s B and C draw ‖s BL, CH, to AE or DF, and cutting ED in the points K and P;
4	„ „	also through . s K and P, draw | MGKN and | XPRO ‖s to AD or EF;
5	Concl.	then AK + MR + (HR + BN) + NL + XH, fill up the figure AEFD.



DEM. 1	*by* C.1.&34.I.	∵ CB = BD, CB = GK, and BD = KN ;
2	Ax. 1.	∴ GK = KN,
3	*Sim.*	In like manner PR = RO :
4	C. 1 & D. 2.	And ∵ CB = BD, and GK = KN ;
5	36. I.	∴ □ CK = BN, and GR = □RN :
6	C. & 43. 1.	But □s CK, RN are compls. of the parallelogram CO, and therefore CK = RN :
7	D. 4.	also rect. BN = rect. GR ;
8	Ax. 1.	∴ the four □s BN, CK, GR, RN, equal one another ;
9	Ax. 6.	and therefore the sum of these four = four times CK.
10	C.1,Def.30 & 31. I.	Again, ∵ CB = BD, BD = BK, and BK = CG ;
11	34.I.Cor.4. II	and also ∵ CB = GK, and GK = GP ;
12	Ax. 1.	∴ CG = GP.
13	D. 12 & 2.	And ∵ CG = GP, and PR = RO ;
14	C. 3 & 4.	and AX ∥ CP, and PO ∥ HF ;
15	36. I.	∴ rect. AG = MP, and rect. PL = RF :
16	Const.	But ∵ □s MP, PL are the compls. of the parallelogram MKLE,
17	43.I.&Ax.1.	∴ □ MP = PL, and also □ AG = RF ;
18	D. 15 & 17.	∴ □s AG, MP, PL, and RF are all equal to one another ;
19	Ax. 1.	and the sum of the four = four times any one, as AG.
20	D. 9.	But (BN + CK + GR + RN) = four times CK :
21	*Add.* & Ax. 2.	∴ the eight rectangles in the gnomon AOH = four times AK.
22	Const.	Now rect. AK is contained by AB . BK, and AB . BK = AB . BC ;
23	D. 21.	∴ four times AK = four times AB . BC :
24	,,	But four times AK = the gnomon AOH ;
25	Ax. 1.	∴ four times AB . BC = AOH :
26	*Ad.* Cor. 4. II.	Add to each XH, that is the square on AC ;
27	34. I. & Ax. 2.	then four times AB . BC + AC^2 = AOH + XH :
28	Ax. 8.	But AOH + XH make up AEFD, the square on AD ;
29	Concl.	∴ 4AB . BC + AC^2 = AD^2 = $(AB + BC)^2$.
30	Recap.	*Wherefore, if a st. line be divided,* &c. Q.E.D.

Alg. & Arith. Hyp.—Given AB = a = 16 ; AC = m = 10, and CB = n = 6.

Alg.	*Arith.*
Then $m+n=a$; taking n from each, $m=a-n$;	$10+6=16$; (-6) and $10=16-6$;
Squaring, $m^2=a^2-2an+n^2$	Squaring, $100=256-192+36$,
$(+4an)$ then $4an+m^2=a^2+2an+n^2$;	$(+4\times96)$ then $384+100=256+192+36$;
But $a^2+2an+n^2=(a+n)^2$,	But $256+192+36=(16+6)^2$,
Therefore $4an+m^2=(a+n)^2$.	therefore $384+100=(22)^2=484$.

We obtain the same result in another form :

Let AB = a = 16, BC = b = 6, and AC = $a-b$ = 16 − 6 = 10 ;
then $4\,AB.BC=4ab$; And $(AB+BC)^2=a^2+2ab+b^2$

	$AC^2=a^2-2ab+b^2$			
Sum	$a^2+2ab+b^2$	=	Sum	$a^2+2ab+b^2$

Sch.—1. The Proposition may be otherwise expressed :—"*The square of the sum of two st. lines is equal to four times the rectangle under them together with the square of their difference ;*" thus $(AB+CB)^2=4AB.CB+(AB-CB)^2=(16+10)^2=4(16\times10)+(16-10)^2$; or $676=640+36$.

2. Or, "*four times the square of half the sum is equal to four times the rectangle under the lines together with four times the square of half the difference ;*" thus, $4\frac{(AB+CB)^2}{2}=4AB.CB+4\frac{(AB-CB)^2}{2}=4\times169=(4\times160)+(4\times9)=676$.

Use and App.—The principles established in Props. 6, 7, and 8 are applied to Algebra and to various operations connected with the extraction of the Square root.

Prop. 9.—Theor.

If a st. line be divided into two equal parts, and also into two unequal parts, the squares of the two unequal parts are together double of the square of the half line and of the square of the line between the points of section.

Cons.—11. I. To draw a st. line at rt. angles to a given st. line at a given point in it.
3. I. From the greater of two st. lines to cut off a part equal to the less.
Pst. 1. A st. line may be drawn, &c.
31. 1. Through a given point to draw a st. line parallel to a given st. line.

Dem.—5. I. The angles at the base of an isosc. triangle are equal.
32. I. If a side of any triangle be produced, the ext. angle equals the two int. and opp. angles ; and the three int. angles of every triangle equal two rt. angles.

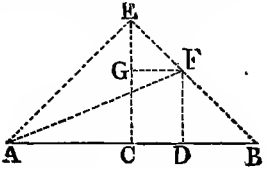
29. I. A st. line falling on two parallel st. lines makes the alt. angles equal, and the ext. angle equals the int. and opp. angles upon the same side, and the two int. angles upon the same side equal two right angles.

6. I. If two angles of a triangle be equal to one another, the sides opposite the equal angles shall be equal to one another.

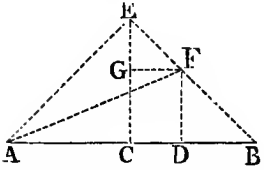
47. I. In every rt. angled triangle the square on the side subtending the rt. angle is equal to the sum of the squares on the sides containing the rt. angle.

34. I. The opposite sides and angles of parallelograms equal one another, and the diagonal bisects them.

Ax. 1. Magnitudes equal to the same are equal.



Exp. 1	Hyp. 1.	Let st. line AB be divided into two equal
	„ 2.	parts at . C, and into two unequal parts at . D;
2	Concl.	then $AD^2 + DB^2 = 2\,(AC^2 + CD^2)$.
Cons. 1	*by* 11. I.	From . C draw CE at rt. angles to AB:
2	3. I. & Pst. 1	make CE = AC or CB, and join EA, EB;
3	31. I.	through . D draw DF ‖ CE, and meeting EB in F;
4	31. I. & Pst. 1	and through . F draw FG ‖ AB, and join AF;
5	Concl.	then the squares on AD and DB = twice the squares on AC and CD.
Dem. 1	*by* C. 2 & 5. I.	∵ AC = CE, ∴ ∠ EAC = ∠ AEC;
2	C. 1 & 32. I.	and ∵ ACE is a rt. angle, ∴ ∠s AEC and EAC = a rt. angle;
3	D. 1 & 32. I.	and ∠ AEC being = ∠ EAC, each is half a rt. angle.
4	*Sim.*	So, ∵ ∠ CEB = ∠ EBC half a rt. angle, ∴ (AEC and BEC) or the whole ∠ AEB is a rt. angle:
5	D. 4, 29. I. C. 4	And ∵ ∠ GEF is half a rt. angle, and ∠ EGF = ECB a rt. angle;
6	32. I.	∴ rem. ∠ EFG = half a rt. angle, and ∠ GEF = ∠ EFG;
7	6. I.	and ∴ the side EG = the side GF,



DEM. 8	D. 4. 29. I.	Again, ∵ ∠ FBD = half a rt. angle, and FDB = the int. ∠ ECB a rt. angle;
9	32. I.	∴ rem. ∠ BFD is half a rt. angle, and ∠ FBD = ∠ BFD;
10	6. I.	and ∴ the side DF = the side DB
11	C. 2.	And ∵ AC = CE, ∴ $AC^2 = CE^2$;
12	Concl.	and ∴ $AC^2 + CE^2$ = twice AC^2.
13	C. 1. & 47. I.	But ∠ ACE being a rt. angle, $AE^2 = AC^2 + CE^2$, or twice AC^2:
14	D. 7.	Again, since EG = GF, the square on EG = that on GF; and ∴ $EG^2 + GF^2$ = twice GF^2:
15	47. I.	but $EF^2 = EG^2$ and GF^2;
16	34. I.	∴ EF^2 = twice GF^2 which = twice CD^2;
17	D. 13.	and from above AE^2 = twice AC^2;
18	D. 13 & 15.	∴ $AE^2 + EF^2$ = twice $(AC^2 + CD^2)$:
19	D. 4. & 47. I.	But ∠ AEF being a rt. angle, $AF^2 = AE^2 + EF^2$;
20	Ax. 1 & D. 18	and ∴ AF^2 = twice $(AC^2 + CD^2)$:
21	C. & 47. I.	But ∠ ADF being a rt. angle, $AF^2 = AD^2 + DF^2$;
22	Ax. I.	and ∴ $AD^2 + DF^2$ = twice $(AC^2 + CD^2)$;
23	D. 10.	and side DF equalling side DB, $AD^2 + DB^2$ = twice $(AC^2 + CD^2)$.
24	Recap.	*Wherefore, if a st. line be divided into two,* &c. Q.E.D.

Alg. & Arith. Hyp.—Let AC, CB each $= a = 10$; $AB = 2a = 20$; and $CD = m = 4$.

Also, let $AD = a + m = 10 + 4$ and $DB = a - m = 10 - 4$.

Alg. Then $(a+m)^2 = a^2 + 2am + m^2$
and $(a-m)^2 = a^2 - 2am + m^2$
Adding $(a+m)^2 + (a-m)^2 = 2a^2 + 2m^2$

Arith. Then $(10+4)^2 = 100 + 80 + 16 = 196$
and $(10-4)^2 = 100 - 80 + 16 = 36$
Adding $(10+4)^2 + (10-4)^2 = 200 + 32 = 232$

Another form of the same Algebraical result is,

$$\begin{array}{rl} AD^2 = & a^2 + 2am + m^2 \\ DB^2 = & a^2 - 2am + m^2 \\ \hline AD^2 + DB^2 = & 2a^2 \qquad + 2m^2 \end{array} \qquad \begin{array}{rl} \text{And } 2AC^2 = & 2a^2 \\ 2CD^2 = & \qquad 2m^2 \\ \hline = & 2a^2 + 2m^2 \end{array}$$

SCH.—1. The Proposition may be expressed, "*the sum of the squares of any two lines is equal to twice the square of half their sum together with twice the square of half their difference.*"

$$\begin{array}{ll} \text{Because } AD^2 = (AC + CD)^2 & = AC^2 + 2AC.CD + CD^2 \\ \text{And } BD^2 = (BC - CD)^2 = (AC - CD)^2 & = AC^2 - 2AC.CD + CD^2 \\ \text{By addition } AD^2 + BD^2 & = 2AC^2 \quad + \quad 2CD^2 \end{array}$$

$$\text{But } AC = \frac{AD + DB}{2} \text{ and } CD = \frac{AD - DB}{2}$$

Therefore, *the sum of the squares of any two lines*, &c. Q. E. D.

2. Or, *the sum of the squares is equal to half the square of the sum together with half the square of the difference;* thus $AD^2 + BD^2 = \frac{(AD + DB)^2}{2} + \frac{(AD - DB)^2}{2}$; or in numbers, $14^2 + 6^2$ *i.e.*, $196 + 36 = \frac{20 \times 20}{2} + \frac{8 \times 8}{2}$ *i.e.*, $200 + 32 = 232$.

PROP. 10.—THEOR.

If a st. line be bisected and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line and of the square of the line made up of the half and the part produced.

CONS.—11. I., 3. I., 31. I., PST. 1.

DEM.—29. I., AX. 9.

AX. 12. If a st. line meet two st. lines, so as to make the two int. angles on the same side of it taken together less than two rt. angles, these two st. lines, being continually produced, shall at length meet on that side on which the angles are less than two rt. angles.

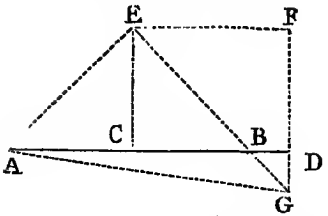
5. I., 32. I.

15. I. If two st. lines cut one another, the opp. or vertical angles shall be equal.

6. I., 34. I.

36. I. Parallelograms upon equal bases and between the same parallels are equal.

47. I.



Exp. 1	Hyp.	Let | AB be bisected in C and produced to . D ,
2	Concl.	then the squares on AD and DB = twice ($AC^2 + CD^2$).
Cons. 1	*by* 11. I.	From . C draw CE at rt. angles to AB ;
2	3. I. & Pst. 1.	make CE = AC or CB, and join AE, EB ;
3	31. I.	through E and D draw EF ∥ AB, DF ∥ CE ;
4	Pst. 2 & 1.	produce FD and EB to meet in . G, and join AG ;
5	Concl.	then $AD^2 + DB^2$ = twice ($AC^2 + CD^2$).
Dem. 1	*by* C. 3 & 29. I.	∵ st. line EF meets the ∥s EC and FD, the ∠s CEF and EFD = two rt. angles.
2	Ax. 9.	and ∴ the ∠s BEF and EFD < two rt. angles.
3	Ax. 12, Pst. 1.	the lines EB and FD ∴ will meet as in G. Join AG ;
4	C. 1 & 5 I.	then ∵ AC = CE, the ∠ AEC = ∠ CAE ;
5	C. 1.	and ACE is a rt. angle ;
6	32. I.	∴ each ∠ AEC, ∠ CAE = half a rt. angle :
7	*Sim.*	And ∠s CEB, CBE being each half a rt. angle, AEB is a rt. angle.
8	D. 7. & 15. I.	And EBC being half a rt. ∠, its vertical ∠ DBG is half a rt. ∠ :
9	29. I. & C. 1.	but ∠ BDG = the alt. ∠ DCE, which is a rt. ∠ ;
10	32. I.	∴ the rem. ∠ DGB = half a rt. angle and also equals ∠ DBG ;
11	6. I.	and ∴ the side BD = the side DG.
12	D. 10 & 34. I.	Again, ∵ ∠ EGF = half a rt. ∠, and ∠ EFG = ECD a rt. ∠ ;
13	32. I. & D. 12.	∴ rem. ∠ FEG = half a rt. ∠, and ∠ FEG = ∠ EGF ;
14	6. I.	and ∴ the side FG = the side FE.

DEM. 15	C. 1. & 36. I.	And ∵ side EC = side AC, ∴ the squares on EC and AC are equal,
16	Ax. 1 & 6.	and ∴ the squares on EC and AC = twice the square on AC:
17	47. I. & D. 16.	but the square on AE = the two squares on EC and AC; and ∴ the square on AE = twice the square on AC.
18	D. 14 & 36. I.	Again, ∵ FG = FE, FG sq. = FE squared;
19	Ax. 1 & 6.	and ∴ the squares on FG and FE = twice the square on FE:
20	47. I. & 34 I.	but the square on EG = the two squares on FG and FE; and ∴ the square on EG = twice the square on FE or on CD:
21	D. 17.	And from the above, the square on AE = twice the square on AC;
22	D. 20 & 21.	∴ $AE^2 + EG^2$ = twice $(AC^2 + CD^2)$.
23	47. I.	But $AG^2 = AE^2 + EG^2$;
24	Ax. 1.	∴ AG^2 = twice $(AC^2 + CD^2)$:
25	47. I. & D. 11.	but AG^2 also = $(AD^2 + GD^2)$ = $(AD^2 + BD^2)$;
26	Concl.	∴ $AD^2 + BD^2$ = twice $(AC^2 + CD^2)$.
27	Recap.	*Wherefore, if a st. line be bisected and produced,* &c. Q.E.D.

Alg. & Arith. Hyp.—Let $AB = 2a = 20$; $\frac{AB}{2} = AC = CB = a = 10$; and $BD = m = 2$.

Alg. Then $AB + BD = AD = 2a + m$
and $CB + BD = CD = a + m$
therefore $(2a + m)^2 = 4a^2 + 4am + m^2$
$(+m)^2$ and $(2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2$
Again $(a + m)^2 = a^2 + 2am + m^2$
$(+a^2)$ and $(a + m)^2 + a^2 = 2a^2 + 2am + m^2$
$(\times 2)$ and $2(a + m)^2 + 2a^2 = 4a^2 + 4am + 2m^2$
But $(2a + m)^2 + m^2 = 4a^2 + 4am + 2m^2$
therefore $(2a + m)^2 + m^2 = 2a^2 + 2(a + m)^2$

Arith. Then $AB + BD = AD = 20 + 2 = 22$
and $CB + BD = CD = 10 + 2 = 12$
therefore $(20 + 2)^2 = (4 \times 100) + (4 \times 20) + 4$
$(+4)$ and $(20 + 2)^2 + 4 = (4 \times 100) + (4 \times 20) + (2 \times 4)$
Again $(10 + 2)^2 = 100 + 40 + 4$

$(+100)$ and $(10+2)^2+100 = 200+40+4$
$(\times 2)$ and $2\ (10+2)^2+200 = 400+80+8$
But $(20+2)^2+4 = 400+80+8$
therefore $(20+2)^2+4 = 200+2\ (10+2)^2$
or $484+4 = 200+288 = 488$

In another form the Algebraical and Arithmetical illustration may be stated.

Let AC or CB $= a = 10$; BD $= m = 2$; CD $= a+m = 10+2 = 12$;
and AD $= 2a+m = 20+2 = 22$

Then $AD^2 = 4a^2+4am+m^2$ And $2\,AC^2 = 2a^2$
$DB^2 = m^2$ $2\,CD^2 = 2a^2+4am+2m^2$

$AD^2+DB^2 = 4a^2+4am+2m^2 = 2\,AC^2+2\,CD^2 = 4a^2+4am+2m^2$

Or, $22 \times 22 = 484$ And $2 \times 10 \times 10 = 200$
$2 \times 2 = 4$ $2 \times 12 \times 12 = 288$
Sum 488 $=$ 488

Sch.—1. Propositions 9 and 10 are applicable to Algebra, and like Prop. 5 and 6 are identical; for the different enunciations arise from the two ways of representing the differences between two lines.

Gen. Use and App. of Prop. I.–X., Bk. II.—These ten Propositions contain the whole theory of the relations of rectangles and squares formed by lines and their parts. As we have shown, all these relations may be expressed Algebraically, and may also be applied equally well to numbers.

"*When lines are expressed numerically, various problems may be proposed respecting them, the solution of which may be derived from the preceding propositions.* We shall here subjoin some of those problems, which will probably be sufficient to familiarise the student with such investigations."

1°. *Given the sum* S, *and difference* D, *of two magnitudes,* AB *the greater and* CD *the less; to find the magnitudes themselves.* The formulas are,—
$\frac{S}{2}+\frac{D}{2}$ = AB the greater; and $\frac{S}{2}-\frac{D}{2}$ = CD the less.

2°. Since the area of a rectangle expressed in numbers is equal to the product of its sides,—*if the Area be divided by one side, the quotient will give the other:* thus, let the sides be AB and BC; the rectangle is expressed by AB . BC; and $\frac{AB.BC}{BC} = AB$; $\frac{AB.BC}{AB} = BC$; or let the area in numbers be 144, and the sides 9 and 16; then $\frac{144}{9} = 16$; and $\frac{144}{16} = 9$.

3°. "There are five quantities depending on a rectangle, any two of which being given, the sides of the rectangle can be found:
1°. The sum of the sides. 2°. The difference of the sides. 3°. The area. 4°. The sum of the squares of the sides. 5°. The difference of the squares of the sides.

These being combined, 1° and 2°; 1° and 3°; 1° and 4°; 1° and 5°; 2° and 3°; 2° and 4°; 2° and 5°; 3° and 4°; 3° and 5°; and 4° and 5°, produce ten Problems,—of which the combination 3° and 5° alone presents any difficulty. The solutions will form useful exercises for the learner."—Lardner's *Euclid*, pp. 79 and 80.

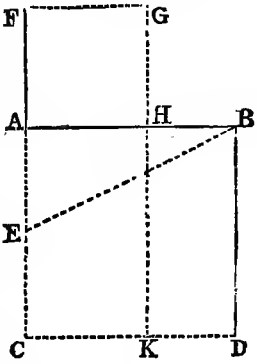
PROP. 11.—PROB.

To divide a given st. line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

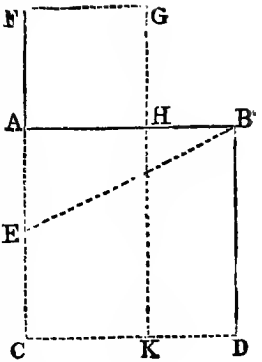
SOL.—46. I. To describe a square on a given st. line.
10. I. To bisect a given finite st. line.
3. I. From the greater st. line to cut off a part equal to the less.
Psts. 1 and 2.

DEM.—6. II. If a st. line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square of half the line bisected, is equal to the square of the st. line which is made up of the half and of the part produced.
47. I. In any rt. angled triangle, the square of the hypotenuse is equal to the sum of the squares of the base and perpendicular.
Ax. 3 and 1.
Def. 30. I. A square has all its sides equal, and its angles rt. angles.

EXP. 1	*Datum.*	Let AB be the given st. line;
2	*Quæs.*	to divide it into two parts, so that rect. AB . BH = AH square.
CONS. 1	*by* 46. I.	On AB describe a square ABDC;
2	10. I. & Pst. 1	bisect AC in E, making AE = EC, and join EB;
3	Pst. 2, & 3. I.	produce CA to F, making EF = EB;
4	46. I.	and on AF describe the square AHFG;
5	Pst. 2.	Produce GH to K;
6	Sol.	the line AB is divided in H, so that AB . BH = the square on AH.
DEM. 1	*by* C. 2 & 3.	∵ st. line AC is bisected in E, and produced so that EF = EB;
2	6. II. C. 2.	∴ □ CF . AF + $AE^2 = EF^2$ and $EF^2 = EB^2$;
3	47. I.	also the square on EB = the squares on AB and AE:



DEM. 4	*Sub.* Ax. 3.	Take away AE^2 from each; then $\square CF \cdot AF = AB^2$.
5	Const.	But $\square FK = CF \cdot FA$, and $AD^2 = AB^2$,
6	Ax. 1.	∴ the square AD = the rect. FK:
7	*Sub.* Ax. 3.	Take away AK, and the square FH = the rect. HD:
8	C. & Def. 30	But $\square$ HD is contained by HB. BD, or HB . AB;
9	C. 4.	and FH is the square on AH;
10	Concl.	∴ the $\square AB \cdot BH = AH^2$.
11	Recap.	*Wherefore, the given line has been divided, &c.* Q.E.F.



N.B.—A line thus divided *is* (in 30. vi.) *said to be cut in extreme and mean ratio.*

The Algebraical Solution may be given thus:

Take $AB = a$,—*required the point* H, *so that* $AB \cdot BH$ *shall equal* AH^2.

Let $AH = x$; then $HB = a - x$.

Now by the prob. $a(a-x) = x^2$; or $x^2 = a^2 - ax$; *i.e.*, $x^2 + ax = a^2$.

Solving the quadratic, we have $x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$

extracting the root, $x + \frac{a}{2} = \frac{a\sqrt{5}}{2}$. Hence $x = \frac{\pm a\sqrt{5} - a}{2}$

Thus AH the one part $= x = \frac{\sqrt{5}-1}{2} \cdot AB$,

and HB the other part $= a - x = a - AH = \frac{3-\sqrt{5}}{2} \cdot AB$.

Or, "*required how far a given line must be produced so that the rectangle contained by the given line and the line made up of the given line and the part produced, may be equal to the square of the part produced.*"—POTT'S *Euclid*, p. 73.

Or, "*to find two lines having a given difference, such that the rectangle contained by the difference and one of them may be equal to the square of the other.*"

The Arithmetical Solution cannot be expressed in whole numbers; we can, however, *approximate* to the values of AH and HB by extending the root of

5 to any number of decimal places desired. Thus, supposing $AB = a = 10$, $AH = x = \frac{\sqrt{5}-1}{2} . a$ and $HB = a - x = \frac{3-\sqrt{5}}{2} . a$

1st. $\frac{\sqrt{5}-1}{2} . a = \frac{2{\cdot}236068 - 1}{2} \times 10 = \frac{1{\cdot}236068}{2} \times 10 = 6{\cdot}18034 = AH$;

2nd. $\frac{3-\sqrt{5}}{2} . a = \frac{3 - 2{\cdot}236068}{2} \times 10 = \frac{{\cdot}763932}{2} \times 10 = 3{\cdot}81966 = HB$;

And $AH + HB = AB = 6{\cdot}18034 + 3{\cdot}81966 = 10{\cdot}00000$.

COR. I.—*To cut a st. line in extreme and mean ratio, it must first be produced in extreme and mean ratio;* that is, CF . FA must equal AB^2.

COR. II.—When a st. line EF, or its equal, is cut in extreme and mean ratio, the rectangle AC . (AC − AF) is equal to the square of AH, or AF, or AC . HB = AH^2.

Hence, if a st. line CF be cut in extreme and mean ratio, the greater segment AC will be cut in the same manner, by taking in it a part equal to the less AF; and the less AF will be similarly cut by taking in a part equal to the difference (AC − AF) or HB; and so on.

COR. III.—A st. line CF being cut in extreme and mean ratio, the rect. AC . AF under its segments = $AC^2 - AH^2$, the difference between their squares; thus, 10 × 6·18034 = 100 − 38·1966 = 61·8034.

SCH.—Let A be a st. line cut in extreme and mean ratio, G the greater segment, L the less, and D the difference: then 1st. $A^2 + L^2 = 3\,G^2$; 2nd. $(A + L)^2 = 5\,G^2$; 3rd. A . D = G . L; and 4th. L^2 = G . D.

USE AND APP.—This 11th Prop. is applied in 10. iv., to the drawing of an isosceles triangle, of which each of the angles at the base is double of the third angle; and in 30. vi., to the cutting of a st. line in extreme and mean ratio;—that is, so that the whole st. line is to the greater segment as the greater is to the less. The construction of pentagons in bk. iv. also depends on this problem, and of regular bodies, as described in bks. xiii., xiv., and xv., called also the Platonic Solids.

PROP. 12.—THEOR.—*(Important.)*

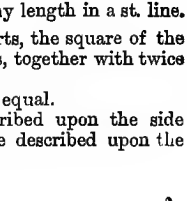
In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides

containing the obtuse angle by twice the rectangle contained by the side upon which when produced the perpendicular falls, and the st. line intercepted without the triangle between the perpendicular and the obtuse angle.

CONS.—12. I. To draw a st. line perpendicular to a given st. line of an unlimited length from a given point without it.
PST. 2. A terminated st. line may be produced to any length in a st. line.

DEM.—4. II. If a st. line be divided into any two parts, the square of the whole line is equal to the square of the two parts, together with twice the rectangle contained by the parts.
AX. 2. If equals be added to equals, the wholes are equal.
47. I. In any rt. angled triangle, the square described upon the side subtending the rt. angle is equal to the square described upon the side containing the right angle.

EXP. 1	*by* Hyp.	Let ABC a triangle have an obtuse ∠ ACB;
CONS. 2	*by* 12. I. & Pst. 2	From . A to BC produced draw a perp. AD;
3	Concl.	then $AB^2 > (AC^2 + BC^2)$ by twice the rect. BC. CD.
DEM. 1	*by* Hyp.	For, since | BD is divided into two parts at . C;
2	4. II.	$\therefore BD^2 = BC^2 + CD^2 + 2\,BC.CD$:
3	*Add.* & Ax. 2.	adding AD^2 to each, $BD^2 + AD^2 = BC^2 + CD^2 + AD^2 + 2BC.CD$.
4	C. 2 & 47. I.	But ∠ ADB being a rt. ang., $AB^2 = BD^2 + AD^2$;
5	47. I.	and also $AC^2 = CD^2 + AD^2$;
6	Concl.	therefore $AB^2 = BC^2 + AC^2 + 2\,BC.CD$;
7	Explan.	*i.e.*, $AB^2 > AC^2 + BC^2$ by twice BC. CD.
8	Recap.	*Wherefore, in obtuse angled triangles, if a perpendicular*, &c. Q.E.D.



Alg. & Arith. Hyp.—Let BC = $a = 5$; CA = $b = 6{\cdot}708$; AB = $c = 10$; and let CD = $m = 3$; DA = $n = 6$;
then BD = $a + m = 5 + 5 = 8$.
the triangles ABD and ACD are rt. angles, viz., at ADB and ADC.

Alg. Since $c^2 = (a+m)^2 + n^2$; and $b^2 = m^2 + n^2$;
therefore $c^2 - b^2 = (a+m)^2 - m^2 = a^2 + 2\,am + m^2 - m^2 = a^2 + 2\,am$.
and $c^2 = b^2 + a^2 + 2\,am$; *i.e.*, c^2 is greater than $b^2 + a^2$ by twice am.

Arith. Since $10^2 = (5+3)^2 + 6^2 = 64+36$; and $6{\cdot}708^2$ or $45 = 3^2 + 6^2 = 9+36$; therefore $10^2 - 6{\cdot}708^2$, or $100 - 45 = (5+3)^2 - 3^2 = 64-9 = 25+30$; and $10^2 = 6{\cdot}708^2 + 5^2 + (2\times5\times3) = 45+25+30$: *i.e.*, 10^2 or 100 greater than $45+25$ by 30.

USE AND APP. — 1. By this Proposition *the Area of a triangle may be ascertained when the three sides are known.*

Let the sides AB = 20, AC = 13, and BC = 11.

Now (12. I.) $AB^2 = AC^2 + BC^2 + 2\,BC.CD$;

therefore $AB^2 - AC^2 - BC^2$; or $AB^2 - (AC^2 + BC^2) = 2\,BC.CD$; *i.e.*, $2\,BC.CD = 400 - (169+121) = 400-290 = 110$;

therefore $BC.CD = \frac{110}{2} = 55$, and $\frac{BC.CD}{BC} = CD = \frac{55}{11} = 5$:

But (47. I.) $AC^2 - CD^2 = AD^2$; *i. e.*, $169 - 25 = 144$; and $AD = \sqrt{144} = 12$.

We have now ascertained AD, the altitude of the triangle ABC, and BC, the base, is given :

Again (41. I.) the Area of a triangle equals half that of a parallelogram on the same base and of the same altitude :

Thus the Area of triangle $ABC = \frac{AD.BC}{2} = \frac{12\times11}{2} = \frac{132}{2} = 66$.

Or, by bisecting the line BC in G, we obtain the same result ; thus,

Since $\frac{AB^2 - AC^2}{2l} = BC.DG$; therefore $\frac{BC.DG}{BC} = DG$;

And $DG + \frac{BC}{2} = DB$; then (47. I.) $AB^2 - BD^2 = AD^2$; whence we find AD itself.

Numerically, $\frac{400-169}{2} = 115.5$, therefore $\frac{115.5}{11} = 10.5 = DG$;

And $10.5 + \frac{11}{2} = 16$; then (47. I.) $400 - 256 = 144$; and $\sqrt{144} = AD$ as before.

PROP. 13.—THEOR.

In every triangle, the square of the side subtending either of the acute angles is less than the squares of the sides containing that acute angle by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

CONS. 12. I. Pst. 2.

DEM.—7. II. If a st. line be divided into any two parts, the squares of the whole line and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

AX. 2. 47. I.

16. I. If one side of a triangle be produced, the exterior angle is greater than either of the int. and opposite angles.

12. II. In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

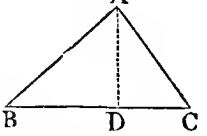
3. II. If a st. line be divided into any two parts, the rectangle contained by the whole and one of the parts equals the rectangle contained by the two parts together with the square of the aforesaid part.

AX. 6. Doubles of the same are equal.

EXP. 1	Hyp.	Let $\triangle$ ABC have the $\angle$ B acute,
CONS. 2	*by* 12. I.	On BC one of the sides let the perp. AD fall from $\angle$ BAC;
3	Concl.	then $AC^2 < (AB^2 + BC^2)$ by twice BC . BD.

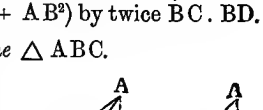
CASE I.—*Let the perp.* AD *fall within the triangle* ABC.

DEM. 1	*by* C. 2 & 7. II.	Since BC is divided at D, $CB^2 + BD^2 = CD^2 + 2\,BC . BD$;
2	*Add.* & Ax. 2.	On adding AD^2, $CB^2 + BD^2 + AD^2 = CD^2 + AD^2 + 2\,BC . BD$:
3	C. 2 & 47. I.	But $\angle$ s at D being rt. $\angle$ s $AB^2 = BD^2 + AD^2$, and $AC^2 = CD^2 + AD^2$;
4	D. 2 & 3.	$\therefore CB^2 + AB^2 = AC^2 + 2\,BC . BD$;
5	Concl.	that is, $AC^2 < (BC^2 + AB^2)$ by twice BC . BD.



CASE II.—*Let the perp.* AD *fall without the* $\triangle$ ABC.

DEM. 1	*by* C. & 16. I.	Since $\angle$ D is a rt. $\angle$, $\angle$ ACB > a rt. $\angle$.
2	12. II.	$\because AB^2 = AC^2 + BC^2$ + twice BC . CD;
3	*Add.* Ax. 2.	add BC^2 to each, and $AB^2 + BC^2 =$ AC^2 + twice $(BC^2 + BC . CD)$.
4	C. & 3. II.	But st. line BD being divided in . C, the $\square$ BD . BC = $BC^2 + \square$ BC . CD;



DEM.		
5	Ax. 6.	Now the doubles of equals being equal,
6	Ax. 6.	twice $\square$ BD . BC = twice $\square$ BC . CD + twice BC^2:
7	D. 3 & 6.	$\therefore AB^2 + BC^2 = AC^2 +$ twice $\square$ BD . BC ;
8	Concl.	$\therefore AC^2$ alone $< AB^2 + BC^2$ by 2 BD . BC.

CASE III.—Lastly, *let the side* AC *be perpendicular to* BC.

DEM.		
1	47. I.	Here $AB^2 = AC^2 + BC^2$;
2	*Add* & Ax. 2.	and adding BC^2, then $AB^2 + BC^2 = AC^2 + 2\,BC^2$, or 2 BC . BC.
3	Recap.	*Wherefore, in any triangle, the square subtending, &c.* Q.E.D.

CASE I.—*Alg. & Arith. Hyp.* Let $BC = a = 10$; $AB = c = \sqrt{61} = 7{\cdot}8102$; $AC = b = \sqrt{41} = 6{\cdot}4031$; $BD = m = 6$; $AD = n = 5$; and $DC = a - m = 10 - 6 = 4$.

Alg. (47. I.) $c^2 = n^2 + m^2$ and $b^2 = n^2 + (a-m)^2$
Subtracting, $c^2 - b^2 = m^2 - (a-m)^2 = m^2 - (a^2 - 2am + m) = 2am - a^2$
Transpose, $a^2 + c^2 = b^2 + 2am$; or $b^2 + 2am = a^2 + c^2$
therefore, b^2 is less than $a^2 + c^2$ by $2am$.

Arith. (47. I.) $61 = 25 + 36$, and $41 = 25 + 16$
Subtracting, $61 - 41 = (2 \times 10 \times 6) - 100 = 120 - 100 = 20$
Transposing, $100 + 61 = 41 + 120$; or $41 + 120 = 100 + 61$
therefore, 41 is less than $100 + 61$ by 120.

CASE II.—The perpendicular AD here passes out of the triangle, and the positions of C and D are changed ; so that we have $BC = a = 2$; $AB = c = \sqrt{61}$; $AC = b = \sqrt{41}$; $BD = m = 6$; $AD = n = 5$; and $DC = m - a = 6 - 2 = 4$.

Alg. (47. I.) $c^2 = m^2 + n^2$ and $b^2 = (m-a)^2 + n^2$
Subtracting $c^2 - b^2 = m^2 - (m-a)^2 = m^2 - m^2 + 2am - a^2 = 2am - a^2$
Transpose $a^2 + c^2 = b^2 + 2am$; or $b^2 + 2am = a^2 + c^2$
therefore b^2 is less than $a^2 + c^2$ by twice am

Arith. (47. I.) $61 = 36 + 25$, and $41 = 16 + 25$
Subtracting $61 - 41 = 36 - 16 = 36 - 36 + 24 - 4 = (2 \times 12) - 4$
Transpose $4 + 61 = 41 + (2 \times 12)$; or $41 + 24 = 4 + 61$
therefore 41 is less than $4 + 61$ by 24.

CASE III.—The perpendicular AD and the side AC coincide, and the points D and C ; so that we have $BC = a = 4$; $AD = AC = b = 5$; and $AB = c = \sqrt{41}$.

Alg. (47. I.) $b^2 + a^2 = c^2$	*Arith.* (47. I.) $25 + 16 = 41$
Add a^2 and $b^2 + 2a^2 = c^2 + a^2$	Add 16 and $25 + 32 = 41 + 16$
i.e., b^2 is less than $c^2 + a^2$ by $2a^2$ or $2aa$	*i. e.*, 25 is less than $41 + 16$ by 2×16

Cor. I. — *If in the figure to* Case II. *a perpendicular* CG *be drawn from the angle* C *to* AB, *the rectangle under the side* AB *and the part* GB *intercepted between the perpendicular and* B *is equal to the rectangle of* BC . DB. The two rectangles AB . BG and BC . DB each equal half the difference between the square AC and the squares AB and BC; *i.e.*, $\frac{(AB^2 + BC^2) - AC^2}{2}$

$= BC \cdot BD = AB \cdot BG$. As $\frac{(61 + 4 - 41)}{2} = \frac{65 - 41}{2} = 12 = 2 + 6$

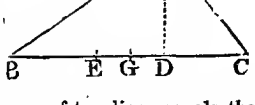
$= 7{\cdot}8012 + 1{\cdot}5364$.

Sch.—The Propositions 12 and 13 are of high importance,—*for they contain the elements of Trigonometrical Analysis, or the Arithmetic of Sines.*

Use and App.—1. In finding the Area of a Triangle, it is of the greatest advantage to obtain the perpendicular either by calculation or by measurement. *When the three sides of a triangle are given, the perpendicular may be obtained,* on the foregoing principles, in either of the ways following:

First Method.—*When the perpendicular,* AD, *falls within the base.*

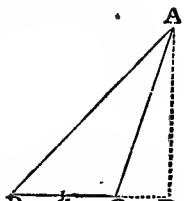
By 12. I., from ∠ A draw AD perpen. to BC;
47. I., in triangle ABD, $AB^2 = AD^2 + BD^2$
47. I., and in triang. ACD, $AC^2 = AD^2 + CD^2$
Subtract, and $AB^2 - AC^2 = BD^2 - CD^2$
By 3 & 10. I. From BC cut off BE equal to DC, and bisect ED in G;
ED is evidently bisected in G, so that 2 GD = ED.



Now, since, by Cor. 5. II., the difference of the squares of two lines equals the rectangle under their sum and difference;
$AB^2 - AC^2 = (BA + AC)(BA - AC) = (BD + DC)(BD - DC) = BC \times 2\,GD = 2\,BC \times GD$.
Dividing now both sides of the equation by 2 BC, we have
$GD = \frac{(BA + AC)(BA - AC)}{2\,BC}$; and since $BG = \frac{BC}{2}$ $\therefore BD = \frac{BC}{2} + GD$,
and $AD = \sqrt{AB^2 - BD^2}$ the perpendicular required.

Ex. *Given* BC = 5, AB = 4, *and* AC = 3; *required the perpendicular* AD.
Bisect BC in G; then $GD = \frac{7 \times 1}{2 \times 5} = \frac{7}{10}$; and $DB = \frac{5}{2} + \frac{7}{10} = 3\frac{1}{5}$
therefore, by 47. I., $AD = \sqrt{16 - \frac{256}{25}} = \sqrt{\frac{400}{25} - \frac{256}{25}} = \sqrt{\frac{144}{25}} = \frac{12}{5} = 2\frac{2}{5}$

SECOND METHOD.—*When the perpendicular, A D, falls without the base.*



By 12. I., from $\angle$ A draw AD perpendicular to BC produced;

47. I., in triangle ABD, $AB^2 = AD^2 + BD^2$

47. I., and in triangle ACD, $AC^2 = AD^2 + CD^2$

Subtract, and $AB^2 - AC^2 = BD^2 - CD^2$

By 10. I., let BC be bisected in G; then BD + CD = 2GD, because BD = 2GB + CD, and BD + 2CD = 2GB + 2CD = 2GD. Also BD − CD = BC.

But, by Cor. 5. II., $AB^2 - AC^2 = (AB + AC)(AB - AC) = (BD + CD)(BD - CD)$

Thus $BC \times 2GD$, or $2BC \times GD = (AB + AC)(AB - AC)$

Divide by 2BC,—and $GD = \frac{(AB + AC)(AB - AC)}{2BC}$

Now CD = GD − GC; and (47. I.) $AD = \sqrt{AC^2 - (GD - GC^2)}$

Ex. *Given the three sides* BA = 5, AC = 4, *and* BC = 2; *required the perp.* AD.

By the formula $GD = \frac{(5+4)(5-4)}{2 \times 2} = \frac{9}{4} = 2\frac{1}{4}$

$$CD = 2\tfrac{1}{4} - 1 = 1\tfrac{1}{4} \text{ and } AD = \sqrt{16 - \frac{25}{16}} = \frac{\sqrt{256 - 25}}{16} = \frac{\sqrt{231}}{4}$$

$$= \frac{15{\cdot}198684}{4} = 3{\cdot}799671.$$

2. When the intercepts between the foot of the perpendicular and the angles on the base are ascertained, the *Area of the Triangle* is by the First Method $\frac{BD + DC}{2} \times AD$; and by the Second Method $\frac{BD - DC}{2} \times AD$; that is, half the rectangle of the base multiplied by the perpendicular.

PROP. 14.—PROB.

To describe a square that shall be equal to a given rectilineal figure.

SOL.—45. I. To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.

Def. 30. A square, &c.

Psts. 2 and 3; 3. I. 10. I. To bisect a given finite st. line.

Pst. 1.

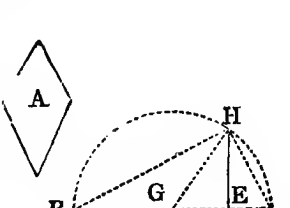
DEM.—5. II. If a st. line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts together with the square of the line between the points of section, equals the square of half the line.

Def. 15 and 16. A circle is a plane figure contained by one line called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. And this point is called the centre of the circle.

47. I. The square of the hypotenuse, &c.

Ax. 3. If equals be taken, &c.

Ax. 1. Things equal, &c.



EXP. 1	*Datum.*	Let A be the given rectil. figure,
2	*Quæs.*	to make a square equal to the fig. A.
CONS. 1	*by* 45. I.	Describe a right angled $\square$ BCDE = A :
2	Def. 30.	If the side BE = ED, the fig. is a square ;
3	*Sup.*	but if BE $\neq$ ED,
4	Pst. 2 & 3. I.	produce BE to F, and make EF = ED :
5	10. I. Pst. 3.	Bisect BF in G, and from centre G with distance GB describe the semicircle BHF ;
6	Pst. 2 & 1.	produce DE to meet circumference in H, and join GH :
7	Sol.	the square on EH = the figure A.
DEM. 1	*by* C. 5.	$\because$ BF is bisected in G, and cut unequally in E ;
2	5. II. Def. 16	$\because$ $\square$ BE.EF + EG^2 = GF^2, and GF^2 = GH^2 :
3	47. I.	But the square on GH = EH^2 + EG^2 ;
4	*Sub.* & Ax. 3	taking away EG^2, then $\square$ BE.EF, = EH^2 ;
5	C. 1. Ax. 1.	But $\square$ BD = BE.ED, or BE.EF ;
6	Concl.	$\therefore$ $\square$ BD = the square on EH ;
7	C. 1 & D. 6	But $\square$ BD = the fig. A : $\therefore$ EH^2 = fig. A.
8	Recap.	*Wherefore, a square has been made equal,* &c.

Q. E. F.

Alg. & Arith. Hyp.—Given the fig. A = *ab* = 36, in area ; required the side EH = *x* of a square equal in area. Let the lines EH = *x* ; BE one side of the rectangle = *b* = 9, and ED the other side = *a* = 4.

Alg. By the problem, $ab = x^2$	*Arith.* $4 \times 9 = x^2$
therefore, ($\surd$) $\sqrt{ab} = x$ the side of the square.	and $\sqrt{36} = x = 6$ the side of the square.

USE AND APP.—1. By this proposition we may find a *mean proportional to two given lines,* as demonstrated in Prop. 13, bk. vi.

Given the lines BE and EF; their sum $= 2r = 13$; $BE = x = 9$; $EF = 2r - x = 13 - 9 = 4$: required to find y, a mean proportional to x, and $2r - x$; *i.e.*, so that $x : y :: y : 2r - x$.

Bisect BE in G; $GF = r = 6{\cdot}5$	$(9 \times 4) + 6{\cdot}25 = 6{\cdot}5^2 = 42{\cdot}25$
By 5. II. $x.(2r-x) + (x-r)^2 = r^2$	$42{\cdot}25 = y^2 + 6{\cdot}25$
But $GF^2 = r^2 = y^2 + (x-r)^2$	$9 \times 4 = 36 = y^2$
Subtract $(x-r)^2$ And $x.(2r-x)$ $= y^2$	

Extract the $\surd$; $y = \surd 36 = 6$ the mean proportional between 9 and 4.

Briefly, *Multiply the two numbers, and take the square root of their product* for the mean proportional.

2. The same Prop. 14 serves also for *approximating* to the square of curve-lined figures, and even of the circle itself; *for the circle may be regarded* (as in 41. I., Use 4) *as a polygon with an infinite number of sides*;—and if we reduce this polygon to a square, we *nearly* obtain the square of the circle.

3. *The Areas of all plane figures are calculated in squares*; and by means of reducing any right-lined figure, first into a rectangle, and then into a square, we obtain the ready means of finding the Area in the units of surface.

N.B.—A selection of the most useful and remarkable problems and theorems, which may be inferred from the principles developed in the second book, will be found in the APPENDIX to the Gradations in Euclid, SERIES II., *Problems and Theorems*, bk. ii.

REMARKS ON BOOK II.

1. Of the fourteen Propositions of this book, the *ten* first contain the theory of the relations of the rectangles and squares on divided lines; the *twelfth* and *thirteenth*, the theory of the relation between the square of any one side of a triangle, and the squares of the other two sides, whatever be the angles,—and constitute the foundation of the Arithmetic of Sines.

2. Lines cut into *any two parts* are considered in Propositions 2, 3, 4, 7, and 8.

3. Lines cut into *two equal* and *two unequal parts*,—in Propositions 5, 6, 9, and 10.

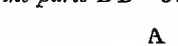
SYNOPSIS OF BOOK II.

CASE 1.—*Given two st. lines, as* A *undivided* = 10, *and* BC = 60 *divided into the parts* BD = 30, DE = 20, *and* EC = 10.

PROP. I. $AB.BC = AB.BD + AB.DE + AB.EC$; or $10 \times 60 = (10 \times 30) + (10 \times 20) + (10 \times 10) = 600$.

Cor. $2\,AB.\frac{1}{2}BC$; or $3\,AB.\frac{1}{3}BC$; or $4\,AB.\frac{1}{4}BC$, &c. $= AB.BC$.
20×30; 30×20; $40 \times 15 = 10 \times 60 = 600$.

CASE II.—*Let a st. line be divided into any two parts;* AB = 10, *representing the whole line;* AD = 7, *and* DB = 3, *the unequal segments.*



PROP. II. $AB.AD + AB.DB = AB^2$; or $(10 \times 7) + (10 \times 3) = 10 \times 10 = 100$.

PROP. III. $AB.BD = AD.DB + DB^2$; or $(10 \times 3) = (7 \times 3) + (3 \times 3) = 30$.

Cor. 1. $AB^2 - DB^2 = (AB + DB)(AB - DB)$; or $100 - 9 = 13 \times 7 = 91$.

2. $AD^2 - DB^2 > (AD - DB)^2$ by $2\,DB.(AD - DB)$; or $49 - 9 > 16$ by $2 \times 3 \times 4$ or by 24.

PROP. IV. $AB^2 = AD^2 + DB^2 + 2\,AD.DB$; or $100 = 49 + 9 + (2 \times 7 \times 3) = 49 + 9 + 42$.

Cor. 1. Parallelograms about the diameter of a square are also squares.

2. $AB^2 = 4\left(\frac{AB}{2}\right)^2$; or $100 = 4 \times 25$.

3. $\frac{AB^2}{2} = 2\left(\frac{AB}{2}\right)^2$; or $\frac{100}{2} = 2 \times 25$.

4. When a line is divided into any number of parts, as $x = 5$; $y = 3$ $z = 2$;
then $AB^2 = x^2 + y^2 + z^2 + 2(x.y + x.z + y.z)$
$100 = 25 + 9 + 4 + 2(15 + 10 + 6)$

PROP. VII. $AB^2 + BD^2 = 2\,AB.BD + AD^2$, or $100 + 9 = (2 \times 10 \times 3) + 49 = 109$.

Cor. 1. $AB^2 + BD^2 = 2\,AB.BD + (AB - BD)^2$; or $100 + 9 = 60 + (10 - 3)^2 = 109$.

2. $(AB + BD)^2$; $(AB^2 - BD^2)$; and $(AB - BD)^2$ are in Arithmetical Progression, the common difference being $2\,AB.BD$; or,

169, 109, and 49; the common difference being $2 \times 10 \times 3 = 60$.

PROP. VIII. $4\,AB.BD + AD^2 = (AB + BD)^2$; or $4\,(10 \times 3) + 49 = (10 \times 3)^2 = 169$.

Cor. 1. $(AD + DB)^2 = 4\,AD.DB + (AD - DB)^2$; or $(7 + 3)^2$ $4\,(7 \times 3) + (7 - 3)^2 = 100$.

2. $4\left(\frac{AD+DB}{2}\right)^2 = 4\,AD.DB + 4\left(\frac{AD-DB}{2}\right)^2$; or $4 \times 25 = (4 \times 21) + (4 \times 4) = 84 + 16$.

CASE III.—*Let a st. line be divided equally and unequally,* AB = 10 *representing the whole line;* AC *or* CB = 5, *the half line; and* AD = 7; DB = 3, *the unequal parts.*

A C D B

PROP. V. $AD.DB + CD^2 = CB^2$; or $(7 \times 3) + (2 \times 2) = 5 \times 5$; *i.e.*, $21 + 4 = 25$.

N.B.—The arith. mean $= \frac{AD+DB}{2} = AC$; the com. dif. $= \frac{AD-DB}{2} = CD$.

$\therefore \left(\frac{AD+DB}{2}\right)^2 = AD.DB + \left(\frac{AD-DB}{2}\right)^2$; or $25 = 21 + 4$.

Or, Let AD and BD be regarded as two independent lines;

then $AD.DB + \left(\frac{AD-DB}{2}\right)^2 = \left(\frac{AD+DB}{2}\right)^2$; or $(7 \times 3) + 4 = \left(\frac{7+3}{2}\right)^2 = 25$.

Again, $(AD + DB).(AD - DB) + DB^2 = AD^2$; or $(10 \times 4) + 9 = 7 \times 7 = 49$.

or, $AD^2 - DB^2 = (AD + DB).(AD - DB)$; or $49 - 9 = 10 \times 4 = 40$.

Cor. $AC^2 - CD^2 = (AC + CD)(AC - CD)$; or $25 - 4 = 7 \times 3 = 21$.

LARDNER'S *Cor.* 1. The rectangle is a *maximum* when AB is bisected by D: its *maximum* value $= \left(\frac{AB}{2}\right)^2$

Cor. 2. The sum of the squares of the parts is a *minimum* when A B is bisected ; the *minimum* value being $2\left(\frac{A B}{2}\right)^2$

3. Of all rectangles having the same perimeter, the square contains the greatest area.

4. Of all rectangles equal in area, the square is contained by the least perimeter.

5. If a perpendicular be drawn from the vertex of a triangle, as in Prop. 13, to the base (A B + A C) . (A B ~ A C) = (B D + D C) . (D C ~ B D) ; or (7·8102 + 6·4031(× (7·8102 – 6·4031) = (6 + 4) × (6 – 4) = 20.

6. The difference between the squares of the sides of a triangle, as in Prop. 13, is equal to twice the rectangle under the base and the distance of the perpendicular from the middle point, x, of B C : *i.e.*, $A B^2 - A C^2 = 2 B C . D x$; or 61 – 41 = 2 × 10 × 1 = 20; *or*, 61 – 41 = (2 × 2) × 5 = 20.

N.B.—If the perpendicular A D fall within the base,

$$D x = \frac{B D \sim D C}{2} ; \text{ and } B C = B D + D C :$$

but if the perp. A D fall without the base,

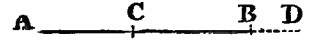
$$D x = \frac{B D + D C}{2} ; \text{ and } B C = B D - D C.$$

PROP. IX. $A D^2 + D B^2 = 2 (A C^2 + C D)^2$; or 49 + 9 = 2 (25 + 4) = 58.

Or $A D^2 + D B^2 = 2\left(\frac{A D + D B}{2}\right)^2 + 2\left(\frac{A D - D B}{2}\right)^2$; or (2 × 25) + (2 × 4) = 58.

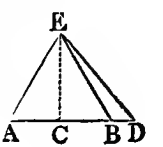
Or $A D^2 + D B^2 = \frac{(A D + D B)^2}{2} + \frac{(A D - D B)^2}{2}$ or $\frac{100}{2} + \frac{16}{2} = 50 + 8$.

CASE IV.—*Let a st. line be bisected and produced,* A B = 10 *representing the original line ;* A C *or* C B = 5, *its half ; and* B D = 3, *the part produced.*



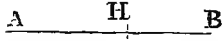
PROP. VI. $A D . D B + C B^2 = C D^2$; or (13 × 3) + (5 × 5) = 64.

Cor. Let A B = 10, the base of an isosceles triangle, be bisected in C, and produced to D ; join E D = 11·7898 ;
then $E D^2 - E B^2 = A D . D B$; or 139 – 100 = 13 × 3 = 39.

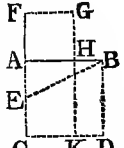


PROP. X. $AD^2 + DB^2 = 2\,(AC^2 + CD^2)$; or $169 + 9 = 2 \times (25 + 64) = 2 \times 89 = 178$.

CASE V.—*A st. line is cut in extreme and mean ratio, when the rectangle of the whole line and one segment equals the square of the other segment.*



PROP. XI. AB so cut in H that $AB \cdot HB = AH^2$
$10 \times 3{\cdot}81966 = 38{\cdot}1966 = 6{\cdot}18034 \times 6{\cdot}18034$.



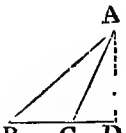
Cor. 1. To cut a line in extreme and mean ratio, it must first be produced in extreme and mean ratio; *i.e.*, in fig. Prop. 11, $CF \cdot FA$ must equal AB^2.

2. When a line CF, or its equal, is cut in extreme and mean ratio, the rectangle $AC \cdot (AC - AF) = AH^2$ or AF^2; or $AC \cdot HB = AH^2$.

3. A line CF being cut in extreme and mean ratio, $AC \cdot AF = AC^2 - AH^2$; *i.e.*, $10 \times 6{\cdot}18034 = 100 - 38{\cdot}1966 = 61{\cdot}8034$.

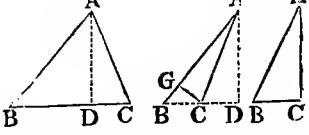
CASE VI.—*The measure of the square of the side of an obtuse-angled triangle opposite to the obtuse angle.*

PROP. XII. $AB^2 = BC^2 + AC^2 + 2\,BC \cdot CD$,
$100 = 25 + 45 + (2 \times 15)$.
or, $AB^2 > BC^2 + AC^2$ by $2\,BC \cdot CD$,
$100 > 25 + 45$ by 30.

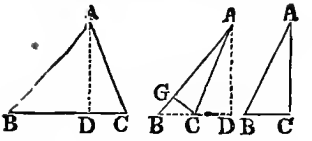


N.B.—The Area of a triangle may be ascertained when the three sides are known; for Area $= \frac{AD \cdot BC}{2}$.

CASE VII.—*The measure of the square of the side subtending an acute angle.*



PROP. XIII. 1°. $CB^2 + AB^2 = AC^2 + 2\,BC \cdot BD$; $100 + 61 = 41 + 120$.
or, $AC^2 < (CB^2 + AB^2)$ by $2\,BC \cdot BD$; $41 < 161$ by 120.

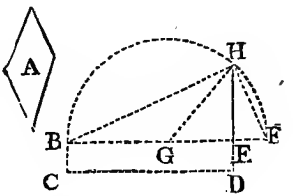


2°. $AB^2 + BC^2 = AC^2 + 2\,BD \cdot BC$; $61 + 4 = 41 + (2 \times 12)$; or $AC^2 < (AB^2 + BC^2)$ by $2\,BD \cdot BC$; $41 < 65$ by 24.

3°. $AC^2 < (AB^2 + BC^2)$ by $2\,BC \cdot BC$; $25 < 41 + 16$ by 2×16.

If in 2°. a perpendicular CG be drawn from $\angle$ C to AB, the $\square$ AB . GB = BC . DB; or $7{\cdot}8012 \times 1{\cdot}5364 = 12 = 2 \times 6$.

CASE VIII.—*A square is found equal to a given rectilineal figure.*



PROP. XIV. By Const., $\square$ BCDE = A;

$BE \cdot EF + EG^2 = GF^2 = GH^2$; $9 \times 4 + 6{\cdot}25 = 42{\cdot}25$;

$\square BD = BE \cdot ED$, or $BE \cdot EF$; $9 \times 4 = 36$;

$\square BD = EH^2 = A$. $\sqrt{36} = 6 = EH$ side of the square.

PRACTICAL RESULTS.

The Practical Results at which we arrive from studying the principles demonstrated in the first and second Books of Euclid, are among the most useful and important of any belonging to Geometry: when united to a few leading theorems of the third and sixth books, they constitute by far the most fruitful source of scientific progress; they are, in fact, the foundation on which our after reasonings depend respecting all magnitudes, space, and number,—the forms and motions of the heavenly bodies,—and the mechanical laws by which the material universe is governed.

Nearly all the Problems in Euclid's Six Books of Plane Geometry may be derived directly and immediately from the first and second books. Indeed, for the simple construction of Geometrical Figures, scarcely any principles are demanded which are not to be found in, or naturally inferred from, those books. More fully, therefore, to give the learner an idea of the wide application of the knowledge he may have acquired, I will briefly exhibit the PRACTICAL RESULTS to which we have attained: they may be arranged under four leading divisions:—

1st. *The Problems for the Construction of Geometrical Figures both stated and demonstrated in Books* I. *and* II.;

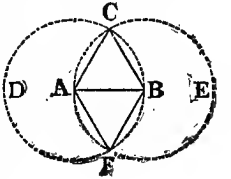
2nd. *Problems in the other books which, though demonstrated under their respective Propositions, are yet most intimately connected with the first and second, for their construction and solution;*

3rd. *Principles of Construction for Geometrical Instruments to measure st. lines and angles; and for Geometrical Figures to exhibit the representative values of actual magnitude and space;* and

4th. *Principles which without requiring that we should measure all the boundaries of a Surface, yet enable us accurately to calculate distances, magnitudes, and areas.*

I.—The Problems for the Construction of Geometrical Figures both stated and demonstrated in Books I. and II.

Problem 1. *On a given st. line* AB *to describe an equilateral triangle.*

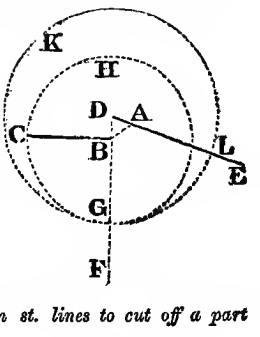


From the extremities of AB, with the distance AB for radius, describe two circles intersecting in C and F; join the point C to the points A and B, and the required triangle is drawn. (1. I.)

Practically, it is sufficient to draw the arcs intersecting in C, and to join the points A, B, and C.

An isosceles triangle may be drawn by taking the length of the equal sides, and using it as the radius from A and B.

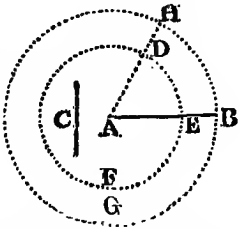
Prob. 2. *From a given point* A *to draw a st. line equal to a given st. line* CB.



Join A and B, and on AB construct an equil. triangle; produce its sides DA and DB indefinitely; from B, with the radius BC, describe a circle HCG; and from D, with DG, the circle GKL: AL is the line of the required length. (2. I.)

Practically, the distance CB will be transferred by the compasses or by some other instrument.

Prob. 3. *From the greater,* AB, *of two given st. lines to cut off a part equal to* C, *the less.*



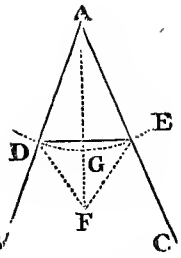
From A, with a radius equal to C, cut the line AB in E: AE will be equal to C. (3. I.)

A less line may be made equal to a greater, by describing an arc with the radius of the greater, and by producing the less until it touches the circumference.

Prob. 4. *To bisect a given rectilineal angle,* BAC; *that is, to divide it into two equal parts.*

From the angular point A, describe an arc, cutting the sides AB and AC in D and E; from D and E, with the same radius, draw arcs intersecting in F: on joining A and F, the angle BAC is bisected. (9. I.)

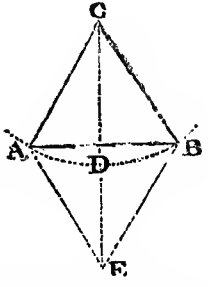
By following the same method, the bisected parts may also be further bisected. *Practically*, the division of an angle into any of the powers of 2 will be effected by measuring the angle BAC with a protractor, and dividing the angular distance by the required power of 2; as, 4, 8, 16, &c.



PROB. 5. *To bisect a given finite straight line*, AB, *and to continue the bisection of the bisected parts.*

From A and B, with the same radius, describe arcs intersecting both above and below the line in C and E; join CE: and the line AB is bisected in D. Follow the same method with AD or DB, and for all the subsequent bisections. (10. I.)

As in the angle, the division by 2, or by any power of 2, will be performed *practically* by measuring the given line, and by dividing the lineal measurement by 2, or the required power of 2, 4, 8, 16, &c.

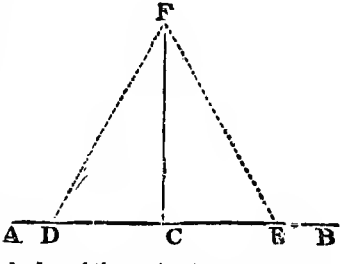


PROB. 6. *To draw a perpendicular*, i.e., *a straight line at right angles to a given straight line* AB, *from a given point* C, *in the same.*

By 3. I., take CD equal to CE; and from D and E, with the same radius, describe arcs intersecting in F: the st. line from F to C is the perpendicular required. (11. I.)

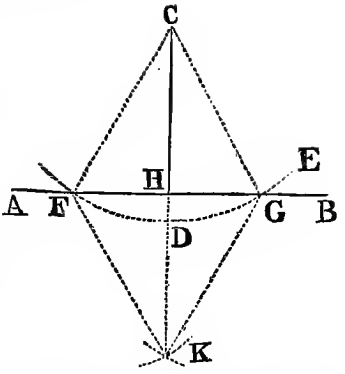
The drawing of a perpendicular from the extremity of a line is shown in § II., Prob. 5, Prac. Res.

The *practical method* is by means of an instrument called a square; or by placing a protractor with its edge along the given line AB, and its centre at C, the angular point; and then joining C with 90° on the graduated edge of the protractor.



PROB. 7. *To draw a perpendicular to a given straight line*, AB, *of an unlimited length, from a given point*, C, *without it.*

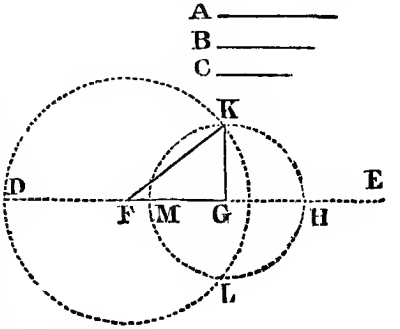
From C, with a radius CD, extending below or beyond the line AB, intersect AB in F and G; and from F and G, with the same radius, describe arcs intersecting in K; join CK: and CH is the perpendicular required. (12. I.)



The *easiest way is to use the square, or the protractor:* lay one edge of the square along the given line, and slide the square, keeping that edge on the given line, until the other edge just covers the given point:—a line along the edge will be the perpendicular. The protractor is used in a similar way.

PROB. 8. *To make a triangle of which the sides shall be equal to three given straight lines,* A, B, *and* C, *but of which any two whatever must be greater than the third.*

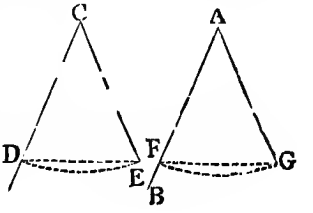
Draw a line of an indefinite length DE, and on DE, from D, in succession lay lines, by 3. I., DF equal to A, FG to B, and GH to C: from F, with radius FD, describe a circle: and from G, with radius GH, another circle; join their point of intersection K, with F and G; and the figure FGK is the triangle required. (22. I.)



In the same way one triangle may be made equal to another. *In practice,* a line FG is taken equal to B; and from F and G, arcs, with radii equal to A and C, are drawn intersecting in K, and K joined to F and G.

PROB. 9. *At a given point* A, *in a given st. line* AB, *to make a rectilineal angle equal to a given rectilineal angle,* DCE.

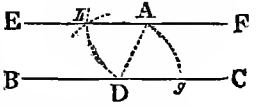
From C and A, with the same radius, describe arcs DE and FG; take DE as a radius, and with it from F describe another arc intersecting arc FG in G; join GA: and BAC is the angle required. (23. I.)



Or, the given angle DCE *may be measured by the protractor;* and by placing the centre of the protractor at the given point A, with its edge along AB, and noting the same number of degrees

from AB as equal the measurement of angle DCE, and joining that note or mark with A, the required angle will be drawn. The same may also be done by the line of chords.

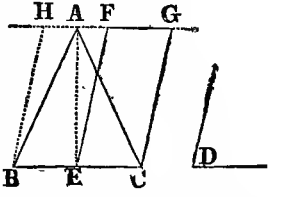
PROB. 10. *To draw a straight line through a given point* A, *parallel to a given straight line* BC.



Join A and any point D, in BC; from D with DA describe the arc A*g*; and from A, with the same radius AD, describe the arc D*h*; by 23. I., make the angle DAE equal to the angle ADC: the line EA will be parallel to the given line BC. (31. I.)

In *practice*, the parallel ruler, or the triangular square, is used.

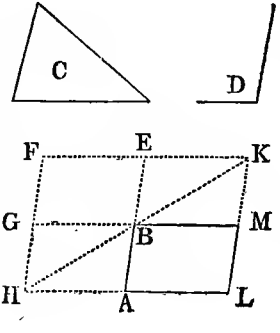
PROB. 11. *To describe a parallelogram that shall be equal to a given triangle* ABC, *and have one of its angles equal to a given rectilineal angle* D.



By 10. I. bisect BC in E; by 23. I. at E make an angle CEF equal to D; through C and A, by 31. I., draw CG parallel to EF, and AG parallel to BC: the figure FECG is the parallelogram required. (42. I.)

If it were required *to describe a triangle equal to a given parallelogram* FECG, *and having an angle equal to a given angle* D, the method would be,—Produce the parallels GF and CE; and, by 3. I., make EB equal to EC; and by 23. I., make the angle CBA equal to D; join CA;—and the triangle ABC will equal the parallelogram FECG, and have an angle equal to D.

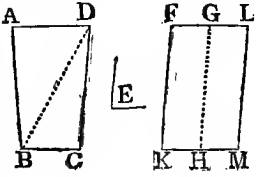
PROB. 12. *To a given line*, AB, *to apply a parallelogram which shall be equal to a given triangle* C, *and have one of its angles equal to a given angle* D.



By 42. I. make a parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D; produce EB; and by 3. I. make BA equal to the given line; produce FG to H, and, by 31. I., through A draw HL parallel to GB or FE; join HB, and produce it until it meets FE produced in K; through K draw KL parallel to EA or FH; and produce GB to M: the figure ABML is the parallelogram required. (44. I.)

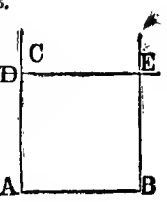
PROB. 13. *To describe a parallelogram equal to a given rectilincal figure* ABCD, *and having an angle equal to a given rectilineal angle* E.

Divide the given figure into triangles, and draw an indefinite line KM; and at K, by 23. I., make the angle FKM equal to angle E; to FK apply, by 42. I., a parallelogram FH equal to the triangle ADB; and to GH the parallelogram GM equal to the triangle DBC: the whole figure FKML having an angle K equal to E, is a parallelogram equal in area to the rectilineal figure ABCD. (45. I.)



PROB. 14. *To describe a square on a given straight line,* AB.

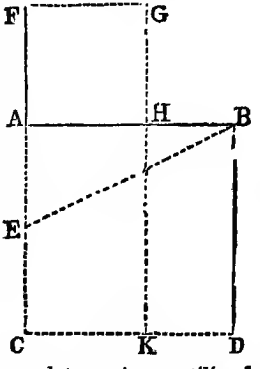
From A, by 11. I., draw the perpendicular AC, and on it make, by 3. I., AD equal to AB; through B and D, by 31. I., draw the parallels, BE to AD, and DE to AB: the figure ABED is the square required. (46. I.)



PROB. 15. *To divide a given st. line,* AB, *into two parts, so that the rectangle contained by the whole line and one of its parts,* AB . BH, *shall be equal to the square of the other part,* AH.

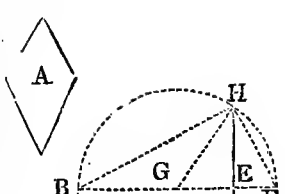
On AB, by 46. I., describe the square AD; bisect, 10. I., CA in E, and join EB; produce CA indefinitely, and, by 3. I., make EF equal to EB; on AF, by 46. I., construct the square FH, and produce GH to K: the line AB is divided at H, so that AB . BH equals the square on AH. (11. II.)

N.B.—In Prop. 30, bk. vi., a line thus divided is said to be cut *in extreme and mean ratio; i.e.*, so that the whole line AB shall be to the greater segment AH as the greater segment AH to the less segment HB.



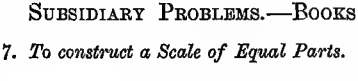
PROB. 16. *To describe a square that shall be equal to a given rectilineal figure* A.

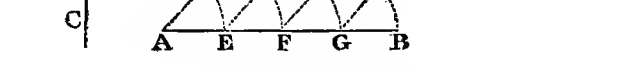
Describe, by 45. I., a rectangle B C D E equal in area to the given figure A: if the sides B E and E D are equal, the construction is finished; but if not, produce B E until, by 3. I., E F equals E D; by 10. I. bisect B F in G, and with G B as radius, describe the semicircle B H F; produce D E to meet the semicircle in H; the square on E H is the square equal to the rectangle B D and to the figure A. (14. II.)



Subsidiary Problems.—Books I. and II.

Prob. 17. *To construct a Scale of Equal Parts.*





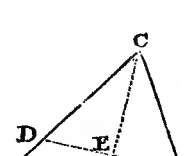
Take a st. line A B of indefinite length towards B; and let C be the given st. line that is to be cut off from A B: from A B, by 3. I., cut off a part equal to C, as A E; then again from E B another part equal to C, as E F; and so on: the parts in A B are each equal to C and to one another; and A B is a scale of equal parts. (3. I.)

On the same principle the line K L is divided. Each of the parts numbered 1, 2, 3, 4, is representative of 10 equal parts, and those between 0 and K of units; or if the smaller divisions be tenths, the larger will be units.

By such a scale *the comparative lengths of lines are ascertained.*

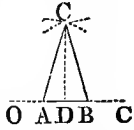
Prob. 18. *Given* A B *the base, angle* A *the less angle at the base,* and A D *the difference of the two sides of a triangle, to construct it.*

On A D the line forming with A B the angle A, by 3. I., set the difference A D of the two sides; join D B; and, by 10 and 11. I., bisect D B by the perpendicular E C; and produce E C and A D to meet in C: A B C is the triangle required. (19 and 4. I.)



PROP. 19. *On a given st. line* A B, *to describe an isosceles triangle, having each of the equal sides double of the base.*

Produce AB both ways; and by 3. I., make BC, AO each equal to AB: OB and AC each equals twice AB. With AC from A, and with BO from B, describe arcs cutting in C; join C to A and B: ABC is the isosceles triangle with sides each the double of the base. (22. I.)

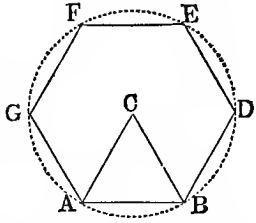


PROB. 20. *To describe an isosceles triangle, the base* AB *being given, and* F *a st. line equal to each of the sides.*

Bisect the base in D, and raise a perpendicular at D; from A with radius equal to F, draw an arc cutting the perpendicular in C; join CA and CB: the required isosceles triangle is ACB. (22. I.)

PROB. 21. *To construct a regular polygon,* as ABDEFG.

By 32. 1. the formula for the angular magnitude of each angle in a regular polygon is, $\angle A = \frac{(S \times 180) - 360}{S}$, the angle between any two sides; $\angle C = \frac{360}{S}$, for the angle at the centre between two radii to the angular points. (*Note*, 32. I., p. 112.)

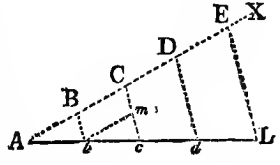


1°. *When the side* AB *is not given.* With any radius, as CA, describe a circle; at C, the centre, make an angle equal to $\frac{360}{S}$, and produce CB to the circumference; join AB: the line AB will, if set round the circumference, divide it into as many parts as there are sides to the polygon; join the points, and the polygon is made.

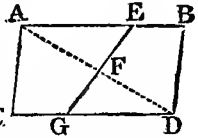
2°. *When the side* AB *is given.* By formula, $\angle A = \frac{(S \times 180) - 360}{S}$; ascertain the angles GAB, ABD, and construct them; by 9. I., bisect each of the angles; and the lines of bisection, AC, BC, intersect in C; from C, with radius CA, describe a circle; if now, with AB as radius, successive arcs be cut off from the circumference, there will be as many arcs as sides, and the drawing of the chords to those arcs will complete the polygon. (*Use and App.* 32. I., p. 112.)

PROB. 22. *To divide a finite straight line* AL, *into any given number of equal parts, as four.*

At A, draw a line of indefinite length, AX, making with AL the angle XAL; take a line AB on AX; and, by 3. I., set along AX three other lines BC, CD, DE, each equal to AB; join EL; and through the points D, C, and B, by 31. I., draw D*d*, C*c*, B*b*, each parallel to EL: the line AL will be divided into four equal parts in the points *b*, *c*, *d*. (*Use and App.* 34. I., p. 115.)

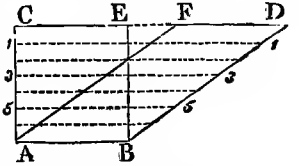


PROB. 23. *From a point* E *in the side* AB *of a parallelogram, to divide the parallelogram into two equal portions.*



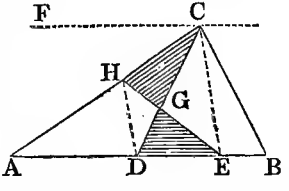
Draw the diagonal A D, and, by 10. I., bisect it in F; join E F, and produce E F to G: the line E G makes the portion A E G C equal to the portion E G D B. (*Use and App.* 6, 34. I., p. 117.)

PROB. 24. *To convert a parallelogram,* ABDF, *into a rectangle,* ABCE, *of equal area.*



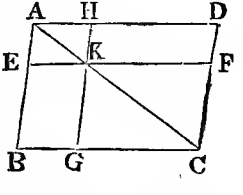
Produce indefinitely the parallel DF; at B and A, the extremities of the other parallel A B, by 11. I., raise the perpendiculars BE and A C to meet the parallel DC; then ABCE is a rectangle equal in area to the parallelogram ABDF. (*Use and App.* 35. I., p. 119.)

PROB. 25. *Given a triangle,* ABC, *and a point,* H, *in one side,* AC; *from that point to bisect the triangle.*



Through C the vertex, by 31. I., draw CF parallel to AB; by 10. I. bisect AB in D, and join DC; join also DH; and from C draw CE parallel to HD, and join HE: then the triangle AHE will equal the trapezium CHEB. (*Use and App.* 2, 38. I., p. 124.)

PROB. 26. *On a given line,* EK, *to draw a parallelogram equal to a given parallelogram,* KD.

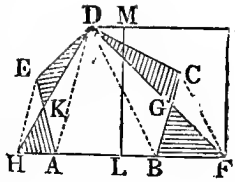


Produce FK, and by 3. I. from K set off a distance equal to the given line; produce indefinitely the sides DF, HK, and DH; and by 31. I. draw through E, AB parallel to HG or DC; draw the diagonal AK, and produce it until it cuts DF produced in C; through C draw a parallel to DA or FE, meeting HK and AE produced in G and B: the parallelogram BK will be equal to the given parallelogram KD. (*Use and App.* 43. I., p. 131.)

PROB. 27. *To change any right-lined figure,* ABCDE, *first into a triangle, and then into a rectangle of equal area.*

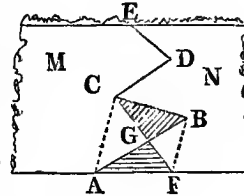
Divide the figure into triangles by joining DA, DB; produce AB indefinitely; through E, by 31. I., draw EH parallel to DA, and through C, CF par. to DB; join DH and DF: then the triangle DHF is equal in area to ABCDE.

Next, draw DN parallel to HF; by 10. I. bisect HF in L; at L, 11. I., raise the perpendicular LM, and draw FN parallel to LM: the fig. LMNF is a rectangle equal in area to DHF, which is equal to ABCDE. (*Use and App.* 2, 45. I., p. 132.)



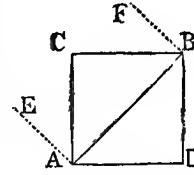
PROB. 28. *To make straight a crooked boundary,* ABCDE, *between two fields,* M *and* N.

Draw AC, the subtend to angle B; and through B, 31. I., FB parallel to AC, and join CF: the crooked boundary AB, BC is now converted into the single boundary CF. In a similar way FC and CD will be converted into one boundary; and this last and DE into a single boundary: and thus the crooked boundries AB, BC, CD, DE, will be changed into one straight boundary without affecting the areas of the two fields. (*Use and App.* 3,45. I. p. 136.)



PROB. 29. *Given the diagonal,* AB, *to construct a square.*

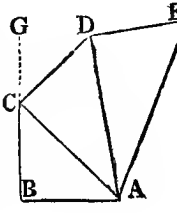
At A and B, by 11. I., draw the perpendiculars AE, BF; by 9. I., bisect the right angles by AC and BC meeting in C; and by 31. I., draw through A and B, parallels, AD to BC, and BD to AC: then the figure ABCD is the square required. (46. I., p. 137.)



PROB. 30. *To find a square equal to the sum of any number of given squares, as on* AB, BC, CD, DE; *or a square that is a multiple of any given square,* AB; *or a square that is equal to the difference of two squares.*

1°. Set the lines AB, BC, representative of the two squares or AB, BC, at right angles, and join AC; then $AC^2 = AB^2 + BC^2$; at C place CD, representative of the square on CD, at right angles to AC; and $AD^2 = AC^2 + CD^2$: and at D, forming a right angle with AD, place DE representative of the square on DE; join AE: and the square on AE equals the squares on DE, DC, CB, and BA. (*Cor.* 3, 47. I., p. 141.)

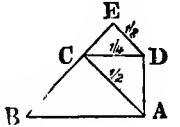
2°. Supposing AB to be representative of the line on which the given square is constructed, its multiple square will be obtained in a similar way;



for in this case, BC, CD, DE, being each equal to AB, the square on AE is the multiple of the square on AB. (*Cor.* 3, 47. I., p. 141.)

3°. Let AB be the less line, and AC the greater; at B, the extremity of the less, raise a perpendicular BG, and from A at the other extremity, with AC as radius, inflect on BG the greater line: the square of the intercept CB will equal the difference of the squares on AC and AB. (*Cor.* 3, 47. I., p. 141.)

PROB. 31. *To make a square that shall be the half, fourth, &c., of a given square on* AB.

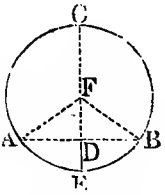


At A and B make the angles each equal to half a right angle; C being a right angle, the square on AC will be one half of the square on AB. Again, at A and C make the angles CAD, ACD each equal to half a right angle: then the square on CD will be half that on AC, or one-fourth of the square on AB. By a similar process a square may be obtained that is $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c., of the original square. (*Cor.* 3, 47. I., p. 142.)

II.—THE PROBLEMS IN BOOK 3, 4, AND 6, WHICH THOUGH DEMONSTRATED UNDER THEIR RESPECTIVE PROPOSITIONS, ARE, FOR THEIR CONSTRUCTION AND SOLUTION, MOST INTIMATELY CONNECTED WITH THE FIRST AND SECOND BOOKS.

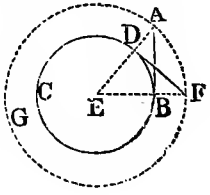
BOOK III.

PROB. 1. *In a given circle,* ABC, *to find the centre.*



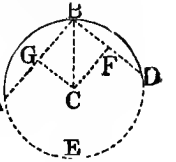
Draw any chord AB; and, 10. I., bisect it at D; from D, 11. I., raise a perpendicular produced both ways to meet the circumference in C and E; bisect CE in F; and F is the centre of the circle. (1. III.)

PROB. 2. *To draw a tangent to a given circle* BCD, *from a given point, either without as* A, *or in the circumference as* D.



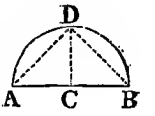
Find the centre E of the circle BCD, and join AE; from E with EA describe the circle AFG, from D, 11. I., draw DF at right angles to EA, and join EBF and AB: then AB shall touch the circle in B, and DF in D. (17. III.)

PROB. 3. *To describe a circle through three given points*, A, B, D, *not in the same st. line; or to complete the circle of which a segment*, ABD, *is given.*



Join A, B, and B, D; 10 and 11. I., bisect AB and BD by perpendiculars GC, FC, meeting in C: C is the centre of the circle, and a radius from C to either A, B, or D, will be the means of completing the circle. (25. III.)

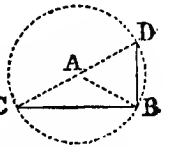
PROB. 4. *To bisect a given circumference*, ADB.



Join the extremities of the arc A, B; 10 and 11. I., and bisect AB by a perpendicular from C: the line CD, will bisect the given circumference in the point D. (30. III.)

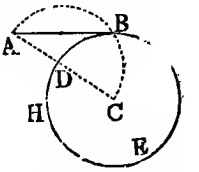
LEMMA.—31. III. *The angle in a semicircle is a right angle.*

PROB. 5. *To draw a perpendicular*, 1°. *from point* B *in a line* CB, *or at its extremity*; 2°. *from a point* D *out of a line*; *and*, 3°. *so as to be a tangent to a given circle* HBE.



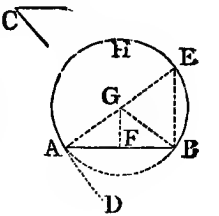
1°. Take any point A, above the line, and with a radius equal to AB describe an arc to cut the given line in C; join C and A, and produce CA to meet the arc in D: DB is the required perpendicular.

2°. Join D and C; and, 10. I., bisect CD in A; and from A with AC describe the semicircle CBD, and join DB: DB is the perpendicular.



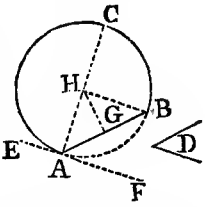
3°. Join the given point A, out of the circle, and C the centre of the circle; 10. I., bisect AC in D; and with DA describe a semicircle: where the semicircle cuts the circle, B, is the tangent point, and AB the tangent from A. *Or*, from B, a point in the circle, draw a line BC to the centre; and at B, 11. I., draw a right angle: BA is the tangent from B.

PROB. 6. *Upon a given straight line* AB, *to describe a segment of a circle containing an angle equal to* C, *a given rectilineal angle.*



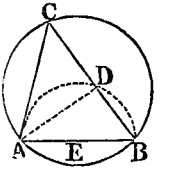
At A, 23. I., make the angle BAD equal to angle C; and 11. I., at A make AE at right angles with DA; 10 and 11. I., bisect AB in F by the perpendicular GF; with GA or GB describe a circle: and in the segment AHB the angle AEB equals BAD, which also equals angle C. (33. III.)

PROB. 7. *Given the angle* D *equal to the vertical angle of a triangle, and the base* AB, *to find the* locus *of the vertex.*



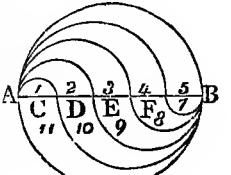
At A, by 23. I., make the angle BAF equal to angle D, and, 11. I., angle FAH equal to a right angle; bisect. 10 and 11. I., AB in G by the perpendicular GH; and from H, with radius HA, or HB describe a circle ABC: the *locus* of the vertex will be at any point in the arc of the segment ACB. (33. III.)

PROB. 8. *Given* AB *the base, angle* ACD *the vertical angle, and* AD *the perpendicular from the extremity of the base* A *on the opposite side* BC, *to construct the triangle.*



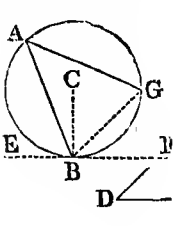
Make on AB, by Prob. 6, a segment containing the given angle ACD; bisect AB, 10. I., in E; and with rad. EA or EB describe the semicircle ADB; and from A, inflect the perpendicular AD upon the semicircle in the point D: BDA is a right angle, 31. III., and BD produced to C, and CA joined, give the triangle required. (33. III.)

PROB. 9. *To divide a given circle of which the diameter is* AB, *into any number of equal parts, of which the perimeters also are equal.*



By Use and App. 34. I., divide the diameter AB into the required equal parts at C, D, E, F; then on one side, from A describe the semicircles 1, 2, 3, 4, 5, &c.; and on the other side from B, the semicircles 7, 8, 9, 10, 11. &c.,—of which the diameters are BF, BE, BD, BC; so shall the parts 1 and 11, 2 and 10, 3 and 9, 4 and 8, and 5 and 7, be equal each to the other, both in Area and Perimeter.—LESLIE'S *Geometry.*

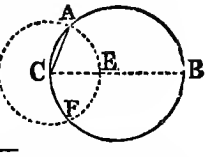
PROB. 10. *From a given circle,* ABG, *to cut off a segment which shall contain an angle equal to a given rectilineal angle* D.



Draw any radius CB; and at B, 11. I., draw EF making right angles with CB; at B, 23. I., make the angle FBG equal to angle D: the segment BAG contains an angle BAG equal to the angle GBF, which equals angle D. (34. III.)

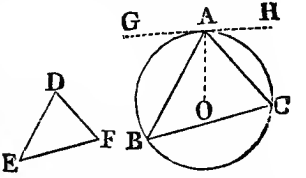
BOOK IV.

PROB. 1. *In a given circle,* ABC, *to place a st. line equal to a given st. line* D, *not greater than the diameter of the circle.*



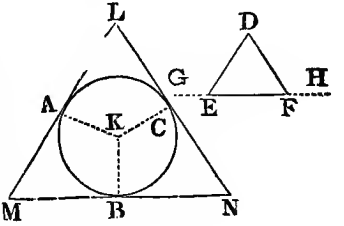
Draw BC the diameter of ABC: 3. I., make CE equal to D; and with CE describe the circle AEF: CA is the line required. (1. IV.)

PROB. 2. I. *In a given circle,* ABC, *to inscribe a triangle equiangular to a given triangle,* DEF.



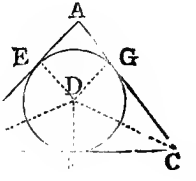
Draw OA any radius; and at A, 11. I., GH a tangent; also at A, 23. I., make the angles HAC and GAB equal to the angles DEF and DFE, and join BC: the triangle ABC is equiangular with the given triangle DEF. (2. IV.)

PROB. 3. *About a given circle,* ABC, *to describe a triangle equiangular to a given triangle,* DEF.



Produce EF both ways to G and H; find K the centre of the circle, and draw a radius KB; at K, 23. I., make the angle BKC equal to angle DFH, and angle BKA equal to angle DEG; at the points A, B, and C draw, 11. I., lines at right angles to AK, BK, CK, and produce them both ways until they meet: the fig. LMN is equiangular with the triangle DEF, and described about the circle ABD. (3. IV.)

PROB. 4. *In a given triangle* ABC, *to inscribe a circle.*

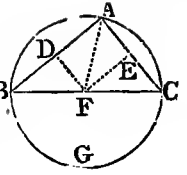


By lines BD, CD, meeting in D, bisect 9. I., the angles B and C; from D, 12. I., drop perpendiculars DE, DF, DG, to AB, BC, and CA; with radius DE, describe a circle: the circle EFG is inscribed in the given triangle. (4. IV.)

PROB. 5. *About a given triangle,* ABC, *to describe a cir. e.*

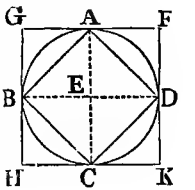
By 10 and 11. I., bisect AB and AC by perpendiculars meeting in F, and join FA; with FA as radius describe the circle ACGB; it is described about the given triangle. (5. IV.)

This is the same as Prob. 3, from bk. III., *to describe a circle through three given points*, A, B, and C.



PROB. 6. *In a given circle*, ABCD, *to inscribe a square.*

Draw BD and AC two diameters at right angles to each other, and join A, B, C, and D: the figure ABCD is a square in the circle. (6. IV.)

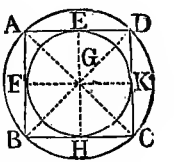


PROB. 7. *About a given circle*, ABCD, *to describe a square.*

Draw two diameters, AC and BD, making right angles at E; at the extremities A, B, C, D, draw lines 11. I., at right angles to EA, EB, EC, and ED, and produce them until they meet in G, H, K, F: the fig. GHKF is a square about the circle. (7. IV.)

PROB. 8. *To inscribe a circle in a given square*, ABCD.

By 10 and 11. I., bisect each of the sides by the perpendiculars EG, FG, HG, KG, intersecting in G; and with GE as radius describe the circle EFHK in the square. (8. IV.)



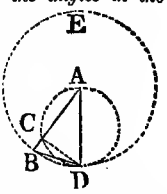
PROB. 9. *To describe a circle about a given square*, ABCD.

Draw the diagonals AC, BD, and from their point of intersection G, with GA, describe a circle: the circle ABCD is about the given square. (9. IV.)

PROB. 10. *To describe an isosceles triangle having each of the angles at the base double of the third angle.*

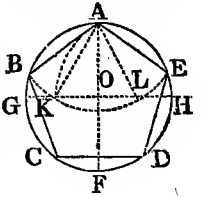
Take any line AB, and, by 11. II., divide it so that the rectangle AB . BC shall equal the square on CA; from A, with radius AB, describe the circle BDE, and in it, by 1. IV., place the line BD equal to AC; and join DA: ABD is the isosceles triangle required. (10. IV.)

For the proof, join DC, and about the triangle ADC, 3. III., describe the circle ACD.

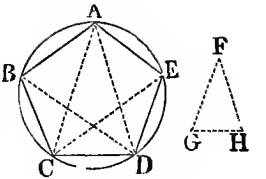


PROB. 11. *In a given circle*, ABCDE, *to inscribe an equilateral and equiangular pentagon.*

Draw two diameters, AF and GH, at right angles; 10. I., bisect the radius OH in L; and with AL as radius describe an arc cutting OG in K: the distance AK is equal to one side of the pentagon; and if set round the circle will complete the figure. If the arcs are bisected, and the points of bisection joined, a decagon will be drawn. (P. 10, bk. XIII.)

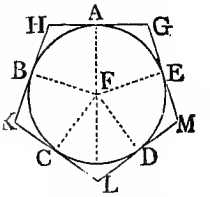


Or, 10. IV., describe an isosceles triangle, FGH, having each of the angles at the base double of that at the vertex; and, 2. IV., in the given circle inscribe a triangle ACD, equiangular with FGH; 9. I., bisect the angles ACD, ADC, and let the bisecting lines be produced to meet the circumference in B and E: the points A, B, C, D, E, are the angular points of the required pentagon. (11. IV.)



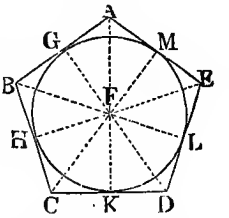
PROB. 12. *About a given circle*, ABCDE, *to describe an equilateral and equiangular pentagon.*

By 11. IV., let A, B, C, D, E, be the angular points of a pentagon inscribed in the circle ABCDE; and from F, the centre of the circle, let lines be drawn to those points; at A, B, C, D, and E draw lines, 11. I., at right angles to FA, FB, FC, FD, and FE: the fig. HGMLK is the pentagon about the given circle. (12. IV.)



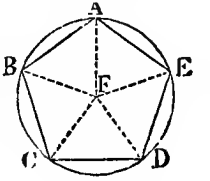
PROB. 13. *In a given equilateral and equiangular pentagon*, ABCDE, *to inscribe a circle.*

By 9. I. bisect the angles BCD, CDE, &c., by lines CF, DF, meeting in F; from F, 12. I., draw perpendiculars FH, FK, FL, to BC, CD, DE; and with any one, as FH, as radius, describe a circle: this circle GHKLM will be inscribed in the pentagon ABCDE. (13. IV.)



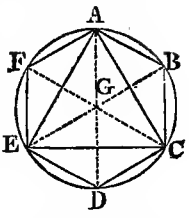
PROB. 14. *To describe a circle about a given equilateral and equiangular pentagon*, ABCDE.

By 9. I. bisect the angles ABC and BCD by the lines BF and CF; and from their intersection in F draw FD, FE, FA; with either of these as radius, describe a circle, as ABCDE: this circle will be about the given pentagon. (14. IV.)



PROB. 15. *To inscribe a regular hexagon in a given circle,* ABCDEF.

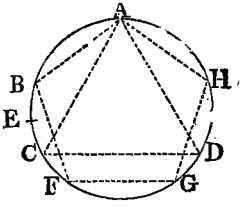
Draw a diameter of the circle, as AD, and bisect it in G; from A with a distance equal to the radius AG, draw arcs cutting the circle in F and B; and from D with the same radius, arcs cutting in E and C; join the points A, B, C, D, E, and F,—and the hexagon is inscribed in the circle. (15. IV.)



N.B. By joining the alternate points, A, C, E, *an equilateral triangle* ACE is inscribed; and by bisecting each of the arcs of the hexagon, and joining the points, *a dodecagon is formed* and inscribed.

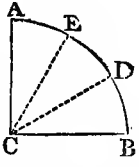
PROB. 16. *In a given circle,* ABCD, *to inscribe an equiangular and equilateral quindecagon.*

By 2 and 11. IV. inscribe in the circle an equilateral triangle ACD, and a regular pentagon; AB being a side of the pentagon, and AC a side of the triangle: 30. III.. bisect BC in E, and BE or EC will be the fifteenth part of the circumference,—from which *the quindecagon* may be constructed. (16. IV.)



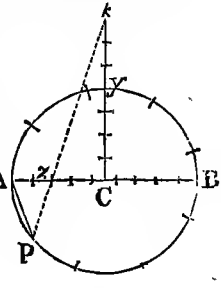
PROB. 17. *To trisect a quadrant.*

From A and B, with the radius of the circle, describe arcs cutting the quadrant in D and E; join EC, DC, and the quadrant is trisected.



PROB. 18. *In a given circle,* ABP, *to inscribe any regular polygon; or to divide the circumference of a circle into any assigned number of equal parts.*

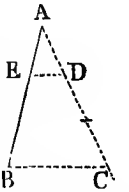
By Use and App. 34. I., divide the diameter AB into the same number of equal parts as the figure has sides, suppose 9; from the centre C, 11. I., draw a perpendicular C*k*; divide the radius C*y* into four equal parts; and set off three of those parts from *y* to *k*; join *k* and *z* the second of the divisions from A, and produce *kz* to the circumference in P: the line AP will be the side of the required polygon.—*See a Treatise on Mensuration, Irish National Schools,* p. 19; *Demonstrations,* p. 53.



BOOK VI.

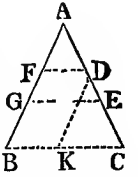
LEMMA.—2. VI. *If a straight line be drawn parallel to any side of a triangle, it divides the other sides, or those sides produced, into proportional segments.*

PROB. 1. *To cut off from a given st. line,* AB, *any part required.*



At A place any line AC, making with AB the angle BAC; in AC take any point D, and in succession, by 3. I. cut off from AC as many parts each equal to AD as there are to be parts in AB; join BC; and, by 31. I., through D draw DE parallel to BC; whatever part AD is of AC, the same part is AE of AB. (9. VI.)

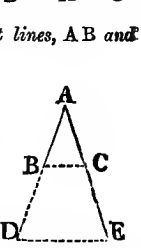
PROB. 2. *To divide a given straight line,* AB, *similarly to a given divided st. line,* AC.



Set AB and AC so as to make an angle BAC, and join BC; through D and E, the points of division on AC, by 31. I., draw DF and EG parallel to BC; in the points F and G, AB is similarly divided to AC.

N.B. *When one line is cut similarly to another, the several segments of the one are proportional to the several segments of the other.*

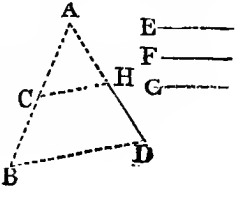
PROB. 3. *To find a third proportional to two given straight lines,* AB *and* AC.



Place the two lines so as to form an angle BAC; produce AC indefinitely, and AB so that BD, 3. I., is equal to AC; and join BC; through D, 31. I., draw DE parallel to BC; CE is the third proportional; *i.e.*, AB : AC :: AC : CE. (11. VI.)

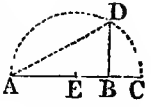
N.B. A repetition of the same construction will avail for continuing the progression.

PROB. 4. *To find a fourth proportional to three given straight lines,* E, F, *and* G.



Draw any two lines from A so as to make the angle BAD; by 3, I., on AB make AC equal to E, CB equal to F, and AH equal to G; and, 31. I., through B draw BD parallel to CH : HD is the fourth proportional. (12. VI.)

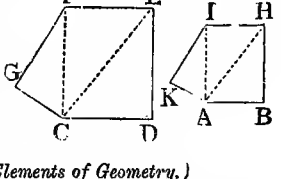
PROB. 5. *To find a mean proportional between two given straight lines,* A B *and* A C.



Set the given lines so as to form one straight line AC; by 11. I., bisect A C in E, and, with E A or E C as radius describe a semicircle; at B, by 12. I., raise a perpendicular B D to D: D B is the mean proportional; *i.e.*, A B : B D :: B D to B C. (13. VI.)

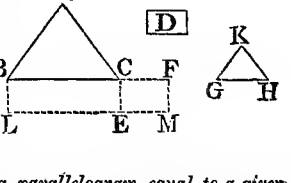
Or, let A C = 2 *r* = 13; A B = *x* = 9; and B C = 2*r* — *x*: then B D^2 = *y*^{2} = 2 *r x* — *x*^{2}.

PROB. 6. *On a given line,* A B, *to construct a rectilineal figure similar and similarly situated to a given rectilineal figure,* C D E F G.



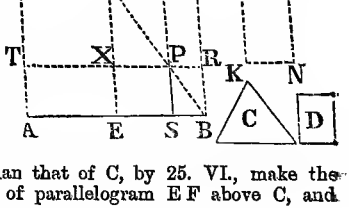
To divide the given figure into triangles, draw C E and C F; by 23. I., at B and A make the angles A B H, B A H, H A I, and I A K equal to the angles E D C, D C E, E C F, and F C G; at H make angle A H I equal to angle C E F; and at I, angle A I K equal to angle C F G: the figure A B H I K is similar and similarly situated to the given figure C D E F G (18. VI.)

PROB. 7. *(The most extensive problem in the Elements of Geometry.)* *To describe a rectilineal figure similar to one given rectil. figure,* A B C, *and equal to another rectil. figure,* D.



By Cor. 45. 1., on B C describe the parallelogram B E equal the figure A B C; and on C E a parallelogram equal to D; and with the angle F C E equal to the angle C B L, by 13. VI., or 14. II., find G H a mean proportional between B C and C F; and on G H, describe by 18. VI., G H K similar to A B C; G H K is the figure required. (25. VI.)

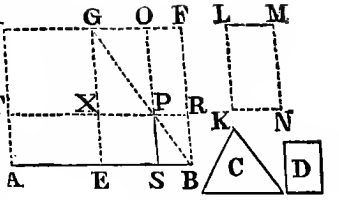
PROB. 8. *To a given st. line,* A B, *to apply a parallelogram equal to a given rectilineal figure* C, *and deficient by a figure similar to a given parallelogram* D. *But the rectilineal figure must not be greater than the parallelogram applied to half the given st. line, whose defect is similar to the given parallelogram* D.



By 10. I., bisect A B in E; and by 18. VI., on E B construct E B F G similar to D; and by 31. I., complete the parallelogram A G, making it either equal to C, or greater than C, If A G is equal to C, the problem is solved,—for on A B has been applied the parallelogram A G equivalent to C, and deficient by the parallelogram E F similar to D.

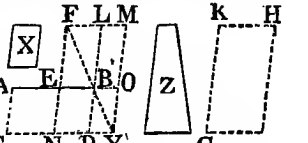
But if the area of A G be greater than that of C, by 25. VI., make the parallelogram K M equal to the excess of parallelogram E F above C, and

similar to D. Make X G, 3. I., equal to K L, and G O to L M ; and complete the parallelogram G O P X ; X O and E F have the same diagonal ; draw that diagonal G P B, and produce X P to R, and O P to S, and complete S R : then the parallelogram T S equal to the given figure C is applied to the given line A B deficient by the parallelogram S R, which is similar to the given parallelogram D. (28. VI.)



PROB. 9. *To a given straight line,* A B, *to apply a parallelogram equal to a given rectilineal figure* Z, *and exceeding by a parallelogram* B X *similar to another given parallelogram* X.

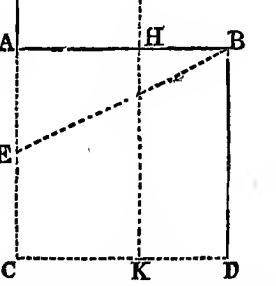
By 10. I., bisect A B in E ; and by 18 VI. on E B construct E B L F similar to X, and K G H, (equalling parallelogram E L, and figure Z) similar to X. Produce F L and F E, making, 3. I., F L M equal to K H, and F E N to K G ; complete the parallelogram M N : E L and M N have the same diagonal F X ; draw that diagonal, and produce E B to O, L B to P, M O and N P to X : then the parallelogram A X equal to the given figure Z, and exceeding by the parallelogram P O which is similar to X, has been applied to the straight line A B. (29. VI.)



PROB. 10. *To cut a given straight line,* A B, *in extreme and mean ratio,* i.e., *so that the whole line is to the greater segment as the greater segment to the less.*

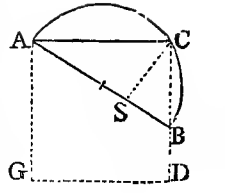
On A B, by 46. I., describe the square B C ; and, by 29. VI., apply to A C the parallelogram C G equal to the square B C, and exceeding by the square A G similar to the square B C : the line A B is cut in H in extreme and mean ratio. (30. VI.)

Or, by 11. II., divide A B in H, so that A B . H B equals the square on A H · then also A B is cut in extreme and mean ratio in the point H.



PROB. 11. *To make a square approximating in area to the area of a given circle on the diameter* A B.

In Cor. 8. VI., it is proved that $AB \cdot AS = AC^2$. Divide AB the diameter into fourteen equal parts, and of these set off eleven from A to S; at S, by 11. I., raise the perpendicular SC terminated in the circumference; join AC, and the square on AC will closely approximate to the area of the circle of which AB is the diameter.—*Mensuration, Irish National Schools*, p. 20.



III.—Principles of Construction.

§ 1. For Geometrical Instruments to Measure Straight Lines, *i.e.*, Distances,—and Angles.

1. *Instruments for measuring st. lines* are constructed by following out the Applications given in pages 55 and 115, 116, when treating of Propositions 3, and 34, bk. I.

Any st. line may be assumed as the unit: if it is the *tenth* of an inch, then, by the repitition of that tenth *ten* times, an instrument to measure an inch will be made; if an *inch* be the unit, and that inch be repeated *twelve* times, we have the measure for a foot; if a *foot*, and there be *three* repetitions of it, we have a yard; and if a *yard* be taken 1760 times, we have a mile. A very usual unit is *the link*, containing 7·92 inches, and 100 of these being taken, the *Gunter's chain* equal to 66 feet is formed.

The instruments commonly employed for the measuring of Lines, are the Inch, Foot, &c.; the Offset Staff and Chain; the Perambulator,—a wheel of which the circumference is $\frac{1}{8}$ of a chain, or 8·25 feet; and Screws of various sizes. The threads on the screw are at exactly equal distances, and a single turn of the screw measures one of those equal distances. The spaces between the threads are generally small; if there are 100 threads in the lineal inch, each turn of the screw represents $\frac{1}{100}$ of an inch, and the hundredth part of a turn $\frac{1}{100}$ of $\frac{1}{100}$ or $\frac{1}{10000}$ of an inch.

2. *Instruments for measuring angles* are only in part constructed on strictly Geometrical principles. At p. 79, Cor. 1 and 2, 15. I., we find that if a circle be round a point, all the possible angles from the centre to the circumference taken together equal four right angles: the circumference being always divided into 360°, each quadrant contains 90°; the quadrant may be trisected—$\frac{2}{3}$ of the quadrant, or 60°, measures the sextant; the 30° being bisected give two fifteens; but the division of 15° into *fifteen* equal parts can only be effected mechanically;—see p. 68, where it is shown that if the method of successive bisections had been followed, the degrees of the circle would be in strict accordance with the principles established in Plane Geometry.

The instruments generally employed for measuring angles, are the Cross Staff for right angles,—the Theodolite for any angle,—the Quadrant for an angle not exceeding 90°,—and the Sextant for an angle not exceeding 60°. The Goniometer is used for measuring the angles of crystals, &c.; and Halley's

REFLECTING QUADRANT for taking the angles between luminous points, as the stars, &c. *Spirit levels* and a *plummet* enable us to find a perpendicular, or a line at right angles to the horizontal line; and a *Graduated* or *Geometrical Square* fastened on a staff and placed in the true vertical position, to measure angles of elevation.

An angle may also be measured by any line of equal parts, by simply measuring the same distance along the lines that form the angle, and from the extremities of the distances thus measured, measuring the subtend to the angle. The *Magnetic Compass*, fixed on a circle called a *Circumferenter*, is often used for surveying mines or large tracts of land where much accuracy is not required.

The knowledge and use of such instruments are only to be attained by a practical acquaintance with them.

3. *There is a great difference between Lineal Magnitude and Angular Magnitude.* The unit in *Lineal Magnitude* is a fixed quantity,—three feet is always the same amount of space: but the unit in *Angular Magnitude* is a variable quantity, changing with the circumference of the circle. Whatever may be the size of the circle, its circumference is divided into 360 degrees; so that a degree may at one time represent the tenth of an inch,—at another time comprise ten million miles. This is familiar in the degrees of longitude: at the equator the degree of longitude measures 60 geographical miles; at Cairo, lat. 30° 6′, about 51 miles; at Petersburgh, lat. 60°, about 30 miles; and at the Magnetic Pole, lat. 70° 5′, only 20 miles:—thus gradually diminishing, until at the Pole itself, the circle having dwindled to a point, all magnitude has ceased.

§ 2. FOR GEOMETRICAL FIGURES *to exhibit the representative values of actual magnitude and space.*

When lines have been measured and the angles taken which are formed by the lines drawn from any point or station to two or more objects in the field of view, we may enter the measurements and the angular magnitudes in a book: but besides this, we need some method of representing the relative positions and distances of object. Geometrical Drawing,—as in the plans of the Architect and of the Engineer, or in the Maps of an Estate, a Parish, or a Kingdom,—is the means by which such a representation is effected. For the actual truth of a Demonstration in Geometry, it is of very little consequence whether we employ any figure at all, or whether the figure we do construct is perfectly accurate; but for the representation of distances and of forms *it is of great importance that we should draw well all the lines and angles, in their due proportion, and in their proper positions*;—indeed, without this, a map is a deception, and a plan given to a workman for his guidance, would only mislead and perplex him.

To ensure the accuracy so essential for all maps, plans, and working drawings, *Scales of equal parts*, and especially the *Diagonal Scale*, are constructed and employed: see pp. 55, 121. The principle acted on is a very simple one,—that in the same Map or Plan, whatever scale is made use of for any one line or distance, shall be adopted for all the lines and distances in that particular Map or Plan. Let the scale fixed on be an inch to represent a mile; then every inch of space on the plan or map, to which the scale is attached, will represent the value or distance of a mile: and again in a working drawing given to a mechanic for his guidance, let the scale agreed on be half-an-inch

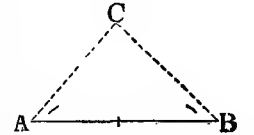
to represent a foot, then every half-inch in the drawing will denote a foot in the thing to be constructed.

The Lines in a Map or Plan *are representative only* of the values of the real distances of objects ;—but the Angles in a Map or Plan, if correctly drawn, *are identical* with the angles of the objects themselves ; and the reason is, that the size of an angle does not depend on the length of the lines which form it, but on the narrowness or width of the opening between them. The instruments for transferring to a drawing the angles measured between any two objects and the angular point or station of observation, are principally the Circle or Protractor, and the Line of Chords ; pp. 40 and 89.

The Methods for drawing nearly all Geometrical Figures are contained in the several Problems, pp. 192—211, which constitute the *first* and *second* of the leading divisions of the PRACTICAL RESULTS ; but in the Application of them we must remember *that lines and angles of a definite numerical value are given when we are required to construct Figures that shall show the positions, forms, and distances of objects.* We subjoin therefore a few examples of THE WAY IN WHICH SUCH FIGURES ARE TO BE DRAWN.

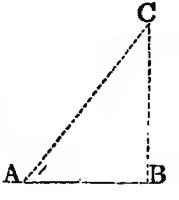
PROB. 1. *Given the two angles of a triangular field, equal respectively to* 50° *and* 45°, *and the interjacent side equal to* 645 *links ; required by Construction the exact figure of the triangle, the amount of the other angle, and the value in links of the other two sides.*

Take from the diagonal scale 645 equal parts, representative of 645 links, and of that length draw a line AB ; at one extremity, A, make an angle of 50° ; at the other extremity, B, an angle of 45° ; produce the lines from A and B until they meet in C ; ABC is the exact fig. required. If the angle at C be measured, it will be found to equal 85° ; and from the same scale of equal parts, AC will be found to equal 458 links nearly, and BC 496 links nearly.

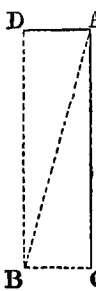


PROB. 2. *The distance* AB *from the foot of a precipice is* 288 *feet ; at* A, *the angle made by a string* CA, *reaching to the top, with* AB, *is* 53° 8′ ; *required by construction the height of the precipice and the length of the string.*

Draw a line AB containing 288 equal parts ; at B erect a perpendicular, and at A draw an angle of 53° 8′ ; produce the lines AC, BC to meet in C :—AC, representing the length of the string, will measure 480 feet ; and BC, the height of the precipice, 384 feet.



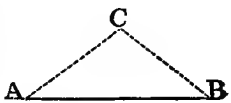
PROB. 3. *The breadth* AD *or* BC, *of the shaft of a well, of which the sides are parallel, is* 10 *feet; the angle of depression,* CAB, 15°: *required by construction,* AC, *the depth of the well.*



Take AD equal to 10 equal parts; and at A and D draw perpendiculars to AD; make the angle CAB 15°; and the point B, where AB and DB intersect, will be at the bottom of the shaft; through B draw BC parallel to DA; and CA equals DB; apply CA or DB to the same scale as AD, and the depth will be found 37·3 feet.

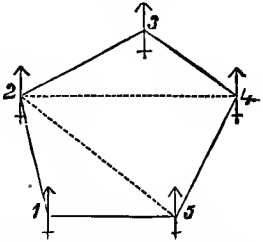
PROB. 4. *Two ships of war intending to cannonade a fort at* C, *separate from each other* 500 *yards; the angles between each ship and the fort and the other ship are* A, 38° 16′. *and* B, 37° 9′: 300 *yards being a convenient distance for a cannonade, have the two ships taken up their stations at a proper distance?*

The line AB being drawn equal to 500, angle A to 38° 16′, and angle B to 37° 9′, and the triangle ACB completed; it will be found from the same scale, that AC measures 312, and BC 320 yards: the ships therefore are *nearly at the exact distance* required.

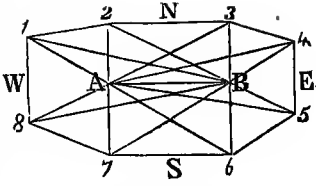


PROB. 5. *In surveying a field, the angles which the sides of the field made with the magnetic meridian, were observed at five stations, and the lengths of the sides measured: required to construct the figure of the field.*

Station 1. North 17° West 262 links.
„ 2. North 62° East 324 „
„ 3. South 51° East 221 „
„ 4. South 28° West 296 „
„ 5. West 242 „



At Station 1 make an angle 17° West of North, and set off 262 equal parts; at Station 2, an angle of 62° East of North, and set off 324; at Station 3, an angle 51° East of South, and set off 221; at Station 4, an angle 28° West of South, and set off 296; and at Station 5, a line due West, with a set off of 242; the figure will be completed: if now the diagonals and perpendiculars were drawn, the area of the field might be computed.



PROB. 6. For more extensive surveys *two stations* A and B are taken, the line A B is measured, and from a scale is set on the Plan; all the angles, which are formed by lines from the stations A and B to the objects at 1, 2, 3, 4, 5, 6, 7, and 8, are also measureed and drawn; and the angular points being joined, an outline of the figure is obtained in the relative size or magnitude of its parts; and by taking any of the distances in the plan, and applying them to the same scale as that from which A B was set out, their equivalent, or representative values will also be obtained. Thus by drawing the diagonals, and dropping perpendiculars from the angular points which the diagonals subtend, the area of the figure may be computed from the scale. It must, however, be observed, that this method of computation *is very liable to error*, for it is difficult to construct an intricate figure so as to make it perfect in all its parts. *Calculations from data obtained by actual measurements, are the only reliable methods of ascertaining with accuracy the areas of figures.*

The foregoing examples will be sufficient to show how by the construction of figures we may exhibit *the representative values* of actual magnitudes and spaces. Other instances are given at pp. 55, 56, 58, 66, 87, 91, 94, 99, 100, 108, 111, 112, 121, 133, &c.

IV.—PRINCIPLES WHICH WITHOUT REQUIRING THAT WE SHOULD MEASURE ALL THE BOUNDARIES OF A SURFACE, YET ENABLE US ACCURATELY TO CALCULATE DISTANCES, MAGNITUDES, AND AREAS.

§ 1. *Lines, or Distances.*

1. The sides of an equilateral triangle, by Def. 24. I., of a square, by Def. 30. I., and of a rhombus, by Def. 32. I., are all equal; *the measurement of one side in each*, therefore, is sufficient for ascertaining the other sides.

2. In an isosceles triangle, by Def. 25. I., two sides are equal; therefore the measurement of *one of those sides* will give the other.

3. In parallelograms, by 34. I., the opposite sides are equal; it is necessary, therefore, only to measure any *two conterminous sides*.

4. Of the three sides, H hypotenuse, B base, and P perpendicular; *any two* being measured, the third may be calculated; for, by 47. I. $H = \sqrt{B^2 + P^2}$; $B = \sqrt{H^2 - P^2}$; and $P = \sqrt{H^2 - B^2}$.

5. In a similar way, in a rectangle, of the two conterminous sides, A B, B C, and the diagonal, A C, *any two being given*, the third is found by 47. I.; for $AC = \sqrt{AB^2 + BC^2}$; $AB = \sqrt{AC^2 - BC^2}$; and $BC = \sqrt{AC^2 - AB^2}$.

6. If C, the circumference of a circle, or D, the diameter, be measured, *the other* may be found; for by 41, I. Use 4, $C = D \times 3{\cdot}1416$, and $D = \frac{C}{3{\cdot}1416}$

7. When, of the three parts of a regular polygon, the side A B, the perpendicular on the side P, and the radius R, or distance from the centre to

the angular points, *any two are measured*, the other is found; for, 41 and 47. I.,

$\frac{AB}{2} = \sqrt{R^2 - P^2}$; $P = \sqrt{R^2 - \left(\frac{AB}{2}\right)^2}$; and $R = \sqrt{P + {}^2\left(\frac{AB}{2}\right)^2}$

8. In any right-lined figure, if diagonals are drawn dividing it into triangles, and from the angular points perpendiculars are drawn to the diagonals,—*on measuring the perpendiculars and the segments of each diagonal, then*, by 47. I., *the sides may be calculated.*

9. In the case of *inaccessible distances*, the amount or calculation is obtained in various ways: as in 1. I., Use 4; 3. I., Use 3; 4. I., Use 2; 6. I., Use; 15. I., Use 1; 20. I., Use 2; 26. I., Use 1, 2, 3; 29. I., Use; 31. I., Use 2; 33. I.; 34. I., Use 4; 46. I., Use; 47. I., Use 4, 5; 6. II. Use.

§ 2. *Angles.*

1. In equilateral triangles, rectangles, squares, and regular polygons, the measurement of *one angle* is *equivalent* to the measurement of all; for, by Cor. 5. I., Def. 31, Def, 30, such figures have all their angles equal.

2. If we measure *any two angles* in a triangle, the third angle by 32. I. will equal the supplement of 180°, or 180°—the sum of the two angles.

3. In a right-angled triangle, with *one acute angle* known, the other acute angle is equal to the complement of 90°, or 90°—the given acute angle.

4. When two st. lines intersect, by 15. I., the measure of any one angle gives the opposite vertical angle.

5. By measuring any two adjacent angles of a parallelogram, 34. I., we obtain the other two angles.

6. When a st. line cuts two parallel lines, the exterior angle equals the interior opposite angle, and the alternate angles are equal, 29. I.; by measuring one angle, therefore, we know the other.

7. At the base of an isosceles triangle the angles are equal, 5. I.; one being measured, the other is known.

8. By approaching to, or receding from, a horizontal mirror, on which a given point from an object is reflected, we can ascertain the acute angle and the base of the right-angled triangle, 20. I., Use 2; and consequently we can calculate the vertical angle.

9. The parallax of a heavenly body is ascertained, 32. I., Use 1, by subtracting the zenith distance at the earth's centre from the zenith distance at the earth's surface.

10. When, by 4. I.: 8. I.; or 26. I.; two or more triangles are proved to have equal angles, the measurement or calculation of the angles of any one triangle is equivalent to the measurement or calculation of the angles of the others.

§ 3. *Magnitudes or Areas.*

1. In *squares*, the measurement of *one side* is sufficient,—for the altitude and base are equal; and the Area = the square of the side: 40. I. 3.

2. In all *parallelograms*, measure the *base and altitude;* for, by 35. I., Use, and 41. I. Use, the Area = the base × the altitude.

3. In all *triangles*, also measure the *base and altitude*, and take half the base; for, by 37. I., and 41. I., Use 1, the Area = $\frac{1}{2}$ the base × the altitude.

4. All *right-lined figures* may be divided into triangles;—then the Area of the figure = *the sum of the areas* of the triangles: 41. I., Use 2.

5. Lines from the centre of a *regular polygon* divide it into equal triangles; its Area, therefore = the Area of one triangle × the number of sides; or, = the perpendicular × ½ the perimeter; 41. I., Use 3.

6. The Area of a *trapezium* is found, 40. I., Use 2, by taking half the sum of the parallel sides and the altitude, and multiplying the two quantities together.

7. In *circles*, measure the diameter D, and ascertain the circumference C; or measure the circumference, and ascertain the diameter; then, by 41. I., Use 4, the Area $= \frac{3{\cdot}1416 \times D^2}{4}$; or, $= \frac{C^2}{4 \times 3{\cdot}1416}$; or, $= \frac{D \times C}{4}$; or, = the radius × the semi-circumference.

By the use of other Geometrical Truths, the Student might have a much more extended view of the Principles which assist Mathematical Calculations; but many of those truths lie out of the limits of an elementary work; and enough has been advanced to show the wide application and Utility of the First and Second Books of Euclid's Plane Geometry.

Indeed, if any defence were required for confining the Examinations of Pupil Teachers, and of Scholars generally, in Elementary Schools, to the two books referred to, it is furnished by the very valuable Practical Results which have just been exhibited. Whoever has mastered and retains his familiarity with the Geometrical Principles now set before him, will possess sufficient knowledge of the subject for all the usual purposes of life,—and, what is more, will possess the means, whenever he chooses to employ them, of advancing with comparative facility to the higher and more abstruse mathematical learning. The right foundation has been laid; and the calls of professional duties and employments may be left to determine, whether the student should remain satisfied with the mark attained, or go beyond it and labour in a wider field. If he is called, or prompted, to try the

more difficult paths, he will never regret that his attention in youth was chiefly confined to the Introductory Books of Plane Geometry. It is the accuracy and the thoroughness of the early training,—and not the wide extent of the subjects, traversed indeed, but not known,—which increase the power of the mind; and the true aim of the teacher is to strengthen power by a smaller quantity well done, than to waste it on a multitude of projects, to none of which it is able to do justice. The steam, that sounds a thousand jerking whistles does not perform half so much useful work as that which keeps in steady motion a single loom.

"By what steps and ways we are to advance in knowledge," observes JOHN LOCKE, *Of Human Understanding*, Book IV., chap. 12, § 7, "is to be learned in the schools of the mathematicians, who, from very plain and easy beginnings, by gentle degrees, and a continued chain of reasonings, proceed to the discovery and demonstration of truths that appear at first sight beyond human capacity. The art of finding proofs, and the admirable methods they have invented for the singling out, and laying in order, those intermediate ideas that demonstratively show the equality or inequality of unapplicable quantities, is that which has carried them so far, and produced such wonderful and unexpected discoveries: but whether something like this, in respect of other ideas, as well as those of magnitude, may not in time be found out, I will not determine."

APPENDIX.

I.—GEOMETRICAL SYNTHESIS AND ANALYSIS.

THE Principles of Plane Geometry, as taught in Euclid's Elements, are established, by proceeding in a regular series from Definitions, Postulates, and Axioms already known, to the consequences which flow from, and which are dependent upon, the Definitions, Postulates, and Axioms. This method is entirely one of building up, or of putting together, and is therefore named *Synthesis:* "it commences with what is given, and ends with what is sought,"—the materials being furnished, out of them it fashions a garment; it takes elementary substances, and forms a compound.

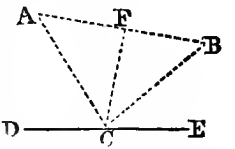
Analysis pursues an opposite course: it takes the compound, and resolves it into its constituent parts; the garment entire and completed is given for examination, and the aim is to discover of what it is made: Analysis begins with the thing sought, as a thing perfected and accomplished, and ends with whatever may have been supplied for the construction.

No aids except those derived from Geometry were admitted by the Ancients in conducting an Analysis; and therefore the term Geometrical Analysis is employed.

In analyzing a Problem, *the solution is assumed* to have been effected; and in analyzing a Theorem, *the truth* of the assertion contained in it is first of all admitted. When a problem is analyzed, the object is—to discover something which, if done, would of necessity lead to the solution of the problem; and when a Theorem is analyzed, the object is—to determine whether the assertion is true or false.

An example will more clearly show the difference between the analytical and the synthetical Methods: for this purpose we take the problem—

In a given st. line, DE, *to find a point,* C, *which shall be at the same distance from two other given points,* A *and* B.



By ANALYSIS.—We assume what was required, namely, that C, in DE is the point equally distant from the given points A and B.

The st. line CA = the st. line CB; and AB being joined, the figure ACB is an isosc. △. If not the st. line AB is bisected in F, we have, in the two triangles, CA = CB, AF = BF, and FC common; ∴, by 8. I., △ AFC = △ BFC, and ∠ AFC = ∠ BFC. But when two points, as A and B, are given, the line joining them, AB, is given; and the line being given, its middle point F, is also given; at F, that middle point, a perpendicular, FC, is given; and consequently, if produced, its point of intersection, C, with the given line DE. But C was the required point, and the production of the perpendicular from F determines it.

By SYNTHESIS.—Join the points A and B, and bisect the st. line AB in the point F; at F raise a perpendicular, and produce it to intersect DE in the point C;—that point C is equi-distant from A and B.

∵ AF = FB, FC common, and ∠ AFC = ∠ BFC.

∴ the side CA = the side CB: which was required.

In the Analysis, we have taken the Problem to pieces; in the Synthesis, we have put the parts together, and completed the purpose at which we aimed. It is by following similar Methods that other Geometrical Propositions may be analyzed and established. Analysis, however, is more suited for Problems;—Synthesis for Theorems.

The Rules for conducting an Analysis are few;—for the mode of procedure depends in a great measure on the knowledge and skill of the Student; and the greater these are, the greater facility and clearness will be manifest in making an analysis. It is a process which calls forth all the resources of his mind,—and therefore a very improving exercise for the young Geometrician.

The suggestions which are contained in RITCHIE'S *Geometry*, p. 50, may be of service to the Learner, and therefore are recommended to his attention.

"1st. As in every other study, endeavour to ascertain what it is you have to do; examine into the nature and meaning of the Proposition, and form a clear, well-defined idea of the quantities concerned in the investigation.

2nd. Construct every figure with exactness, that the *eye* may aid the judgment.

3rd. It will often be necessary to join certain points so as to form equal triangles, isosceles or equilateral triangles, and other right-lined figures.

4th. Often, also, to lengthen lines, to draw perpendiculars from certain points in a line, or to let fall perpendiculars from certain points on straight lines.

5th. Often, straight lines or angles must be bisected, one angle added to another, so as to get the sum of two angles; or a line drawn within an angle, so as to get the difference of two angles.

6th. Also, it will often be necessary to draw st. lines parallel to certain lines through remarkable points, which may be either given or required.

7th. In short, the Learner must form such a combination of lines, angles, and circles, as will, in his judgment, lead to the discovery of the object required. If, after trial, he finds he cannot reach the required point, and take the citadel by the path he has sketched out, he must commence the attack anew by following a different road, and by adopting a different system of *tactics*."

"REMARK.—1. A point is said to be given, when its *position* is either given, or may be determined.

2. A line is given in *position* when its direction is given; in *magnitude*, when its length is given.

3. A line is given in *position and magnitude* when both its direction and magnitude are given.

4. The Position of a *point* can be found only—*first*, by a st. line cutting another st. line; *second*, by a st. line cutting the circumference of a circle; or, *third*, by the intersection of the arcs of two circle.

5. The Position of a *line* is found, when any two points in it are found; and its Magnitude, when the extreme points are found.

A few EXAMPLES, selected from various sources, and restricted to the First and Second Books of the Elements, will show the Learner *how* an Analysis, or a Synthesis may be conducted.

EX. 1.—PROB.—*Given an angle*, A; *a side*, B C, *opposite to it, and the sum*, B D, *of the other two sides of a triangle:—to construct the triangle.*

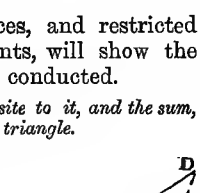
ANALYSIS.—The figure B A C is the required △, ∠ A the given angle, B C the side opposite to ∠ A, and B D the sum of the other two sides.

Join D C: the sides of the triangle B A and A C = the sum B D.

Take away the common part B A, and the rem. A C = the rem. A D; ∴ ∠ A C D = ∠ A D C.

But the ext. ∠ B A C = ∠ A C D + ∠ A D C, or ∠ B A C = 2 ∠ B D C; thus ∠ B D C = ½ ∠ B A C.

Hence the method of construction.

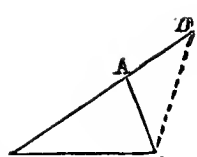


SYNTHESIS.—At D, one extremity of B D, make an angle = ½ ∠ B A C; and from B, the other extremity, draw B C, the given side, to meet D C in C; at C, make ∠ A C D = ∠ A D C, so that C A may meet B D in A: the triangle B A C will have its sides B A + A C = B D the sum; its ∠. = the given ∠, and its side = the given side.

EX. 2.—PROB.—*Given* B C *the base*, ∠ B *at the base, and* B D *the sum of the other two sides of a triangle:—to construct the triangle itself.*

ANALYSIS.—The triangle A B C is the triangle required; for ∠ B equals the given angle, B C is the base, and B D = B A + A C, the sum of the sides.

On joining D C, A C D becomes an isosceles △, and ∠ A C D = ∠ A D C.



SYNTHESIS.—Make B C = the base, ∠ B = the given ∠ at the base, and B D = the sum of the sides.

Join C D, and at C make ∠ D C A = ∠ A D C; let C A meet B D in A; then the figure C A B is the triangle required.

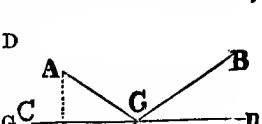
EX. 3.—PROB.—*From two given points,* A *and* B, *on the same side of a st. line given in position,* C D, *to draw two st. lines which shall meet in that line* C D, *and make equal angles with it.*

ANALYSIS.—The point sought is G; and ∠ B G D = ∠ A G C.

Produce B G until G E = G A, and join A E.

The ∠ E G C = ∠ B G D = ∠ A G C;

∴ ∠ E G F = ∠ A G F, and A G = G E, and F G is common, ∴ △ A F G = ∠ E F G; and they are rt. ∠ s, and A F = F E.



But since A and C D are given, the perpendicular A F is given; and the point E is known, and G is determined by the intersection of C D and E B.

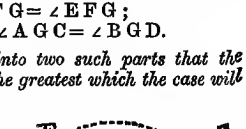
SYNTHESIS.—From A draw a perpendicular to C D, and produce it, till F E = A F; join E, B, cutting C D in G; and draw A G: the lines A G and B G are the lines required.

For A F = F E, F G is common, and ∠ A F G = ∠ E F G;

∴ ∠ A G F = ∠ E G F = ∠ B G D; and ∴ ∠ A G C = ∠ B G D.

EX. 4.—PROB.—*To divide a given st. line,* A B, *into two such parts that the rectangle contained by them may be three-fourths of the greatest which the case will admit of.*

ANALYSIS.—Let A B be the given line, and bisected in D; then □ A D . D B, or the square on D B, is the greatest possible rectangle.



Assume C, in A B, as the point required, so that A C . C B = ¾ of D B².

On A B describe a semicircle, and draw C E perpendicular from A B;

then the square on C E = □ A C . C B.

Again, bisect D B in F; on F B describe the rt. ∠ d. △ in which F G = D B;

then B G is the line the equare of which = ¾ of D B².

Hence, G B = E C; join G and E; ∴ the points E and C are found.

SYNTHESIS.—Let A B be bisected in D, and D B in F; and on F B let the rt. ∠ d. △ F B G be made, having its hypotenuse F G = D B.

On A B describe a semicircle; through G draw G E ∥ A B, and through E, E C ∥ G B;

then C is the point required, the □ A C . C B being equal to ¾ of the sq. on D B, which is half the line A B.

Some Problems admit of only one definite solution, and they are called *determinate;* and some admit of two or more definite solutions, and these are called *indeterminate.*

A Problem is *determinate.* when it relates to the determination of *a single point*, and the data are sufficient to determine the position of that point : it is *indeterminate*, if, on omitting one or more of the conditions, the data which remain may be sufficient for the determination of *more than one point*, each of which satisfies the conditions of the problem. In general, such points are found to be situated in some line ; and hence such line is called the *locus* of the point which satisfies the conditions of the problem.

Some Problems also are possible within certain limits ; and certain magnitudes increase, while others decrease, within those limits, and after having reached a certain value, the former begin to decrease, while the latter increase. This circumstance gives rise to questions of *maxima* and *minima*, or the greatest and least values which certain magnitudes may admit of in indeterminate problems.—POTTS' *Euclid*, p. 289-292.

It may be remarked that Synthesis is well adapted for communicating a knowledge of the first principles of Geometry, or of any Science ; but that Analysis is the great instrument or means of discovery,—it enables the Student to find out the solution of a problem for himself, and makes him more and more self-reliant in his search for Geometrical truth ;—a most valuable power of mind worth many a struggle to attain.

II.—VARIETY OF PATHS.

UNTIL some practice and self-work have given familiarity with the Processes of Geometrical Reasoning and of Geometrical Analysis, students are apt to suppose there is but one unvarying Method of conducting a Demonstration and of arriving at a sound conclusion.

Fixed, rigid, and unalterable indeed is the law, that, beyond Postulates and Axioms, nothing is to be taken for granted, and that each step must rest upon mathematical certainty before another is attempted ; but a *variety of paths*, all free from error, may in many instances be followed. Such a variety of paths is admirably shown in the *Second Book of Geometry*; 8vo, 1863, by Thomas Hill. of Harvard College, Cambridge, Mass., U.S.N.A ; and

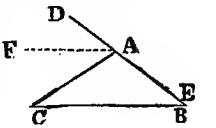
with this candid acknowledgment that chapter of his work is here made use of:—

"1. As there are usually many paths by which we may ascend a hill, so there are usually *many modes by which we may demonstrate a proposition.* In the case of a simple proposition, it is not usually worth while to try more than one mode. But with more difficult problems, it is sometimes worth while to spend a great deal of labour in discovering the simplest mode of demonstration. There are geometrical truths which can be demonstrated in so simple a manner as to require only twenty lines to write down the demonstration: and yet some writers, from ignorance of this simple mode, have written more than twenty pages to prove the same truths.

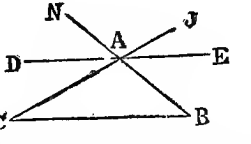
"2. It will therefore be useful to you, to show you by a simple example, such as that of the equality of the sum of the angles in a triangle to two right angles, *the great variety of methods* by which a single proposition can be proved.

"3. In the proofs of this proposition, which I will now give you, I shall not be careful to follow out every step. It will be enough, *for the purpose we now have in view,* simply to show you the general line of the paths, without taking you through every step of the way.

"4. The line D E might be drawn so as to coincide in direction with one of the other sides of the triangle, as A B, which would give us the figure in the margin. And by imagining the dotted line A F parallel to C B, we should have ∠ F A C equal to ∠ A C B, and ∠ F A D equal to ∠ C B A, which would make the three angles at A equal to the three angles of the triangle, as in the proof of Prop. 32, bk. i.

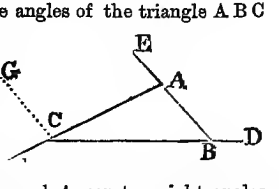


"5. By prolonging the lines C A and B A through the point A, D E being parallel to B C, we shall have ∠ N A D equal to ∠ A B C, ∠ J A E equal to ∠ A C B, and ∠ N A J equal to ∠ C A B. So that the three angles of the triangle A B C will be equal to the three angles D A N, N A J, and J A E, which are manifestly equal to two right angles.



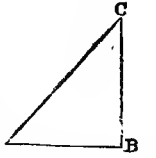
"6. The three methods of proving this proposition that have now been given are strictly Geometrical. Others might be given, that are some thing more like Algebraical reasoning.

"7. Let us, for instance, imagine each of the angles of the triangle A B C prolonged at its right-hand end, as in the figure, and also a st. line C G drawn from one vertex, as C, parallel to the opposite side B A. Now, the external angles, F C B, D B A, E A F, are plainly equal to the three angles F C B, B C G, G C F, and amount in magnitude to four right angles. But each external angle is plainly the supplement of one of the angles of the triangles; that is, it is equal to the difference between two right angles and one angle of the triangle. The sum of the three external angles F C B,



D B A, and E A F must therefore be equal to the difference between six right angles and the angles of the triangle. But as this difference amounts to four right angles, the three angles of the triangle must be equivalent to two right angles.

"8. If we introduce *the idea of motion*, we can devise quite a different sort of demonstration. Suppose, for instance, that I stand at the point A, with my face towards the point C. Let me now turn to the right until I face towards the point B. I have now changed the direction of my face by an amount which is equal to the angle at A. Suppose that I now walk to B, without turning, I shall have my back towards A, and if standing still, I turn to the right, until I have changed my direction by an amount equal to the angle A B C, I shall have my back towards C. Let me now walk backwards, without turning, until I reach C, and I shall have my face towards B. I will now turn a third time to the right, until I face the point A. My three turnings, or changes of direction, have been equal to the three angles of the triangle; they have all been to the right; therefore my whole change of direction is equal to the sum of these angles; I am now looking in exactly the opposite direction to that from which I started; I am looking from C to A, instead of from A to C; I have turned half way round; that is, through two right angles. Whence, *the sum of the three angles of the triangle is equivalent to* two right angles.



"9. Another demonstration, *by means of motion*, may be obtained as follows:—Suppose an arrow, longer than either side of the triangle, to be laid upon the side A C, pointing in the direction from A to C. Taking hold of the pointed end beyond C, turn the arrow round upon the point A, as a pivot, until the arrow lies upon the line A B. Taking now hold of the further end, beyond A, turn the arrow upon B as a pivot, until the arrow lies upon the line B C. Using C as a pivot, turn it now until the point of the arrow is over A. The arrow has thus been reversed in direction, turned half-way round, or through two right angles, It has been turned successively through the three angles of a triangle, and every time in the same direction, like the hands of a watch; so that the total change of direction, two right angles, is equivalent to the sum of the three angles.

"10. You have thus seen how a single proposition, and that a simple one, may be proved in a variety of ways. We have shown, in six different ways, what is the value of the sum of the angles in a triangle; in three, by what is called rigid *Geometry;* in one, by a partly *Algebraical* process; and in two, by introducing *the idea of motion*. And I wish you to observe that every one of the six ways is satisfactory. They are all proofs that are certain, because they lead you from self-evident truths by self-evident steps. One is not more certain than the other, because they are all absolutely certain. The only choice between them is, that some are more purely geometrical; some are better adapted to the peculiar tastes of different students; and some are neuter, and more quickly perceived by untaught persons."

III.—GEOMETRICAL EXERCISES.

THE Skeleton Propositions, which form a continuation or completion of the plan pursued in the Gradations of Euclid, furnish a most useful and improving series of exercises; and, on the ascertained fact in the art of teaching, that repetition is a most important auxiliary, they are recommended to the notice and adoption of the teachers of mathematics. At any rate, where employed, they will accustom the student to be systematic and exact, and not to advance a step without a reason; and progress in this way, though it may chance to be less rapid, will be on a solid basis, and bring into play some of the most valuable qualities of the mind.

POTTS, COLENSO, COOLEY, CHAMBERS, and others, have each published Collections of Geometrical Exercises; but as teachers may not wish to go beyond the limits of the present work, two Series of Exercises are now appended; the *First Series* consisting of Problems and Theorems which are inserted in the Gradations, and of which the General Enunciations may be given out to learners, for analysis, solution, or proof; and the *Second Series* containing Propositions which are not fully proved or not inserted in the Gradations.

SERIES I.

Problems in Book I.

1. By means of an equilateral triangle to measure an inaccessible distance.
2. To construct a scale of equal parts.
3. To ascertain an inaccessible distance when two sides and their included angles have been measured.
4. To show, by observations on the shadows which objects cast, the perpendicular heights of the objects.
5. To determine without a theodolite the angle at a given point made by a st. line from two objects meeting in that point.
6. To construct the Mariner's Compass-Card.
7. From a given point at the end of a st. line to raise a perpendicular.
8. On a given line to describe an isoceles triangle of which the perpendicular height is equal to the base.
9. From a given point over the end of a st. line to let fall a perpendicular to the line.
10. Given an angle of 73°, required its complement; an angle of 96°, required its supplement.

11. By the application of the principle that vertical angles are equal, to find the distance between two objects.
12. From a given point, A, to direct a ray of light against a mirror, so that the ray shall be reflected to another given point.
13. To determine the number and kind of polygons which may be joined so as to cover a given space.
14. To construct a triangle when the base, the less angle at the base, and the difference of the sides, are given.
15. By means of a mirror placed horizontally, to construct a triangle, the perpendicular of which shall be representative of the height of any object.
16. Given three st. lines respectively equal to 40, 50, and 30 equal parts, to form a triangle.
17. On a given st. line to describe an isosceles triangle having each of the equal sides double of the base.
18. On a given st. line and with a given side to construct an isosceles triangle.
19. To construct a Line or Scale of Chords.
20. By aid of a line of Chords,—1°· to make an angle containing a certain number of degrees, as 40°; 2°· to measure a given angle; 3°· from the extremity of a line to raise a perpendicular; and 4°· to construct a triangle of which the base contains 40 equal parts, one of the angles at the base 40°, and its other adjacent side 35 equal parts; and to measure the other side and the other angles.
21. To measure an inaccessible distance, A B, only A being approachable.
22. By the theory of Representative Values to find the distance between two stations.
23. To measure an inaccessible distance, A B, neither A nor B being approachable.
24. Given the vertical angle and the perpendicular height of an isosceles triangle, to construct it.
25. On the principle that two rays of light proceeding from the centre of the sun to two points on the earth are physically parallel, to ascertain the earth's circumference.
26. By the method of parallel lines to ascertain the distance of an inaccessible object.
27. To determine the Parallax of a heavenly body.
28. To construct a figure which will give the representative value of the perpendicular height of a mountain.
29. To construct a regular Polygon,—1°· when the side is given; 2°· when the side is not given.
30. To ascertain both the perpendicular height of a mountain, and the horizontal distance from the foot of the mountain to the foot of the perpendicular.
31. To divide a finite st. line into any given number of equal parts.
32. To construct a Sliding Scale to measure the hundreth part of an inch.
33. To construct a Sliding Scale for measuring the minutes into which a degree on a circle is divided.
34. To ascertain the continuation of a st. line when an obstacle intervenes.
35. From a given point in the side of a parallelogram to bisect the parallelogram.
36. To convert a parallelogram into an equivalent rectangle.

37. By the method of Indivisibles to explain the equality of parallelograms on the same base and between the same parallels.
38. To construct a Diagonal Scale.
39. To divide a triangular space into two equal parts.
40. From any point in the side of a triangle to divide the triangle into two equal parts.
41. To find the Area of a trapezium.
42. To find the Area of a square.
43. To find the Area of a triangle.
44. To find the Area of any right-lined figure.
45. To find the Area of a regular polygon.
46. To find the Area of a circle.
47. Of the Diameter, Circumference, Area, and Ratio of the diameter and circumference, any two being given, to ascertain the others.
48. To describe a triangle equal to a given parallelogram, and having an angle equal to a given angle.
49. A parallelogram being given, to find another parallelogram equal to it, and having one side equal to a given st. line.
50. Given the area of one figure, and the side of another which is to be a parallelogram equal to the given figure, to find the other side of that parallelogram.
51. To change any right-lined figure, first into a triangle, and then into a rectangle of equal area.
52. To straighten a crooked boundary without changing the relative size of two fields.
53. Given the diagonal to construct a square.
54. To ascertain the height of an inaccessible object by aid of the Geometrical Square.
55. Given in numbers the sides of a right-angled triangle, to find the hypotenuse.
56. Given in numbers the hypotenuse and one side, to find the other side.
57. To find a square equal to any number of squares; or a square that is the multiple of a given square; or a square that equals the difference of two squares; or a square that is the half, the fourth, &c., of a given square.
58. To make a rectilineal figure similar to a given rectilineal figure.
59. To make a circle the double, or the half, of another circle.
60. To construct the Chords, Natural Sines, Tangents, and Secants, of Trigonometrical Tables.
61. To find right triangular numbers.
62. To compute Heights and Distances from the curvature of the earth.

Theorems in Book I.

1. By the bisection of the vertical angle of an isosceles triangle to show that the angles of the base are equal; and also that the bisecting line bisects the base at right angles.
2. Only one perpendicular can be drawn from a given point to a given st. line.

3. The perpendicular is the shortest line from a given point to a given st. line.
4. From the same point only two equal st. lines can be drawn to a given st. line.
5. All heavy bodies free to move continually descend, or seek the point which is nearest to the earth's centre.
6. Of all st. lines that can be drawn from one point to a plane surface, and reflected to a third point, those are the shortest which make the angle of incidence equal to the angle of reflection.
7. The chord of 60° is equal to the radius of the circle.
8. A st. line which is perpendicular to one parallel, is also perpendicular to the other.
9. If a st. line falling on two other st. lines make the interior angles on the same side less than two rt. angles, those two lines on being produced shall intersect.
10. A parallel to the base of a triangle through the point of bisection of one side, will bisect the other side.
11. The st. lines which join the middle points of the three sides of a triangle, divide it into four triangles which are equal in every respect.
12. The st. line joining the points of bisection of each pair of sides of a triangle, is equal to half the third side.
13. A trapezium is equal in area to a parallelogram of the same altitude, and of which the base is half the sum of the parallel sides.
14. The squares of equal st. lines are equal; and if the squares are equal, the lines are equal.
15. Every parallelogram having one rt. angle, has all its angles rt. angles.
16. If a perpendicular be drawn from the vertex of a triangle to the base, the difference of the squares of the sides is equal to the difference between the squares of the segments.
17. If a perpendicular be drawn from the vertex of a triangle to the base, or to the base produced, the sums of the squares of the sides and of the alternate angles are equal.

Problems in Book II.

1. From Propositions 1, 2, and 3, deduce various methods for the Multiplication of Numbers, and demonstrate the rule.
2. From Prop. 4, point out a practical way of extracting the Square root of a number, and prove the correctness of the formula.
3. To find the difference between the squares of two unequal numbers without squaring them.
4. To find Quantities in Arithmetical Progression.
5. To find the value of an Adfected Quadratic Equation in Algebra.
6. By aid of Prop. 6, to ascertain the diameter of the earth.
7. Given the sum and the difference of two magnitudes, to find the magnitudes themselves.
8. From the Area of a rectangle and one side given, to obtain the other side.
9. To divide a given Line a, so that its parts x and $a-x$ may make $a(a-x) = x^2$. Let the solution be given both algebraically and arithmetically.

10. To ascertain the Area of a triangle when the three sides are known.
11. From the three sides of a triangle given, to obtain the perpendicular :— 1°. when the perp. falls within the base ; and 2° when it falls without the bass.
12. To find a mean proportional to two given lines.
13. To approximate to the square of any curve-lined figure.
14. To calculate the Area of any right-lined figure.

Theorems in Book II.

1. The difference of the squares of two quantities, equals the rectangle of their sum and difference.
2. The difference of the squares of two quantities is greater than the square of their difference, by twice the rectangle of the less and their difference.
3. The square of the sum of two lines is equal to four times the rectangle under them, together with the square of their difference.
4. Four times the square of half the sum is equal to four times the rectangle under the lines, together with four times the square of half the difference.
5. The sum of the squares of any two st. lines is equal to twice the square of half their sum, together with twice the square of half their difference.
6. The sum of the squares is equal to half the square of the sum, together with half the square of the difference.

SERIES II.

PROPOSITIONS NOT FULLY PROVED, OR NOT INSERTED IN THE GRADATIONS.

Problems.—Book I.

1. To find a point which is equidistant from the three vertical points of a triangle.
2. To bisect a triangle by a st. line drawn from a given point in one of its sides.
3. Describe a circle which shall pass through two given points, and have its centre in a given line.
4. Through a given point to draw a st. line that shall be equally inclined to two given st. lines.
5. Given a triangle A B C, and a point D in A B ; to construct another triangle A D E equal to the former, and having the common angle A.
6. To change a triangle into another equal triangle of a given altitude.
7. To draw a st. line which, if produced, would bisect the angle between two given st. lines, without producing them to meet.
8. To trisect a right angle.

9. To trisect a given st. line.
10. Given the sum of the sides of a triangle, and the angles at the base, to construct it.
11. Given the diagonal of a square, to construct the square of which it is the diagonal.
12. Given the sum and difference of the hypotenuse and a side of a right-angled triangle, and also the remaining side, to construct it.
13. To find the *locus* of all points which are equidistant from two given points.

Theorems.—Book I.

1. In an isosceles triangle, the right line which bisects the vertical angle also bisects the base, and is perpendicular to the base.
2. If four straight lines meet at a point, and make the opposite vertical angles equal, each alternate pair of st. lines will be in the same st. line.
3. The difference of any two sides of a triangle is less than the remaining side.
4. Each angle of an equilateral triangle is equal to one-third of two right angles, or to two thirds of one right angle.
5. The vertical angle of a triangle is right, acute, or obtuse, according as the st. line from the vertex bisecting the base is equal to, greater, or less than half the base.
6. If the opposite sides or opposite angles of a quadrilateral be equal, the figure is a parallelogram.
7. If the four sides of a quadrilateral are bisected, and the middle points of each pair of conterminous sides joined by st. lines, those joining lines will form a parallelogram the area of which is equal to half that of the given quadrilateral.
8. If two opposite sides of a parallelogram be bisected, and two st. lines be drawn from the points of bisection to the opposite angles, these two st. lines trisect the diagonal.
9. In any right-angled triangle, the middle point of the hypotenuse is equally distant from the three angles.
10. The square of a st. line is equal to four times the square of its half.
11. The st. line which bisects two sides of a triangle, is parallel to the third side, and equal to one half of it.
12. If two sides of a triangle be given, its area will be greatest when they contain a rt. angle.
13. Of equal parallelograms that which has the least perimeter is the square.
14. The area of any two parallelograms described on the two sides of a triangle, is equal to that of a parallelogram on the base, whose side is equal and parallel to the line drawn from the vertex of the triangle to the intersection of the two sides of the former parallelograms produced to meet.
15. The vertical angle of a triangle is acute, rt. angled, or obtuse, according as the square of the base is less than, equal to, or greater than, the sum of the squares of the sides.

Problems.—Book II.

1. The sum and difference of two magnitudes being given, to find the magnitudes themselves.
2. To describe a square equal to the difference of two given squares.
3. To divide a given st. line into two parts, such that the squares of the whole line and of one of the parts shall be equal to twice the square of the other part.
4. To divide a given st. line into two such parts that the rectangle contained by them may be three-fourths of the greatest of which the case admits.
5. Given the area of a right-angled triangle, and its altitude or perpendicular from the vertex of the rt. angle to the opposite side, to find the sides.
6. Given the segments of the hypotenuse made by the perp. from the rt. angle, to find the sides.
7. To divide a st. line internally, so that the rectangle under its segments shall be of a given magnitude.
8. To cut a st. line externally, so that the rectangle under the segments shall be equal to a given magnitude, as the square on A.
9. Given the difference of the squares of two st. lines and the rectangle under them, to find the lines.
10. There are five quantities depending on a rectangle,—1°. the sum of the sides; 2°. the difference of the sides; 3°. the area; 4°. the sum of the squares of the sides; and 5°. the difference of the squares of the sides:—by combining any two of these five quantities, find the sides of the rectangle.

Theorems.—Book II.

1. The square of the perpendicular upon the hypotenuse of a right-angled triangle drawn from the opposite angle, is equal to the rectangle under the segments of the hypotenuse.
2. The squares of the sum and of the difference of two st. lines, are together double of the squares of these lines.
3. In any triangle the squares of the two sides are together double of the squares of half the base, and of the st. line joining its middle point with the opposite angle.
4. The square of the excess of one st. line above another, is less than the squares of the two st. lines by twice their rectangle.
5. The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.
6. If a st. line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together equal to twice the rectangle contained by these parts, together with four times the square of the st. line between the points of section.
7. If a st. line be drawn from the vertex of a triangle to the middle point of the opposite side, the sum of the squares of the other sides is equal

to twice the sum of the squares of the bisector and half of the bisected side.

8. The sum of the squares of the sides of a quadrilateral figure is equal to the sum of the squares of the diagonals, together with four times the square of the st. line joining their points of bisection.
9. If st. lines be drawn from each angle of a triangle bisecting the opposite side, four times the sum of the squares of these lines is equal to three times the sum of the squares of the side of the triangle.
10. The square of either of the sides of the rt. angle of a rt. angled triangle, is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side.
11. If from the middle point C, of a st. line AB, a circle be described, the sums of the squares of the distances of all points in this circle from the ends of the st. line A.B. are the same; and those sums are equal to twice the sum of the squares of the radius and of half the given line.
12. Prove that the sum of the squares of two st. lines is never less than twice their rectangle; and that the difference of their squares is equal to the rectangle of their sum and difference.
13. If, within or without a rectangle, a point be assumed, the sum of the squares of st. lines drawn from it to two opposite angles, is equal to the sum of the squares of the st. lines drawn to the other two opposite angles.
14. If the sides of a triangle be as 4, 8, and 10, the angle which the side 10 subtends will be obtuse.
15. If in a rt. angled triangle a perpendicular be drawn from the rt. angle to the hypotenuse, the rectangle of one side and of the non-adjacent segment of the hypotenuse, shall equal the rectangle of the other side and of the other non-adjacent segment of the hypotenuse.

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"Even the Sleeping Geometrician," says old RALPH CUDWORTH, p. 160, "hath at that time, all his Geometrical Theorems and Knowledges some way on him : as also the Sleeping Musician, all his Musical Skill and Songs : and therefore why may it not be possible for the Soul to have likewise some *Actual Energie* on it, which it is not Expressly Conscious of ?"

JOHN HEYWOOD, Excelsior Printing and Stationery Works, Hulme Hall Road, Manchester.

EUCLID'S PLANE GEOMETRY,

BOOKS III—VI.,

PRACTICALLY APPLIED;

OR

GRADATIONS IN EUCLID, PART II.,

WITH

ALGEBRAICAL AND ARITHMETICAL ILLUSTRATIONS, EXPLANATORY NOTES, AND A SYNOPTICAL INDEX TO THE SIX BOOKS, SHEWING THE USES OF THE PROPOSITIONS, &c.,

BY HENRY GREEN, A.M.

"THERE IS (GENTLE READER) NOTHING (THE WORD OF GOD ONELY SET APART) WHICH SO MUCH BEAUTEFIETH AND ADORNETH THE SOULE AND MINDE OF MAN, AS DOTH THE KNOWLEDGE OF GOOD ARTES AND SCIENCES." *Billingsley's Euclid*, A. D. 1570.

MANCHESTER: JOHN HEYWOOD, 143, DEANSGATE.
LONDON: SIMPKIN, MARSHALL, & CO.

1861.

TO WILLIAM FAIRBAIRN, C.E, LL.D., F.R.S, F.G.S.,

CORRESPONDING MEMBER OF THE NATIONAL INSTITUTE OF

FRANCE, AND OF THE ROYAL ACADEMY OF TURIN,

CHEVALIER OF THE LEGION OF HONOUR,

ETC., ETC. ;

WHOSE AIMS IN LIFE

HAVE LED HIM TO ENTERPRISES OF PUBLIC UTILITY,

AND WHOSE WRITINGS AND EXAMPLE

SHOW THE PRACTICAL USES OF SCIENTIFIC KNOWLEDGE;

THIS WORK,

BY HIS PERMISSION, IS INSCRIBED;

AND WITH EVERY SENTIMENT OF RESPECT.

PREFACE.

The Preface to Bks. I and II of the "Gradations in Euclid" contains most of the observations which can be required from the Author. One of them he now repeats;—"At the present day nearly every edition of EUCLID's Elements must be more or less a compilation, in which the Author draws freely on the labours of his predecessors. 'The Gradations' are, in a great degree, of this character; and an open acknowledgment will suffice, once for all, to repel any charge of intentionally claiming what belongs to others. It is affectation to pretend to great originality on a subject which has, like Geometry, for so many centuries exercised men's minds."

Originality has not been the Author's aim, but usefulness. With this purpose before him, he has endeavoured to show how few, if any, Geometrical Truths are destitute of a practical application. The stigma which some have attempted to fasten on what they name, "mere Mathematical Theories," is thus removed, and the Science, instead of being repulsive from its dry abstractions, is invested with the ever-abiding charm of being both the foundation and the builder-up of very many most important practical results. In fact, there is scarcely a branch of human knowledge, from the art of sketching an outline, to that of spanning and measuring the heavens themselves, which does not depend for its vigour and comprehensiveness on the aids which Geometry furnishes.

Step by step man ascends the Himalayas and compasses the earth; step by step is the Mathematician's course. As in the First Part, so in this Second Part of the "Gradations," the same method has been pursued. Prefixed to the Proof are references to the principles that have to be employed, and often quotations of the very words in which Euclid embodies those principles. These are not so fully given, indeed, as in the First Part, for it is presumed that familiarity with leading principles and propositions has been already attained. Occasionally too, in the steps of the Construction and Demonstration, the special reference is not made in the margin to the evidence on which an argument or a conclusion rests; but the Learner will scarcely find this any obstacle, if he has mastered what he has read.

It will be well for the Learner thoroughly to consider the references, before he proceeds to the Particular Enunciation, the Construction and the Demonstration of the Proposition;—indeed, were he of himself to put together the truths with which he is supplied, and to see how the new truth is to be deduced from them, he would derive the best assistance, that from the reasoning of his own mind, to understand and appreciate the fuller proof of the formal demonstration. The Memory, no doubt, is a most valuable power in acquiring any kind of knowledge, but in Mathematics especially it is the understanding and the reasoning faculty that are employed to most advantage and developed with greater exactness.

As an instance of the method recommended, let the Learner take that important Proposition, 35, III, "*If two st. lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.*" After weighing these words of the General Enunciation, let him call to

mind the several truths contained in the references under the heads, "CON.," construction, and "DEM.," demonstration; and if he has forgotten any of them let him turn back to the very propositions numbered, as 10, I., 3, III., 47, I., &c., and carefully think them over. He thus burnishes up his old weapons; and now let him try to trace out the connexion between the propositions referred to, and to ascertain how they lead to the new proposition which he seeks to establish. He will say to himself, here are several undoubted truths and facts presented to me;—I have already accepted them as principles of Geometrical Reasoning,—and they are now given that I may demonstrate some other truth, or solve some other problem. Can I not, with the implements provided, build this new hou se and see how, like the others, it is composed of indestructible materials?

He may rely, that by thus exercising his judgment, he will do more, than any mere effort of memory can do, for really understanding and retaining mathematical truths.

A very full Table of signs and abbreviations is given, and this should be consulted until they have become well known.

Some of the Demonstrations, as in 8, V, and 15, V, have been shortened. By the time the Learner has mastered so much of the Geometry, he will readily perceive the connexion of the argument, and not require the entire fulness of which it is capable; or, if he should, he may be expected to supply it from his own resources.

The Index was a subject of some consideration. An alphabetical Index had been prepared, but it was rejected, because it would have occupied too much space. The advantage of learners appeared to be more promoted by having the whole of the General Enunciations

of Euclid's Geometry brought together under the heads of Problems and Theorems, with their respective illustrations, applications, and uses. An alphabetical Index would have facilitated references to particular truths, but the consecutive or synoptical Index conduces more to the understanding of the whole work, and to the tracing out of the connexions of its parts.

A word or two to those who, from inexperience, do not understand the difficulty of avoiding errors of the press in a work where many signs, abbreviations, and references are used. As a most justly celebrated mathematician* has observed,—The Table of Corrigenda, at the end of the volume, "may convey an impression that the work is incorrectly printed, which is not the case;" and he adds, "If every mathematical work, at its completion, had the fruits of some years of examination presented to the reader, I know of none which would not have lists as large in proportion to their size and the number of symbols contained in them as the present one." On this subject the author will simply remark, that should "the Enclid Practically applied" attain a second edition, these and some other faults will be carefully amended.

In his Mathematical Preface, "written at his poor House at Mortlake, Anno 1570, February 9," "John Dee, of London," addressed himself "to the vnfained Lovers of Truthe and constant Studentes of Noble Sciences;" "he hartely wisheth them grace from heaven and most prosperous successe in all their honest attemptes and exercises." So, with him, I say to all who value good learning, "I commit you vnto God's Mercyfull direction for the rest; hartily beseechyng hym, to prosper your Studyes and honest Intentes to His Glory and the Commodity of our Country."

October 1, 1861.

* De Morgan, in his *Differential and Integral Calculus*, A.D. 1842.

SYMBOLICAL NOTATION AND ABBREVIATIONS.

I.—*Signs common to Arithmetic, Algebra, and Geometry.*

∵ because.	+ *plus*, add, together with.
∴ therefore.	− *minus*, subtract, take away.
∴ wherefore.	~ difference between.
= equals, *or* equal.	× *into*, multiply.
≠ not equal to, or unequal.	÷ *by*, divide.
> greater than.	√ root.
⋡ not greater than.	: ratio.
< less than.	:=: equality of ratios.
⋠ not less than.	:::: proportion.

: : : Numbers or Quantities in Progression

The signs >, ⋡, <, ⋠, between ratios, as A : B > C : D, or A : B ⋡ C : D, or A : B < C : D, or A : B ⋠ C : D, denote that the one ratio *is greater than*, or *not greater than*, *less than*, or *not less than*, the other ratio, according to the sign.

II.—*Geometrical Signs.*

• a point. *	△ triangle.
| straight line.	▱ parallelogram.
∥ parallel, parallel to.	□ square, ▭ rectangle.
∠ angle.	⊙ circle.
⊥ perpendicular to, at rt. ∠s.	⊙ce circumference.

* When an *s* is added to a sign, or to an abbreviation, the plural is denoted

A *single* capital letter, as A, or B, denotes the point A, or the point B; but sometimes, as in Bks. V and VI, the quantity, or magnitude, A, B, C, &c.

Two capital letters, as A B, or C D, denote the straight line A B, or C D; but when the letters indicate opposite angles, they denote a parallelogram, or a rectangle, or a polygon, as the figure will show.

A capital letter, or two capital letters, with the *numeral* ^{2} just above to the right hand, as A^2, or AB^2, denote not the square *of* A, A B, but the square *on* A or A B.

Capital letters, with a point between them, as A B . C D, denote, not the product of A B multiplied by C D, but the rectangle formed by two of its sides meeting in a common point.

III.—*Additional Algebraic Expressions.*

M	Magnitude.	n or p	another multiple.
m	multiple.	$m + n$	the sum of the quantities m & n.
m A &c.	multiple of A &c.		
m A, m B, &c.	equimultiples of A, B &c.	mn	the product of $m \times n$.
		mn A	a multiple of A by mn
m (A + B)	multiple of (A+B)	$(m + n)$ A	a multiple of A by $m + n$.
m (A − B)	multiple of (A−B).		
m (A+B−C)	multiple of the excess of (A+B) above C.	*pt.*	part.
		sub-m	submultiple.

III.—*Abbreviations.*

Add . . . . . *Addendo*, by adding.
App. . . . . Application of a Prop.
Appl. . . . . *Applicando*, by applying
C. or Con. . Construction.
C. 1 &c. . . Step 1 &c. of the Construction.
Conc. . . . . Conclusion, inference.
Cor. . . . . Corollary.
Dat. . . . . *Datum*, or *data*.
D. or Dem. Demonstration.
D. 1 &c. . . Step 1 &c. of the Dem.
E. or Exp. . Exposition, or Particular Enunciation of a Prop.
Ex. . . . . Example.
Gen. . . . . General Enunciation.
H. or Hyp. Hypothesis of a Prop.
H. 1 &c. . . Step 1 &c. of the Hyp.
L . . . . . Line.
M. or Mag. Magnitude.
P. or Prop. Proposition.
Pon. . . . . *Ponendo*, by placing, by position.
Prob. . . . . Problem.
Proced . . . *Precedendo*, by going on.
Prel . . . . . Preliminary.
Prod. . . . . Prodnct.
Pst. or Psts. Postulate, or Postulates
Quæs. . . . . *Quæsitum* or *Quæsita*.
R. . . . . Ratio.
Rec. . . . . Recapitulation.
Remk. . . . Remark to be made.
Sch. . . . . *Scholium* or *Scholia*.
Sim. . . . . So, similarly, by similar reasoning.
S. or Sol. . . Solution of a Problem.
Sum. . . . *Sume*, take away.
Sup. . . . . Suppose, *or* Let.
Super. . . . *Superponendo*, by superdosition.
Theor. . . . Theorem.

Q. E. D. *quod erat demonstrandum*, which was the thing to be proved.

Q. E. F. *quod erat faciendum*, which was the thing to be done.

adj. . . . . .adjacent.
ad impossibile, to an impossibility.
a fortiori, by a stronger reason.
alt. . . . . .altitude.
altr. . . . . .alternate.
antec . . . .antecedent.
ang .....angular.
assumendo, by adopting.
bis. . . . . .bisects, *or* bisect.
bisd. . . . . .bisected.
bisg. . . . . .bisecting.
cen. . . . . .centre.
ch. . . . . .chord.
com. . . . . .common.
comp- . . . .compound.
compl. . . . .complement.
con. sup. . .contrary supposition.
c. scr. . . . . .circumscribe.
c. scg. . . . .circumscribing.
conseq. . . . .consequent.
cont. . . . . .continued..
conterm . . .conterminous.
contn. .....contain, or contained.
descr. . . . .describe, or described.
descg. . . . .describing.
diag. . . . . .diagonal.
diam. . . . . .diameter.
diff. . . . . .difference.
dist. . . . . .distance, or distant.
div. . . . . .divide, or divided.
dupl. . . . .duplicate.
eq. . . . . .equal or equally.
eq. ang. . . .equiangular.
eq. lat. . . . .equilateral.
ex. ab. . . . .ex absurdo, by an absurdity.
ex gr,. . . . . .exempli gratiâ, for example's sake.
ext. . . . . .exterior, or exteriorly.
extn. . . . . .externally.
extr . . . . . .extremity, or extremities.
fig. . . . . .figure.
gr. . . . . .greater.
homol. . .homologons.
hyp. . . . . .hypotenuse.
incl. . . . . .include, included.
indef. . . . .indefinitely.
inscr. . . . .inscribe, inscribed.
int. . . . . .interior.
inters. . . . .intersect, intersection.
intr .....internal, internally.
m or mult. . . } multiple
magn. . . . .magnitude.
meas. . . . .measure.
mid. . . . . .middle.
ob. . . . . .obtuse.
opp. . . . . .opposite.
par. . . . . .parallel.
parlm. . . . .parallelogram.
pent . . . . .pentagon.
perp. . . . . .perpendicular.

pos. . . . . .position.
prod. . . . .produce, produced.
propl. . . . .proportional.
pt. . . . . .part.
qu. ang. . .quadrangular.
qu. lat. . . .quadrilateral.
rad. . . . . .radius.
rat. . . . . .ratio.
recip. . . . .reciprocal.
rect. . . . . .rectangle.
rectl. . . . . .rectilineal.
rectr. . . . . .rectangular.
reg. . . . . .regular.
rem. . . . . .remaining.
resp. . . . . .respective.
rt. . . . . .right.
sect. . . . . .sector.
seg. . . . . .segment.
sem. c. . .semicircle.
sem. cirf. . .semicircumference.
sim. . . . . .similar to, similarly.
sim. sit. . . .similarly situated.
sq. . . . . .square.
st. . . . . .straight.
suppl. . . . .supplemental.
tang. . . . . .tangent.
trap. . . . . .trapezium.
undiv. . . . .undivided.
uneq. . . . .unequal, or unequally.
vert. . . . . .vertex, vertical.

GRADATIONS IN EUCLID.

BOOK III.

TREATING OF THOSE PROPERTIES OF THE CIRCLE, AND OF STRAIGHT LINES IN AND ABOUT IT, WHICH CAN BE DEDUCED FROM THE FIRST AND SECOND BOOKS.

A circle, strictly speaking, signifies the space bounded by a circumference, but in this book the term is employed sometimes to denote that space, and at other times, the circumference itself.

Euclid, too, occasionally assumes from experimental knowledge, certain properties of the circle, which a more rigid and exact method of reasoning would have established before using them. This is the case in the first Proposition itself, where it is taken for granted that the perpendicular to the chord of an arc will meet the circle in two points. In some instances also the method of *indirect* demonstration is adopted, when the more satisfactory method of *direct* proof is available; examples of this occur in Props. 2, 13, 16 and 36.

By restricting the meaning of the term angle to an opening formed by two conterminous lines, and less than two right angles,

Euclid renders some of his demonstrations, as that of Prop. 21, more cumbersome than they need be.

The Properties of the right-angled triangle, of the circle, and of certain lines in and about a circle, as the radius, the sine, the tangent and the secant, have laid the foundations of by far the most extensive branch of Mathematics. Trigonometry, Plane and Spherical, resting on these properties and at first "confined to the solution of one general problem, has now spread its uses over the whole of the immense domains of the mathematical and physical sciences."—LARDNER'S Trigonometry, p. 3.

The Learner may therefore enter on the study of this Third Book with the assurance, that he is about to cross the threshold of one of the most important parts of Plane Geometry. "The influence, indeed, of the properties of the circle upon abstract mathematical analysis has been so great that an attempt to describe the manner in which the means of expression derived from this figure has been used, would fill a volume."

The quaint English *Editio Princeps* of Euclid, published in 1570, thus opens to the Reader the Summary of Bk. III.

"This third booke of Euclide entreateth of the most perfect figure, which is a circle. Wherefore it is much more to be estemed then the two bookes goyng before, in which he did set forth the most simple proprieties of rightlined figures. For sciences take their dignities of the worthynes of the matter that they entreat of. But of al figures the circle is of most absolute perfection, whose proprieties and passions are here set forth, and most certainly demôstrated. Here also is entreated of right lines subtended to arkes in circles: also of angles set both at the circumference and at the centre of a circle, and of the varietie and difference of them. Wherfore the readyng of this booke, is very profitable to the attayning to the knowledge of chordes and arkes. It teacheth moreover which are circles contingêt, and

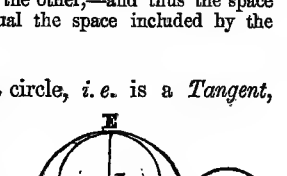
which are cutting the one the other: and also that the angle of contingence is the least of all acute rightlined angles: and that the diameter in a circle is the longest line that can be drawen in a circle. Farther in it may we learne how, three pointes beyng geuen how soever (so that they be not set in a right line) may be drawen a circle passing by them all three. Agayne, how in a solide body, as in a Sphere, Cube, or such lyke, may be found the two opposite pointes. Whiche is a thyng very necessary and commodious, chiefly for those that shall make instrumentes seruyng to Astronomy and other artes."—BILLINGSLEY'S EUCLID, fol. 81.

DEFINITIONS.

1. *Equal circles* are those of which the diameters are equal, *or*, from the centres of which the straight lines to the circumferences are equal.

The *criterion* of the equality of circles is that their diameters or their radii are equal;—but this is neither a Definition nor an Axiom; properly it is a Theorem, the truth of which may be proved by *superposition*;—for if centre be placed on centre and the equal radii or diameters on each other, the circumference of the one will in each point coincide with the circumference of the other,—and thus the space included by one circumference will equal the space included by the other.

2. A straight line is said to touch a circle, *i. e.* is a *Tangent*, when it meets the circle, and being produced does not cut it; as AB tangent to EFC in C.



The point in which the straight line meets the circle is the *point of contact;* the straight line "does not cut," *i. e.* does not pass into the circle. A *Secant* is a straight line which, when it meets the circle and is produced, passes into the circle, *i. e.*, cuts or crosses the circumference; as BHD, secant to EFC in H.

The terms Tangent and Secant are often restricted in meaning;—the *first* to the line which by one extremity touches an extremity of the diameter at right angles to it, and which has its other extremity terminated by a straight line, the Secant, from the centre of the circle across the circumference; the *second* to the line from the centre, across the circumference, and terminated by the tangent. Thus BC is the tangent and DB the secant of the arc CH, or of the angle CDB measured by the arc; the name Cosine being given to the space DK cut off between the centre D and the sine; and Versed Sine to the space KC between the sine and the tangent point C.

The term *Sine* denotes a perpendicular to a diameter from the point where the secant crosses the circumference; as HK.

The terms Tangent, Secant, Sine, &c., thus restricted, were of continual use in Trigonometry; and with a widely extended meaning are now constantly employed. The process is instructive, by which extension has been given to Trigonometrical Symbols, and may thus be briefly stated;

1. Sine, cosine, &c., at first denoted lines so named drawn in and about a circle, with reference to an angle at the centre, and measured by its arc, each angle having a different sine, &c., according as the radius of the circle was increased or diminished in length.

2. To avoid these continual diversities, *that* radius was supposed always to be a unit, or rather the unit of measurement for the other lines; and the secant, sines, &c., to be multiples or fractional parts of that unit; thus the sine of 60°, being equal to the radius, was unity, or 1, and the sine of 30° was $\frac{1}{2}$, or ·5. The names sines, cosines, &c., in this way lost their first meaning; they denoted, not lines, but the numerical ratios of those lines to the radius, and were abstract numbers.

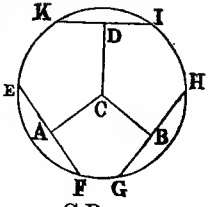
3. Another step was to represent the angle itself by an abstract number. Degrees and minutes had been the measure of the central angle, *that* angle was measured by its arc, and the arc bore a numerical ratio to the unit of measurement, the radius.

4. A fourth step made the process perfect. Hitherto the sum of the angles could not exceed four right angles, but this limit also was to be passed. The idea of a line revolving round a point, and continuing its rotation after a revolution had been completed, originated the method of using angles consisting of more than four right angles.

Thus angles, sines, &c., were all represented by numbers; and though the old names were retained, Trigonometry which at first was a simple application of Geometrical truths, and which still rests on Geometry for its foundation, became a branch of the higher Arithmetic, and has its operations conducted on arithmetical and algebraical principles.

8. Circles are said to touch one another, which meet, but do not cut one another. Thus the circle of which L is the centre, touches EFC in E, and circle G touches it in F.

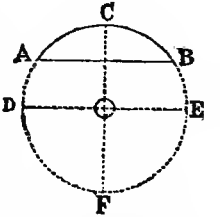
4. Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal; thus E F and G H are equally distant from C, when perp. C A = perp. C B.



5. And the straight line on which the greater perpendicular falls, is said to be farther from the centre; thus I K is farther from C than G H is, because the perp. C D > perp. C B.

As the distance from the vertex of a triangle to its base is measured by a perpendicular, so the distance of a straight line from the centre of a circle is the perpendicular drawn to it from the centre. A Proposition analagons to Props. 7 and 8, Book III., would explain "why the perpendicular from a point on a straight line is called the distance from that line."

6. A *segment* of a circle is the figure contained by a straight line, and the circumference which it cuts off; as the fig. A B C A.

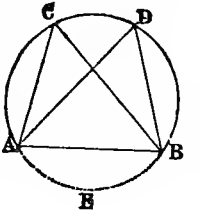


"A figure included by an arc and its chord is a segment."—LARDNER.

The straight line of a segment, as A B, is named the *chord*, and the circumference it cuts off the *arc*, as A C B. Every chord, except a diameter, divides the circle into unequal portions. The division of the circle by a diameter makes *semi-circles*, D C E, and D F E; by two diameters, bisecting at right angles, *quadrants*, as D F, F E. Two arcs A C B, A F B, having the same chord A B, evidently make up the whole circle.

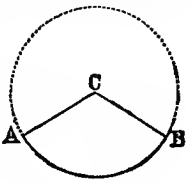
7. The *angle of a segment* is that which is contained by the straight line and the circumference; as angle A B C contained by A B and the arc B C A.

8. An *angle in a segment* is the angle contained by two straight lines drawn from any point of the circumference of the segment, to the extremities of the straight line which is the base of the segment; as ∠ A C B, or ∠ A D B.



9. An *angle* is said to *insist*, or, *stand upon* the circumference intercepted between the straight lines that contain the angle; as ∠ A C B on arc A E B.

10. A *sector* of a circle is the figure contained by two straight lines drawn from the centre, and by the circumference between them; as A C B.



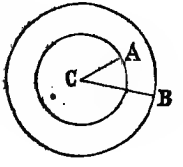
Sectors are equal when they have equal radii and equal angles; for by superposition their boundaries coincide in every respect.

Certain Sectors of a definite size are known by definite names; as the *Quadrant*, a sector of which the arc is 90°; the *Sextant*, of 60°; the *Octant*, of 45°.

11. Similar segments of circles are those in which the angles are equal, or which contain equal angles; as ∠ A C B = ∠ D F E.

The idea of similarity here introduced belongs to all regular figures: all squares, equilateral triangles, hexagons, &c. are similar, though not equal,—the similarity depending on the equality of the angles. Figures become *identical* when their sides as well as their angles are respectively equal each to each.

12. Concentric circles are such as have a common centre; thus, circle A and circle B have the same centre C.



Axiom A.

"If the distance of a point from the centre of a circle be less than the radius of the circle, the point is within the circle; and if the distance of a point from the centre of a circle is greater than the radius, the point is without the circle."—Hose, p. 300. See also Sch. 2, Pr. 1, III.

PROPOSITIONS.

PROP. 1.—PROB.

To find the centre of a given circle.

SOL.—Pst. 1. Let it be granted that a st. line may be drawn from any one point to any other point.

10, I. To bisect a given finite st. line.

11, I. To draw a st. line at rt. $\angle$s to a given st. line from a given point in the same.

Pst. 2. A terminated line may be produced to any length in a st. line.

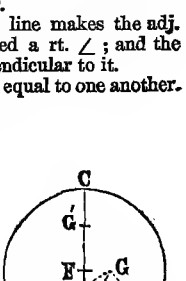
DEM.—Def. 15, I. A $\odot$ is a plane figure contained by one line, which is called the circumference, and is such that all st. lines drawn from a certain point within the figure to the $\odot$ce are equal to one another.

8, I. If two $\triangle$s have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one, shall be equal to the angle contained by the two sides equal to them of the other.

Def. 10, I. When a st. line standing on another st. line makes the adj. $\angle$s equal to each other, each of these $\angle$s is called a rt. $\angle$; and the st. line which stands on the other is called a perpendicular to it.

AX. 1. Things which are equal to the same thing are equal to one another.

E. 1	Dat.	Let A B C be the given $\odot$;
2	Quæs.	to find its centre.
C. 1	Pst. 1. 10, I.	Draw any chord A B, and bis. it in D;
2	11. I.	at D draw D C $\perp$ A B;
3	Pst. 2. 10, I.	prod. C D to E, and bis. C E in F;
4	Sol.	then · F is the cen. of $\odot$ A B C.
5	*Assum.*	If not, take G as the cen;
6	Pst. 1.	and join G A, G D, G B.



D. 1	C.1,Def.15,I	In △s ADG, BDG ∵ DA = DB, GA = GB & GD com.
2	8,I. Def.10,I	∴ ∠ ADG = ∠ BDG, ∴ BDG is a rt. ∠.
3	C 2. Ax. 1.	But ∵ ∠ FDB is a rt. ∠ ∴ ∠FDB = ∠ BDG, *i. e.* the less = the gr.;
4	*ad imposs.*	an impossibility ∴ G is not the cen.
5	*Sim.*	So, no point out of CE is the cen.
6	C. 3,	And ∵ CE is bis. in F,
7	Def. 15, I.	∴ any other point in CE is not the cen.,
8	Conc.	∴ No point but F is cen. of ⊙ ABC.

Q. E. F.

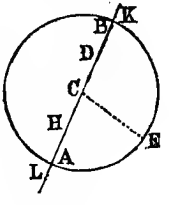
COR.—*If in a* ⊙ *a st. line,* CE, *bisects another,* AB, *at rt.* ∠*s, the cen. of the* ⊙ *is in the line,* CE, *which bisects the other.*

SCH.—1. When the · G′ is taken in the diam. CE, the demonstration holds good only if G′ coincides with F; should this not he the case, it is evident that G′C ≠ G′E ∴ G′ is not the cen.

2. The rigour of the reasoning would have been greatly promoted, if Euclid, previously to the above Problem, had established the following proposition; *Any point,* D, *fig.* 1, F, *fig.* 2. *being assumed within a* ⊙, *a rt. line,* HD *or* HF, *drawn through it and produced indefinitely in both directions, will meet the* ⊙ *in two points, and not in more; and every point of the line between these two points of intersection will be within the* ⊙, *and every point beyond them without it.*—LARDNER'S Euclid. p. 91

I. Let H D through D, also pass through the cen. C. FIG. 1.

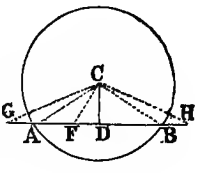
C 1 & 2 D	Pst. 3.	Prod. H D indef. in both directions; & from C make C K, C L each > C B, C A;
3	Ax. A. III.	∴ the · s K & L are *without* the ⊙.
4	3. I.	Also from C make C D, C H each < C B, C A;
5	Ax. A. III.	∴ the · s H & D are *within* the ⊙.
6	3. I.	Lastly, from C make C B, C A each = rad. C E;
7	Def. 15, I.	∴ the · s A & B *on* the ⊙ce.



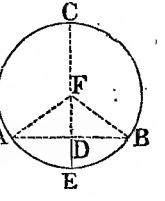
II. Let H F throngh F *not* pass through the cen. C.

C 1 & 2 D	11 I.	From C draw CD ⊥ F H;
	19, I. *a fort*	∵ C D < C F & C F < rad. C A ∴ C D < C A.
3	Cor.3. 47,I.	Find A D² = rad.²−C D², *i. e.* C A² − C D²,
4	3, I. Pst. 1.	from D set D A = D B & join C A, C B.
5	D.3.&47, I.	∵ B D² or D A² + C D² = rad.² ∴ C A = C B = rad.

FIG. 2.



6	Def. 15, I.	$\therefore$ the $\cdot$s A & B each *on* the $\odot$ce.
7	C. & Ax. A	Again $\because$ D F $<$ D A $<$ rad. C A $\therefore$ F is *within* the $\odot$.
8	19, I.	And $\because$ from G or H in AB produced, CG, or CH $>$ rad. C A;
9	Ax. A.	$\therefore$ the $\cdot$s G & H are *without* the $\odot$.
10	Conc.	Hence the st. line H F meets the $\odot$ce only in two points.
11	Recap.	$\therefore$ *Any point being assumed, &c.* Q. E. D.



USE.—1. Practically the centre of a $\odot$ is found, by bisecting any chord, A B with a perp. C E, terminated in the $\odot$ce; and C E being bisd. in F, F is the centre.

2. The First Prop. bk. III. is applicable to all cases in which the centre of any circular object, as of the horizontal section of a tree, may be required. A circular disk of metal, a wheel, a flower-bed, any object possessing the circular form will have its centre found in the same way.

PROP. 2.—THEOR.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

CON. 1, III. Pst. 1 & 2.

DEM. Def. 15, I.

5, I. The angles at the base of an isosc. $\triangle$ are equal to each other; and if the equal sides be produced, the $\angle$s on the other side of the base shall be equal.

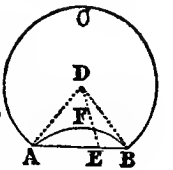
16, I. If one side of a $\triangle$ be produced the ext. $\angle$ is greater than either of the int. opp. angles.

19, I. The gr. $\angle$ of every $\triangle$ is subtended by the gr. side, or has the gr. side opposite to it.

E 1	Hyp.	Let A B C be $\odot$, and A B any two $\cdot$s in the $\odot$ce;
2	Conc.	the st. line from A to B *within* the $\odot$.

SUP.—If not *within*, and it is possible, let it be *without*, as A E B.

C. 1	1. III. Pst. 1.	Find D cen. of ⊙ A B C, & join D B, D A ;
2	*Assum.*	in ⊙ce A F B take any • F;
3	Pst. 1 & 2.	join D F, and prod. it to meet A B in • E.
D. 1	Def. 15, I. 5, I.	Then ∵ D A = D B, ∴ ∠ D A B = ∠ D B A;
2	C. & 16. I.	& ∵ A E in △ D A E is prod. to B; ∴ ext. ∠ D E B > int. & opp. ∠ D A E.
3	D. 1.	But ∠ D A E = ∠ D B E; ∴ ∠ D E B > ∠ D B E;
4	19. I.	and ∴ D B > D E.
5	Def. 15, I. *ad imp.*	Now D B = D F ∴ D F > D E; an impossibility;
6	Conc.	∴ the line from A to B *not without* the ⊙.
7	*Sim.*	So, A B does *not* fall *upon* the ⊙ce;
8	Conc.	∴ A B is *within* the ⊙.
9	Recap.	∵ *If any two points be taken &c.*



Q. E. D.

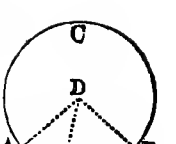
COR. 1.—*A st. line*, A B, *cannot cut the ⊙ce of a ⊙ in more points than two;* for, every st. line joining any two points in the ⊙ce falls within the ⊙, neither co-inciding with any other points in the ⊙ce, nor meeting it except in the two given points.

COR. 2.—*A st. line which touches a circle meets it only in one point.*

COR. 3.—A circle is concave towards its centre.

SCH.—Instead of the *ex absurdo* demonstration of this Proposition, a *direct* method of proof, founded on Axiom A, bk. III. was given by COMMANDINE, who lived between A.D. 1509 and 1575; he applied himself to mathematics at Verona, and in 1572, at Pesaro, published Euclid's Elements in fifteen books, in Latin.

C. 1	*Assum.*	In the given line A B, take any · E;
2	1, III. Pst. I.	find D cen. of the ⊙; and join D A, D E, D B.
D. 1	Def. 15, I. 5, I	∵ in △ D A B, D A = D B; ∴ ∠ D A B = ∠ D B A;
2	C.	and ∵ in △ A E D, A E is prod. to B,
3	10, I.	∴ ext. ∠ D E B > int. and opp. ∠ D A E;
4	D. 1.	but ∠ D A E = ∠ D B E; ∴ ∠ D E B > ∠ D B E.
5	D. 4. 19, I.	But ∠ D E B > ∠ D B E ∴ D B > D E;
6	Remk.	*i. e.* D E the dist. of E from D < D B the rad.
7	Ax. A. III.	∴ the · E is *within* the circle.
8	*Sim.*	So is every · between A and B;
9	Conc.	∴ the line A B, joining A and B falls *within* the ⊙.



Q. E. D.

USE.—On this proposition are grounded those which show, that a circle touches a st. line in only one point; for if the st. line touched two points of the ⊙ce the st. line would be drawn from one point of the ⊙ce to the other, and consequently would fall within the circle, contrary to the very definition of such a line, that it does not cut the circumference. THEODOSIUS of Tripolis, a mathematician who lived some time after the reign of Trajan, compiled a work on the Properties of the Sphere and of the circles described on its surface, an edition of which was published at Oxford in 1675: he used Prop. 2. bk. III. to demonstrate that a Globe resting on a plane surface cannot touch the plane in any but a single point; otherwise the plane would enter the globe.

PROP. 3.—THEOR.

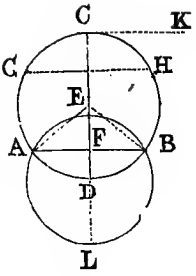
If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and conversely, if it cuts it at right angles, it shall bisect it.

CON. 1, III. Pst. 1.

DEM. Def. 15, I. 8, I. Def. 10, I. 5. I.

26, I. If two △s have two ∠s of the one equal to two ∠s of the other, each to each, and one side equal to one side, viz., either the sides adj. to the equal ∠s in each, or the sides opp. to them, then shall the other sides be equal, each to each, and also the third ∠ of the one to the third ∠ of the other.

E. 1	Hyp. 1.	Let A B C be a circle;
2	„ 2	& C D a straight line through E the cen. bisg. A B a st. line not through E;
3	Conc. 1.	then C D cuts A B ⊥ in F;
4	„ 2.	& if ⊥ in F, A F = F B.
C. 1	1, III. Pst. 1.	Take E cen. of ⊙, & join E A, E B.



I. *The st. line* C D *shall cut* A B *at rt.* ∠*s.*

D. 1	H 2. & C.	∵ A F = F B, and F E com. to △s A F E, B F E;
2	Def. 15, I. 8, I.	& ∵ E A = E B ∴ ∠ A F E = ∠ B F E;
3	Remk. Def. 10, I.	& being adj. ∠s ∴ ∠s A F E & B F E rt. ∠s.
4	Conc.	∴ C D through E, bisg. A B not through E, cuts A B at rt. ∠s.

II. *Also* C D *cutting* A B *at rt.* ∠*s, bis.* A B, *i.e.* A F = F B.

D. 1	Def. 15, I. 5, I.	∵ E A = E B ∴ ∠ E A F = ∠ E B F;
2	Def. 10, I.	and rt. ∠ A F E = rt. ∠ B F E;
3	D. 1, 2 & C.	Thus ∠s E A F, A F E = ∠s E B F, B F E, & E F com.;
4	26, I.	∴ A F = F B, *i.e.* A B is bisd.
5	Rec.	∴ *If a st. line be drawn, &c.* Q.E.D.

Cor. 1.—*A st. line* E F, *bisecting any chord,* A B, *at rt.* ∠*s, passes through the centre* E *of the circle.*

Cor. 2.—*All chords,* A B, G H *par. to the tang.* C K, *at either extremity of the diam.* C D, *are bisected by the diam.*

Cor. 3.—*The st. line which bisects* A B, *the com. chord of two circles,* A C B, A L B, *at rt.* ∠*s, passes through* E *&* D, *the centres of both circles.*

COR. 4.—*When in a circle there are several chords,* AB, GH *par. to each other the* locus *of their points of bisection is in that diam.* CD, *which is at rt.* $\angle$*s to them; and if the line,* CD, *which bisects one chord be a perp. that line bisects all the par. chords at rt.* $\angle$*s.*

SCH.—This Prop. might be more briefly given; "If a diam. cut any other chord at rt. $\angle$s, it shall bisect it; and conversely, if a diam. bisect any other chord, it shall cut it at rt. $\angle$s."

USE AND APP.—The use of this Prop. extends to the various cases, in which arcs, or circles, have to be drawn, and their centres ascertained; as,—

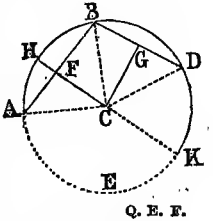
I. *Given a circle,* ABC, *to find its centre,* as in the last figure.

S. 1	Pst. 1. 10, I. 11, I.	Draw any ch. AB; bis. it in F by the perp. EF;
2	Pst. 2.	prod. EF to · s C, D, forming diam. CD; ·
3	10. I.	bis. CD in E; and E is the centre.
D. 1	C. 1. Cor. 3, III.	∵ CD bis. AB ⊥ ∴ DC through the centre;
2	C. 2., Def. 15, I.	∵ CD is a diam. ∴ its mid. · E cen. of ⊙

Q. E. F.

II. *Given an arc* ABD, *to find the cen. of the* ⊙ *of which it is an arc.*

S. 1	*Assum.*	Take any · B, between the extremities of the arc;
2	Pst. 1.	from B draw chords to A & D;
3	10, I. & 11, I.	bis. AB & BD by perps. from F & G, meeting in C;
4	Sol.	then, the intersecting · C is the centre.
D. 1	Cor. 3, III.	∵ the cen. lies in FC & also in GC;
2	Conc.	∴ it must be in the intersecting · C.

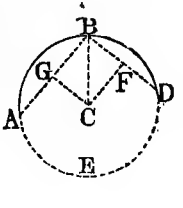


Q. E. F.

N.B. With rad. CA, or CD, the circle, ABDE, may be completed.

III. *Through three points,* A, B, D, *not in a st. line, to draw a circle.*

S. 1	Pst. 1. 10, I.	Join AB, BD, and bis. them in F, G;
2	11. I.	at F & G raise perps. intersecting in G;
3	Cor. 3, III.	C is the centre of ⊙ through A, B, D:
4	Pst. 3.	and with rad. AC, or BC the circle may be drawn.



N.B. The Dem. is given in Prop. 9, III.

IV. In Trigonometry we show 1st, by this Prop., in the last figure but one, that the half chord F B or F A of an arc A H B is perp. to the semidiam. C H, and consequently is the *Sine* of the half arc H B, or H A; 2nd, that the sides of a triangle (4, VI.) have the same ratio as the sines of their opp. angles.

V. In the last figure but one, That part of the perp. to the chord which passes through the centre and is intercepted between the centre and the chord, namely, C F, is called the *versed sine* (see note to Def. 2, p. 4); and the radius, semichord, and versed sine form respectively the hypotenuse, base and perp. of a rt. angled triangle, and by 47, I., when any two are measured, or given, the third may be found;—for,

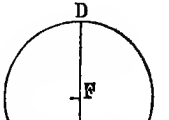
$rad. = \sqrt{semich.^2 + vers.\ sine^2}$; $semich. = \sqrt{rad.^2 - vers.\ sine^2}$; and $vers.\ sine = \sqrt{rad.^2 - semich.^2}$

Prop. 4.—Theor.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Con.—10, I. 1, III. Pst. 1.

Dem.—Def. 15, I. 3, III.
Ax. 11. All rt. ∠s are equal.
Ax. 9. The whole is greater than its part.

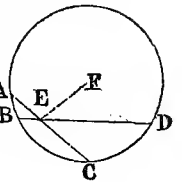


E.1	Hyp.1.	Let A B C be a circle;
2	„ 2.	& A C, B D two st. lines cutting in E, but not both through cen. F;
3	Conc.	then A C, B D do not bis. one another; *i. e.* E not mid. · both of A C & B D.

Sup. I. *Let* B D *pass through the centre and* A C *not.*

C.	10, I. Def. 15, I.	Bis. B D in F, then F cen. of ⊙.
D.	Def. 15, I.	∵ F B = F D ∴ E not mid. · of B D; *i. e.* B D not bisected in E by A C.

SUP. II. *Let neither* A C *nor* B D *pass through the centre.*



E.1	Hyp.	If possible let both A E = E C and B E = E D.
C.	1, III. Pst. 1.	Find F the cen. and join F E.
D.1	H.	∵ F E through cen. F, bis. A C not through F,
2	3. III.	∴ F E cuts A C ⊥, and F E A is a rt. ∠.
3	H.	Again, ∵ F E through cen. F bis. B D not through F,
4	3, III.	∴ F E cuts B D ⊥, and F E B is a rt. ∠.
5	Ax. 11.	∴ F E A a rt. ∠ = F E B a rt. ∠,
6	Remk. Ax. 9.	*i. e.* a part = the whole;—an impossibility;
7	Conc.	∴ A C and B D do not bisect each other.
8	Rec.	∵ *If in a circle two st. lines, &c.* Q. E. D.

COR.—*No parallelogram except a rectangle can be inscribed in a circle.*

D.1	C. & 34. I.	∵ the diags. are diams. ∴ the diags. bis. each other in their centres.
2	Def. 15, I.	∴ the diags. are equal.
3	8, I. 34, I.	And ∵ the suppl. ∠s are equal;
4	Conc.	∴ each ∠ is a rt. ∠, and all the ∠s are rt. ∠s. *i. e.* the inscribed fig. must be a rectangle.

USE.—The fourth Prop. has been employed to determine the eccentricity of the Sun's apparent path, or of the Earth's orbit described in a year.

In an eccentric wheel the distance of the fixed point, or centre of rotation, E, round which the revolution is performed, from F, the centre of the wheel, will be found in the same way.

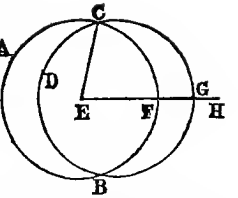
PROP. 5.—THEOR.

If two circles cut one another, they shall not have the same centre.

CON.—Pst. 1.

DEM.—Def. 15, I. AXS. 1 & 9.

E.1	Hyp.	Let the two ⊙s ABC, CDG cut in · s B & C;
2	Conc	then, they have not the same centre.

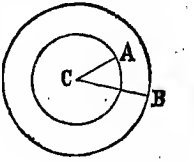


Sup.—*If possible let* E *be the cen. of both circles.*

C.1	Pst. 1.	Join EC, & from E draw a st. line EFGH, meeting the ⊙s in · s F & G.
D.1	Sup. Def. 15, I.	∵ E cen. of ⊙ ABC ∴ EC = EF.
2	,, ,,	∵ E cen. of ⊙ CDG ∴ EC = EG.
3	D. & Ax. 1.	But EC.= EF ∴ EF = EG,
4	Remk. Ax. 9.	*i.e.* the less = gr.; an impossibility;
5	Conc.	∴ E not the com. cen. of ⊙s ABC, CDG.
6	Rec.	∴ *If circles cut one another, &c.*

Q. E. D.

Sch.—"This proposition may be better announced thus: 'Concentric circles cannot meet, and that which has the lesser radius will be included within the other.'"—Lardner, p. 94.



For ∵ CA < CB ∴ ⊙ A *within* ⊙ B.

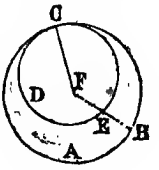
Prop. 6.—Theor.

If one circle touch another internally, they shall not have the same centre.

Con.—Pst. 1.

Dem.—Def. 15, I. Axs. 1 & 9.

E.1	Hyp.	Let ⊙ CDE touch ⊙ ABC internally in C;
2	Conc.	they have not the same cen.



SUP.—*If they have the same centre let it be* · F.

C.	Pst. 1.	Join F G, and from F draw a st. line F E B, meeting the ⊙s in · s E & B.
D.1	Sup. Def. 15, I.	∵ F cen. of ⊙ A B C ∴ F C = F B;
2	,, ,,	∵ F cen. of ⊙ C D E ∴ F C = F E;
3	D. 1. & Ax. 1.	But F C = F B ∴ F E = F B,
4	Remk. Ax. 9.	*i.e.* the less = gr.; which is impossible;
5	Conc.	∴ F not the com'.cen. of ⊙s A B C, C D E.
6	Rec.	∵ *If one circle touch another internally, &c.* Q. E. D.

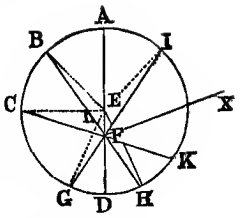
SCH.—Props. 5 & 6 may be combined into one; "circles with a common centre do not touch either externally or internally:" for the circle with the less radius will have every point within the circumference of the other, and consequently does not meet the other in any point whatever.

PROP. 7.—THEOR.

If any point which is not the centre be taken in the diameter of a circle, then, 1st, *of all the straight lines which can be drawn from it to the circumference,* the greatest *is that in which the centre is, and the other part of that diameter is* the least; *and,* 2*nd, of any other st. lines, that which is nearer to the line which passes through the centre is always* greater *than the one more remote; also* 3*rd, those lines which make equal angles with the diameter are* equal; *and,* 4*th, from the same point there can be drawn* only two equal *st. lines, one upon each side of the diameter.*

CON.—Pst. 1. 23, I. At a given · in a given line, to make a rectl. ∠ equal to a given rectl. ∠.
3, I. From the gr. of two given lines to cut off a part equal to the less.

DEM.—20, I. Any two sides of a △ are together gr. than the third side. Def. 15, I. Ax. 9.
24, I. If two △s have two sides of the one equal to the two sides of the other, each to each, but the ∠ contained by the two sides of the one gr. than the ∠ contained by the two sides of the other, the base of that which has the gr. ∠ shall be gr. than the base of the other.
Ax. 5. If equals be taken from unequals the remainders are unequal.
4. I. If two △s have each two sides and the included ∠ of the one equal to two sides and the included ∠ of the other, the △s are equal in every respect.—Ax. 1.



E.1	Hyp. 1.	Let ABCD be a ⊙ and AD its diam.
2	„ 2.	and in diam. AD any · F not the cen.
3	„ 3.	and let E be the centre; *First*, then of all st. lines FB, FC, FD &c. from F to ⊙ce.
4	Conc. 1.	FA through cen. E the greatest,
5	„ 2.	FD the other pt. of diam. AD the least; *Second*, of any other lines,
6	„ 3.	FB nearer to FA > FC more remote and FC > FG; *Third*, lines FB, FI making equal ∠s with diam. AD.
7	„ 4.	the line FB = the line FI. and *Fourth*, of lines from the same · F to the ⊙ce,
8	„ 5.	only two eq. lines, FG, FH, one on each side of diam. AD.
C.1	Pst. 1.	Join BE, CE, GE.

I.—*A line* FA *through cen.* E > *any other line as* BF.

D.1	20. I.	∵ two sides of a △ > third side ∴ BE + EF > BF,
2	Def.15, I.	but AE = BE ∴ AE + EF *i.e.* FA > FB.

II.—*The other part of the diam.*, FD < *any other line* FG.

D.1	Def. 15, I. C.	Again ∵ BE = CE & FE com. to △s BEF, CEF,
2	Ax. 9. 24, I.	but ∠BEF > ∠CEF ∴ BF > CF;
3	*Sim.*	So CF > GF and GF > DF.
4	20. I, Def. 15, I	Also ∵ GF + FE > GE, and EG = ED;
5		∴ GF + FE > ED.
6	*Sub.* Ax. 5.	take away com. pt. FE ∴ rem. GF > rem. FD.
7	Conc.	∴ Of all st. lines from F a · not the cen. to the ⊙ce, FA through cen. E is the greatest, and FD the least; and

III.—*Line* BF *nearer diam.* > CF *more remote &* CF > GF.

IV.—*The lines* FB, FI, *making with* AD ∠ BFA = ∠IFA, *are equal.*

Suppose one to be gr. *i. e.* FB > FI.

C.	3, I. Pst. 1.	Make FL = FI and join EL and EI.
D.1	C. & H.	In △s FLE, FIE ∵ FE com., FL = FI, & ∠ BFA = ∠ IFA,
2	4, I.	∴ EL = EI;
3	Def.15,I.Ax.I.	but EI = EC ∴ EC = EL.
	Ax. 9.	*i. e.* a part = the whole, which is absurd.
4	Conc.	∴ Neither FB nor FI the gr.; *i. e.* FB = FI.

V.—*Also only two equal lines,* FG, FH, *from* · F *to the* ⊙ce.

C.	23. I. Pst. 1.	At E in EF make ∠ FEH = ∠ FEG, and join FH.
D.1	Def. 15, I. C.	∵ GE = HE, EF com. and ∠ GEF = ∠ HEF,
2	4, I.	∴ FG = FH,
3	Remk.	And from F to ⊙ce. no other line = FG,
4	*Sup.*	If possible let FK = FG;
5	C. Ax. 1.	∵ FK = FG = FH ∴ FK = FH,
6	Remk.	*i. e.* a line nearer the diam. = a line more remote,
7	Case III.	which has been proved to be impossible.
8	Rec.	Therefore, *If any point which is not the centre, &c.* Q. E. D.

SCH.—If from a point, within a circle, not the centre, as F, a st. line of indef. length, as FX, revolve so as in each part of its revolution to be terminated or cut off by the circumference, as in A, B, C, G, D; its *maximum* length FA, will be attained when it coincides with that part of the diam. AD, in which E the centre is; and its *minimum* FD, when it coincides with the other part of the diam.; and the nearer FX, is to the *maximum* the greater it is, as FB, and the nearer to the *minimum*, the less, as FG.

USE.—THEODOSIUS, mentioned p. 11, by aid of this proposition proves that, if from the pole of the world, which is not the pole of the horizon, (for the zenith is *its* pole) several arcs of great circles be drawn to the circumference of the horizon, the *greatest* arc shall be that part of the meridian which passes through the zenith.

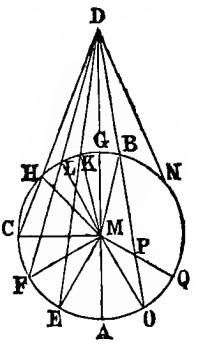
By this proposition we may also prove that the Earth being in Aphelion is at the greatest distance from the Sun, in Perihelion, at the least; and so for all the other planets.

Prop. 8.—Theor.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; 1*st, those which make equal angles with the line passing through the centre are* equal; 2*nd, of those which fall upon the concave circumference,* the greatest *is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre is always* greater *than one more remote; but,* 3*rd, of those which fall upon the convex circumference,* the least *is that between the given point without the circle and the diameter, and of the rest, that which is nearer to the least is always* less *than one more remote; and,* 4*th, only* two equal lines *can be drawn from the same point to the circumference, one upon each side of the line which passes through the centre.*

Con. 1, III., Pst. 1. 3, I. 23, I.

Dem. 4, I. Def. 15, I. Ax. 1 & 2. 20, I. Ax. 9. 24, I. Ax. 5.
21. I. If from the ends of the side of a △ there be drawn two st. lines to a · within the △, these lines shall be less than the other two sides of the △, but shall contain a gr. ∠.



E.1 Hyp. 1. Let ABC be a ⊙ and D any · without it;
2 „ 2. from · D let st. lines DA, DE, DF, DC, and DO be drawn to the ⊙ce;
3 „ 3. Of these let DA pass through cen. M,
4 „ 4. and DE, DO make with DA, ∠ ADE = ∠ ADO;
5 Conc. 1. the line DE = line DO;

1st. *Of lines incident on* AEFC *the concave* ⊙*ce,*

6 „ 2. the greatest is DA passing through cen. M;
7 „ 3. and any line nearer DA is > a line more remote, *i. e.* DE > DF and DF > DC:

2nd. *But of lines incident on* HLKG *the convex* ⊙*ce,*

E.8	Conc. 4.	the least is DG, between · D and diam. GA;
9	,, 5.	and any line nearer DG < a line more remote, *i. e.* DK < DL and DL < DH.
10	,, 6.	Also only two equal lines, DB and DK, from · D to ⊙ce, can be drawn, one upon each side of DA.
C.1	1, III.	Take M the cen. of ⊙ ABC
2	Pst. 1.	and join MO, ME, MF, MC, MH, ML, and MG.

I. *The st. lines* DE *and* DO *are equal, making equal* ∠s *with* DA.

	Sup.	If possible let DO > DE.
C.	3, I. Pst. 1.	Make DP = DE, and join ME, MP.
D.1	C. and H.	In △s MPD, MED ∵ MD com. DP = DE and ∠ ADE = ∠ ADO or ADP;
2	4, I.	∴ MP = ME.
3	Def.15,I. Ax.1	But MQ = ME ∴ MP = MQ,
4	Ax. 9.	*i. e.* a part = the whole, which is absurd ;
5	Conc.	∴ Neither DO nor DE gr., *i. e.* DO = DE.

II. *Of lines incident on* AEFC *the concave* ⊙*ce,* DA *through the cen. is greatest, and* DE, *nearer* DA *is* > DF *more remote, and* DF > DC *:*

D.1	Def. 15,I.*Add.*	∵ AM = ME, to each add MD;
2	Ax. 2.	∴ AM + MD, *i. e.* AD = EM + MD;
3	20, I.	but EM + MD > ED ∴ AD > ED.
4	Def.15,I. Ax.9	Again ∵ ME = MF, MD com. & ∠ EMD > ∠ FMD,
5	24, I.	∴ ED > FD.
6	*Sim.*	In like manner FD > CD.
7	Conc.	∴ DA through cen. M is the greatest, DE > DF & DF > DC.

III. *Of lines incident on* HLKG *the convex* ⊙*ce,* DG, *between* D *and the diam.* AG, *is the least,* DK < DL & DL < DH.

D.1	20, I. Def. 15, I.	∵ MK + KD > MD, & MK = MG;
2	Ax. 5.	∴ rem. KD > rem. GD; *i. e.* GD < KD,
3	C.	& ∵ MLD is a △, and from · s M, D, extrs. of one side MD, MK & DK are drawn to · K within the △;

D.4	21, I.	$\therefore$ MK + KD < ML + LD;
5	Def. 15, I. Ax. 5	but MK = ML $\therefore$ rem. KD < rem. LD.
6	*Sim.*	In like manner DL < DH:
7	Conc.	$\therefore$ DG the least, DK < DL & DL < DH.

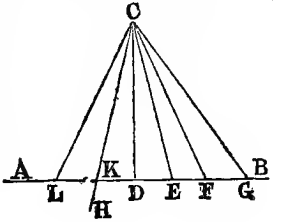
IV. *Also only two equal st. lines can be drawn from the* · D *to the* ⊙*ce, one on each side of* AD *the line which passes through cen.* M.

C.	23, I. Pst. 1.	At M in MD make $\angle$ DMB = $\angle$ DMK, & join DB
D.1	C.	$\because$ in $\triangle$s KMD, BMD, MK = MB, MD com. & $\angle$ KMD = $\angle$ BMD;
2	4, I.	$\therefore$ DK = DB.
3	Remk.	But besides DB no other line from · D to the ⊙ce = DK;
4	Sup.	If there can, let it be DN;
5	D. 4 & 3, Ax. 1.	$\because$ DK = DN & also = DB $\therefore$ DB = DN
6	Remk.	*i. e.* a line nearer the least = one more remote;
7	D. II. 7.	but this has been proved to be impossible;
8	Conc.	$\therefore$ No line but DB = DK.
9	Rec.	Therefore, *If any point be taken*, &c. Q. E. D.

Sch. 1.—The *concave* and *convex* parts of a circumference are determined by the tangents from a point external to the circumference. All the parts of the circumference which are farther from the external point than the tangent points of the circle, are *concave* with respect to that point; and those nearer to the external point than the tangent points are *convex*.

Sch. 2.—A proposition, analagous to Prop. 7 & 8, explains why in Def. 4, III. and in 14 & 15, III. the perp. from a point out of a st. line to the line is called the *distance* of the point from or to the line.—Hose, p. 302.

If any · C *be taken out of a st. line* AB; *then of all st. lines*, CD, CE, CF, CG, *from that* · *to the st. line*, the least, CD, *shall be that which is perp. to the st. line; and of the rest*, CE, CF, CG, *that which is nearer, as* CE, *to the perp., shall always be* less *than that*, CF *or* CG, *which is more remote; and from this* · C *there can be drawn to the st. line* AB, *only one st. line* = *a given st. line*, CE, *drawn from the same* · C, *to the st. line, which shall be on the opp. side of the perp.*



I. *The perp.* CD *is the least line from* C *to* AB ; CE > CD, CF > CE *and* CG > CF.

D.1	H	∵ ∠ CDE is a rt. ∠, ∴ ∠ CED < a rt. ∠ ;
2	H. & Sup.	If not, in △ CDE, ∠s CDE + CED ⩽ two rt. ∠s ;
3	17, I.	which is impossible.
4	D. & 19, I.	∴ ∠ CDE > ∠ CED ∴ CE > CD.
5	D. 4.	Also ∵ ∠ CED < a rt. ∠ ∴ ∠ CEF > a rt. ∠.
6	*Sim.*	and in like manner ∠ CFD < a rt. ∠.
7	19, I.	∴ ∠ CEF > ∠ CFE, and CF > CE.
8	*Sim.*	So also CG > CF ;
9	Conc.	Hence the perp. CD is the least ; CE > CD, CF > CE, & CG > CF.

II. *From* · C *only one st. line,* CK = CE *a given st. line from* C *to* AB *on the opp. side of the perp.* CD.

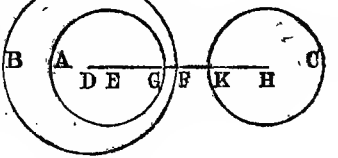
C.1	23, I.	At C in D C make ∠ DCH = ∠ DCE ;
2	Pst. 2.	and let CH cut AD in K ;
3	Conc.	Then CK = CE.
D.1	Ax.11, C.1	∵ rt. ∠ CDK = rt. ∠ CDE, ∠ DCK = ∠ DCE, and CD com.
2	26, I.	∴ CK = CE.
3	Remk.	Also no line from C to AB, but CK = CE ;
4	Sup.	If there can be, let it be CL ;
5	H.D.2Ax.1	Then ∵ CL = CE & CK = CE ∴ CL = CK ; *i. e.* CL, more remote from CD than CK, equals CK ;
6	Case I.	which is impossible.
7	Conc.	Hence from C only one st. line to AB, namely CK = CE on the opp. side of perp. CD.

Sch. 3.—"Thys Proposition," says the Translator of 1570, "is called commonly in old bookes amongest the barbarous *Cauda Pauonis*, that is, the Peacockes taile."—Fol. 88.

Use and App.—I. The Proposition 8 Bk. III. is employed to show, that if a tangent and secant be drawn to the same point, the tang. < sec., but > external part of the secant : Thus in fig. to Prop. 8, let DH represent a tang. and MD a sec. to arc HG, or ∠ HMG, then DH < DM but > DG.

II. By aid of Props. 7 & 8, and Ax. A, bk. III., we may demonstrate ;

1st. When one circle A is contained within another, B, without touching it, the distance between the centres, DE < (DF ∼ EG), the difference of the radii : and *conversely*, when the distance between the centres, DE < (DF ∼ EG), the difference of the radii, the lesser circle will be within the greater without meeting it.



2nd. When two circles, B, C, lie each without the other and do not meet the distance between the centres, DH > (DF + HK,) the sum of the radii, and *conversely*, when the distance between the centres, DH > (DF + HK,) the sum of the radii, the circles lie each without the other and do not meet.

Prop. 9.—Theor.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Con. Pst. 1 & 2.

Dem. Def. 17, I. A diam. of a ⊙ is any st. line through the cen., and terminated both ways by the ⊙ce.—7, III.

E.1	Hyp.1	In ⊙ A B C let · D be taken,
2	,, 2	& from D to ⊙ce more than two equal lines, as D A = D B = D C;
3	Conc.	then · D shall be the cen. of the ⊙.

Sup. *For if not, let* · E *be the centre.*

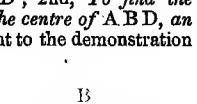
C.	Pst. 1 & 2.	Join D E, & prod. it to the ⊙ce in F & G.
D.1	Def. 17, I.	∵ F G is the diam., & ∵ in F G is taken · D not the cen.
2	7, III.	∴ D G greatest line from D to ⊙ce, D C > D B, and D B > D A;
3	H. *ad imp.*	but they are also equal;—an impossibility;
4	Conc.	∴ · E not the cen. of ⊙ A B C.
5	*Sim.*	So no point but D the cen.
6	Conc.	∴ · D is the cen.
7	Rec.	∵ *If a point be taken, &c.* Q. E. D.

Cor.—1. *From any other point than the centre only two equal st. lines can be drawn to the ⊙ce, whether the point* (7, III.) *be within, or* (8, III.) *without the circle.*

Cor.—2. *From three points given not in the same st. line, the circumference of the circle may be found.*

SCH.—This Prop. gives the *criterion* for determining the centre of a circle; it is, that from a point, supposed to be the centre, to the circumference more than two points in the circumference shall be equally distant from the centre.

USE and APP.—By this Proposition the Problems may be demonstrated; 1st, *To draw a circle through three given points*, A, B, D; 2nd, *To find the centre of a given circle* A B D E; and 3rd. *To determine the centre of* A B D, *an arc of a circle.* The demonstration of the first, is equivalent to the demonstration of the other two.



C. 1	Pst. 1. 10, I.	Join · s A, B, D and his. AB and BD;
2	11, I.	at F and G raise perps. FC, GC meeting in C;
3	Psts. 1. & 3.	join CA, CB, CD and with either as rad. from C desc. a ⊙
4	Sol.	the ⊙ passes through A, B and D.
D1.	C. 1 & 2	∵ AF = FB, FC com. and ∠ AFC = ∠ BFC,
2	4, I.	∴ AC = CB
3	*Sim*	In like manner CD = CB
4	Ax. 1. & 9. III.	∴ CA = CB = CD & ∴ C cen. of ⊙ through · s A, BD,

Q. E. F.

PROP. 10.—THEOR.

One circumference of a circle cannot cut another in more than two points.

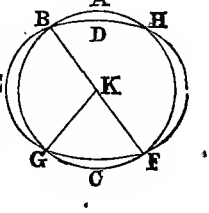
CON. 1, III. Pst. 1.

DEM. Def. 15, I. 9, III.

5. III. If two circles cut one another they shall not have the same centre.

SUP.—*If possible let* ⊙*ce* F A B *cut* ⊙*ce* D E F *in more than two* · *s, as in* B, G, F.

C.	1, III. Pst. 1	Take K cen. of A B C, and join KB, KG, KF
D.1	C. Def. 15, I.	∵ K cen. of ⊙ A B C ∴ KB = KG = KF:
2	C.	and ∵ within ⊙ D E F there fall from K to ⊙ more than two equal st. lines.



D.3	9, III.	∴ • K is cen. of ⊙ DEF;
4	C.	but • K also cen. of ⊙ ABC;
5	Conc.	∴ the same • is cen. of two ⊙s cutting each other;
6	5, III.	which is impossible;
7	Rec.	therefore, *One circumference of a circle, &c.*

Q. E. D.

Sch.—"Two circles cannot have more than two points in common;" if they coincide in three points they will coincide in every point; or, "only one circle can be drawn through three given points."

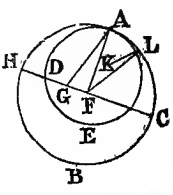
Prop. 11.—Theor.

If one circle touch another internally in any point, the straight line which joins their centres being produced shall pass through the point of contact.

Con. Psts. 1, 2.

Dem. 20, I. Def. 15, I. Ax. 5.

E.1	Hyp. 1.	Let ⊙ ADE touch ⊙ ABC internally in A,
2	,, 2.	and let F be cen. of ⊙ ABC, G cen. of ⊙ ADE;
3	Conc.	then the st. line joining F, G, being produced passes through A, the • of contact.



Sup.—*If* FG *produced do not pass through* A, *let it, if possible, fall as* FGDH.

C.	Pst. 1.	Join AF and AG.
D.1	20, I. Def. 15, I.	∵ FG + GA > FA, but FA = FH,
2		∴ FG + GA > FH,
3	*Sub.* Ax. 5.	take away com. pt. FG ∴ rem. AG > rem. GH;

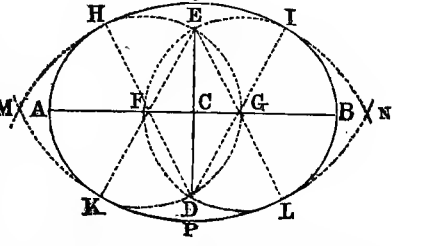
D.4	Def. 15, I.	but AG = GD ∴ GD > GH;
5	Remk. *ad imp.*	*i. e.* the less > the greater;—an impossibility.
6	Conc.	∴ FG joining F and G, being produced cannot fall except upon · A;
7	Remk.	*i.e.* FG prod. must pass through A, the · of contact.
8	Rec.	Therefore, *If one circle touch, &c.*

Q. E. D.

Sch.—When the distance, FK, between F and K, the centres of the two circles ADE and ABC is equal to the difference of the radii, AF and AK, the circles touch *internally*. For

C.	Pst. 1.	Take L, a · in ⊙ ADE, and join KL, FL;
D.1	20,I.Def.15,I.	then ∵ in △ FKL, FL < FK + KL, but KA = KL;
2		∴ FL < FK + KA *i. e.* < FA ∴ L within ⊙ ABC.
3	*Sim.*	So all other · s in ⊙ ADE except · A lie within ⊙ ABC.
4	Conc.	*When the distance between the centres, &c.*

Use and App.—I. By aid of this proposition *an oval may be described on any given major axis, as* AB.

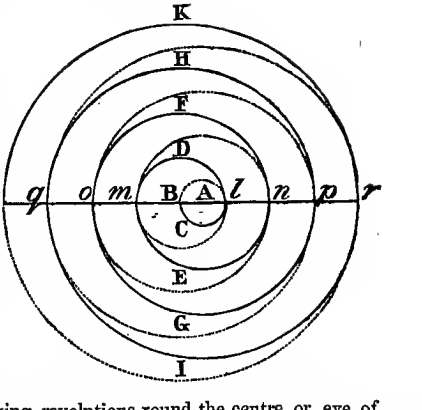


C.1	10,I. 11, 1.	Bis. AB in C by a perp. CD, CE
2	Remk.	the minor axis is OP, *i. e.* DE produced.
3	Pst. 3	If on C with rad. CA or CB a line revolve a circle will be traced;
4	Use2,34,I.	but div. AB into three eq. pts. AF = FG = GB;
5	Pst. 3.	from F and G with FA and GB desc. ⊙s AEGD and BEFD;
6	Pst. 1, 2.	from · s of intsrs. D, E, draw DE, DG, EF, EG, prod. to H, I, K, L;
7	Pst. 3.	next from D with rad. DH or DI desc. arc MHON,
8	Pst. 3.	and from E with rad. EK or EL desc. arc MKPN;
9	11, III.	the arcs AH, HI, IB, BL, LK, and KA touch in · s H,I,K,L,
10	Sol.	and the curve AHCIBLPKA will form an oval.

If the major axis be divided into four or more equal parts, by a similar method ovals more elongated, or with the minor axis in less proportion to the major, may be described.

N.B.—The oval thus described is only an approach to the true ellipse, the method being *practically* useful, not theoretically correct.

II. It is on the same principle that *a Spiral is described*, by successive semicircles taken alternately from two common centres A and B; for the line AB which joins them being produced, passes through the points of contact of the successive semicircles, *l*, *m*, *n*, *o*, *p*, *q*, *r*, *s*.



From A, the eye of the spiral, with rad. AB describe the semicircle C; from B, with B*l*, the semic. D; from A, with A*m*, semic. E; from B, with B*n*, semic. F; from A, with A*o*, G; from B, with B*p*, H; from A, with A*q*, I; and from B, with B*r*, K. The spiral may be continued to any extent in the same way.

A *Spiral* is a curve line making revolutions round the centre or eye of the curve, which do not return into themselves as the revolutions of a circle do. A *Plane Spiral* is generated by a continually increasing radius, and according to the law or rate of increase spirals differ in their curves. Our older English writers considered the circle and the ellipse to be spirals; but they are excluded by the above definition, inasmuch as their revolutions return into themselves, or to the very point from which they started.

Besides the above, the principal plane Spirals are, the Spiral of Archimedes or Conon,—the Hyperbolic or Reciprocal Spiral,—the Lituus, and the Logarithmic Spiral; but they can only be noticed here by way of definition. 1st, When from a given point any number of lines are drawn forming equal angles at that point, and the length of each line increases in succession by an equal quantity, the curve which passes through these points is named *the Spiral of Archimedes*. 2nd, The *Hyperbolic* or *reciprocal Spiral* is a curve passing through the extremities of any number of arcs of circles of equal length measured from a given st. line. 3rd, The *Lituus*, so named from the crooked staff of the Roman augurs, is a Spiral to be thus described;—"Let a variable circular sector always have its centre at one fixed point, and one of its terminal radii in a given direction. Let the area of the sector always remain the same, then the extremity of the other terminal radius as it revolves describes the Litnus." 4th, The *Logarithmic Spiral*, in which the radii make equal angles, and the spiral cuts them all at an equal angle, the length of the successive radii increasing in geometrical progression.

It may be observed that curves are infinite in variety, though only about thirty have received specific names. The Parabola, Hyperbola, Cycloid, Watt's Parallel motion curve, &c., are among them; but it would be out of place to explain them here.

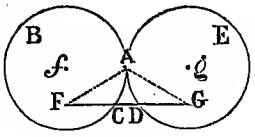
PROP. 12.—THEOR.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point of contact.

CON. Pst. 1.

DEM. Ax. 2. 9. Def. 15, I. 20, I.

E.1	Hyp. 1.	Let the two ⊙s ABC, ADE touch extern. in A;
2	„ 2.	and let F be the cen. of ABC, G of ADE;
3	Conc.	then FG shall pass through A, the · of contact.

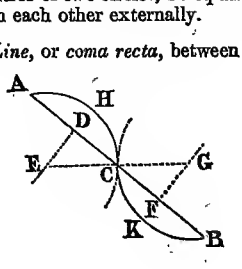


SUP. If not, let it pass through C and D.

C.	Pst. 1.	Join FA and GA.
D.1	H.2. Def.15,I.	∵ F is cen. of ⊙ ABC ∴ FA = FC;
2	H.2. Def.15,I.	and ∵ G is cen. of ⊙ ADE ∴ GA = GD;
3	Ax. 2. 9.	∴ FA + GA = FC + GD, and ∴ FG > FC + DG.
4	20, I. *ad imp.*	But FG < FA + AG; which is impossible.
5	Conc.	∴ the st. line FG cannot pass except through A the · of contact; *i. e.* FG must pass through A.
6	Rec.	Therefore, *If two circles touch, &c.* Q. E. D.

SCH.—Should fg, the distance between the centres of two circles, be equal to $fA + Ag$, the sum of the radii, the circles touch each other externally.

USE AND APP.—The drawing of a *Serpentine Line*, or *coma recta*, between two given points, as A, B, depends on this 12th Prop. For join A B, and bisect it in C; again bis. A C by perp. D E and B C by perp. F G; make D E = F G; and from E with rad. E C desc. arc A H C, and from G with rad. G C, the arc C K B; the arcs touch in the · C, and the two arcs form the serpentine A H C K B; which might be continued to any extent by following up the same process.



Prop. 13.—Theor.

One circle cannot touch another in more points than one, whether it touches it on the inside or the outside.

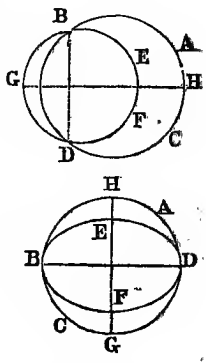
Con. Pst. 1, 11, I.

Dem. 2, III. 11, III. Cor. 1, III.

E.1	Hyp.	If possible let ⊙ E B F touch ⊙ A B C in more · s than one.

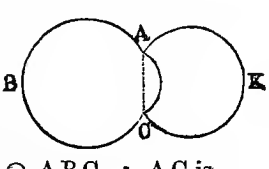
I.—*On the inside in the* · s B, D.

C.	Pst.1.11,I.	Join B D, and draw G H bisecting B D ⊥.
D.1	H. 2, III.	∵ · s B, D are in the ⊙ce of each ⊙ ∴ B D falls within them;
2	Cor.1,III.	∴ their centres are in G H which bis. B D ⊥;
3	11, III.	∴ G H passes through the · of contact;
4	H.	but · s B, D, are without the st. line G H,
5	Conc.	∴ G H does not pass though the · of contact, which is absurd.
6	Rec.	∴ one ⊙ cannot touch another internally in more · s than one.



II.—*If possible, let* ⊙ A C K *touch* ⊙ A B C *externally in* A *and* C.

C.	Pst. 1.	Join A C.
D.1	Hyp.	∵ A & C are in the ⊙ce of the ⊙ A C K,
2	2, III.	∴ A C which joins them falls within ⊙ A C K;
3	H.	But ⊙ A C K is without ⊙ A B C, ∴ A C is without ⊙ A B C;

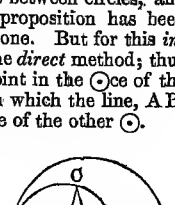


D.4	H.	but ∵ A & C are · s in the ⊙ce of the ⊙ A B C,
5	2, III.	∴ A C must also be within the same ⊙; an absurdity;
6	Conc.	∴ one ⊙ cannot touch another on the *outside* in more · s than one.
7	Case I.	And ⊙s cannot touch on the *inside* in more · s than one;
8	Rec.	Therefore *one circle cannot touch, &c.* Q. E. D.

Sch.—It has been by assuming two points of contact between circles, and by showing the assumption to be impossible, that the proposition has been proved, that two circles cannot touch in more parts than one. But for this *indirect* method of demonstration, it is better to substitute the *direct* method; thus

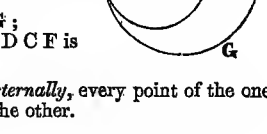
1st. If the circles touch *internally*, as at D, each point in the ⊙ce of the less ⊙, except the common point of contact, D, through which the line, A B, joining their centres, A, B, passes, must be within the ⊙ce of the other ⊙.

C.	Pst. 1.	For take C, a · on the ⊙ce of D C F, & join centres & C.
D.1	7, III.	Here A D > A C.
2	D. 1.	And ∵ A C < A D, the rad. of ⊙ D E G;
3	Ax. A, III.	∴ the · C is within ⊙ D E G;
4	*Sim.*	So every ·, except D, of ⊙ D C F is within the ⊙ D E G.

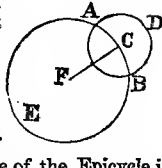


2nd. If the ⊙s D C F and D E G touch *externally*, every point of the one, except the common point of contact is without the other.

C.	Pst. 1.	Join A B, the centres, and draw A C E.
D.1	H. & 8, III.	∵ the ⊙s touch externally, ∴ A E > A C or A D;
2	Ax. A, III.	∴ the · E lies out of the ⊙ D C F;
3	*Sim.*	thus every · of ⊙ D E G, except the · D, lies out of the ⊙ D C F.



Use and App.—The four Propositions, 10, 11, 12 and 13, are employed by astronomers to explain the motion of the Planets in Epicycles. An *Epicycle* is a circle, A B D, the centre of which, C, is carried round upon another circle A B E.



Ptolemy of Alexandria, the celebrated astronomer, A.D. 139, in explaining the motions of the planets on what is called the Ptolemaic System, employs the theory of Epicycles: but the common notion is erroneous that this use of the Epicycle is peculiar to the Ptolemaic Astronomy: "the modern astronomer to this day resolves the same motions into epicyclic ones. When the latter expresses a result by series of sines and cosines (especially when the angle is a mean motion or a multiple of it) he uses epicycles; and for one which Ptolemy scribbled on the heavens, to use Milton's phrase, he scribbles twenty."—A. De M. *Gk. and Rom. Biography*, Vol. III., p. 576.

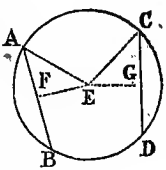
Prop. 14.—Theor.

Equal straight lines in a circle are equally distant from the centre; and conversely, those which are equally distant from the centre, are equal to one another.

Con. 1, III.
12, I. To draw a perp. to a st. line from a point without it. Pst. 1.

Dem. 3, III.
Ax. 7. Things that are halves of the same thing, or of eqnals, are equal.
Def. 15, I.
47, I. In any rt. ∠d △ the square on the side opp. to the rt. ∠ shall be equal to the squares on the sides including the rt. ∠.
Axs. 1, 3. Def. 4, III.
Ax. 6. The doubles of the same thing, or of equal things, are equal.

E.1	Hyp.	I. Let AB = CD in ⊙ ABCD;
2	Conc.	then AB & CD shall be equally distant from cen. E.
C.1	1, III.	Find E the cen. of ⊙ ABCD.
2	12,I.Pst.1	from E draw EF, EG ⊥ AB, CD & join EA, EC.
D.1	C. 2	∵ EF through cen. E cuts AB not through cen. at rt. ∠s;
2	3, III.	∴ EF bis. AB, & AF = FB, and AB = 2AF;
3	*Sim.*	For the same reason CD = 2CG;
4	H. & Ax.7	but AB = CD ∴ AF = CG.
5	Def. 15, I.	And ∵ AE = EC ∴ $AE^2 = EC^2$;
6	C.2 & 47,I.	but ∠ AFE being a rt. ∠ ∴ $AF^2 + FE^2 = AE^2$;
7	*Sim.*	So $EG^2 + GC^2 = EC^2$;
8	Ax. 1.	∴ $AF^2 + FE^2 = EG^2 + GC^2$;
9	D. 4.	but AF = CG ∴ $AF^2 = CG^2$;
10	Ax. 3.	∴ rem. FE^2 = rem. EG^2, & ∴ FE = EG;
11	Def.4,III.	but lines are equally dist. from cen. when the perps. to them from the cen. are equal,
12	Conc.	∴ AB and CD are equally dist. from E.



E.1	H.Df.4,III	II. Let AB & CD be equally dist. from E; *i. e.* FE = EG;
2	Conc.	then AB = CD.
C.	*Sim.*	Let the same construction be made.
D.1	C.&3,III.	∵ EF bis. AB and EG bis. CD; ∴ AB = 2AF and CD = 2CG;
2	47, I.	and $EF^2 + FA^2 = EG^2 + GC^2$;
3	H.	and ∵ FE = EG, ∴ $FE^2 = EG^2$;
4	Ax. 3.	∴ rem. AF^2 = rem. CG^2, and ∴ AF = CG.
5	D.1. Ax.6.	But AB = 2AF, and CD = 2CG; ∴ AB = CD.
6	Rec.	Therefore, *Equal st. lines are equally, &c.* Q. E. D.

SCH.—A principle employed in this and the next proposition is—if two quantities A + B = C + D, then if A = C, B = D; if A > C, B < D; if A < C, B > D.

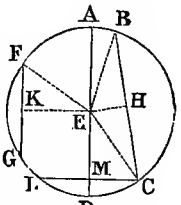
PROP. 15.—THEOR.

The diameter is the greatest straight line in a circle; and of all others that which is nearer to the centre is always greater than one more remote: and the greater is nearer to the centre than the less.

CON. 12, 1. Pst. 1.

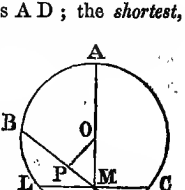
DEM. Def. 15, I. Ax. 1. 20, I. 14, III. 47, I. Ax. 5.
Def. 5, III. The st. line on which the gr. perp. from the cen. falls is farther from the cen.

E.1	Hyp. 1.	I. Let ABCD be a ⊙, AD its diam., and E its cen.
2	„ 2	and let BC be nearer to E than FG is;
3	Conc.	then AD > any line BC not a diam., and BC > FG.
C.1	12, I. Pst. 1.	From E draw EH, EK perp. to BC, FG; and join EB, EC, EF.



D.1	Def.15,I.Ax.2.	∵ AE = EB, and ED = EC; ∴ AD = EB + EC;
2	20, I.	but ∵ EB + EC > BC, ∴ AD > BC.
3	H. 2.	And ∵ BC is nearer cen. E than FG;
4	Def. 5, III.	∴ EH < EK;
5	14, III. D.	but BC = 2BH, FG = 2FK,
6	47, I.	and $EH^2 + HB^2 = EK^2 + KF^2$;
7	D. 3.	and ∵ EH < EK ∴ $EH^2 < EK^2$;
8	Conc.	∴ $BH^2 > FK^2$ and BH > FK; and ∴ BC > FG.
E.1	H.	II. Next, let BC > FG;
2	Conc.	then BC is nearer cen. E than FG is; *i. e.* EH < FK.
D.1	H.&D.7. Case I.	∵ BC > FG, BH > KF, and $BH^2 > FK^2$;
2	D. 5. Case I.	and ∵ $BH^2 + HE^2 = EK^2 + KF^2$;
3	Ax. 5.	∴ rem. $HE^2 < EK^2$, and HE < EK;
4	Def. 5, III.	and ∴ BC is nearer the cen. E than FG is.
5	Rec.	∴ *The diam. is the greatest st. line, &c.* Q. E. D.

Sch. 1.—In a circle the *longest* chord is the diam., as AD; the *shortest*, through a given point, as M, is that, LC, which is perp. to the longest.



Through M, a given · draw a diam. AD, and any other chord, as BE; and to AD, through M, a perp. LC also a chord; and from O the cen. of the ⊙ draw OP perp. to BE.

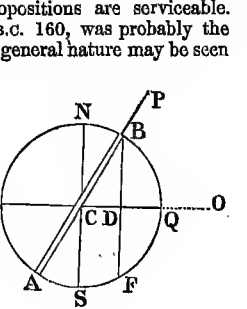
By 47, I. OM > OP; and, 15, III., BE > LC; and the same being true of any other chord through M, ∴ LC at rt. ∠s to AD is the *least* chord, and, 15, III., AD the *longest*, through · M.

2. The *less* the ∠ which a given chord through a given ·, as M, makes with the diam. the *greater* will the chord be;

For, it is evident, that as the ∠ AMB diminishes, the perp. OP, marking the distance of the chord from the cen., will also diminish, and the chord increase. Lardner's Euclid, p. 104.

Use and App.—I. The 14th and 15th Props. were employed by Theodosius to demonstrate that in a Sphere the centres of the least circles on the sphere are the most distant from the centre of the sphere itself; or, in other words, that the circles of latitude diminish as the poles of the earth are approached.

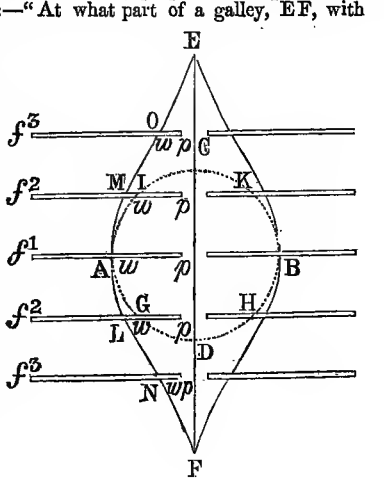
II. In the *Astrolabe*, a circular instrument for *taking the stars, i. e.* for observations on the heavenly bodies, the same propositions are serviceable. HIPPARCHUS, the greatest of Greek astronomers, B.C. 160, was probably the first who constructed an instrument of this kind. Its general nature may be seen from the following representation, in which the approach of the chord BF to the diam. NS may be measured, either by the perp. CD, or the arc BQ.



Let NESQ be a circle fixed to one position, vertical or horizontal; and let AB be a line or tube moveable round the cen. C; through the tube, along the line AB, any object, as P, may be seen; and, by turning the tube round its centre C, another object, as O, may also be seen; the ∠ QCB, subtended by the two objects is their *angular distance*, which may be measured in degrees when the circle NESQ is graduated.

If EQ represent the equator, and if the plane of the circle ESQN pass through the poles N and S; then the ∠ BCN is the north *polar distance*, and the ∠ BCQ, the *declination* of the star or planet P. Again, if the circle ESQN be fixed in the plane of the equator, and EQ point to the vernal equinox at the instant that the tube, AB, points to the star P, then the ∠ BCQ will be the right ascension of the star.

III. It has been said that ARISTOTLE, B.C. 340, propounded a question in mechanics similar to the following:—"At what part of a galley, EF, with rounded sides, EAF, EBF, does an oar, handled from a seat, or station, just above the line of the keel, EF, produce the greatest effect in moving the galley?"



If we examine the conditions of the rounded sides, we shall find that AB, the diam. of the ⊙ ACBD, is the greatest width; then, by 15, III., AB being the greatest line in the circle, all other chords, GH, IK, are less than AB. When the oar is applied at A, the leverage pf^1 of the rower, in proportion to wf^1, the leverage of the weight or resistance, is greater than when the oar is applied at L or at M; where again it is greater than when applied at N or O. The best position therefore for the oars is in the line of the greatest width, AB.

Prop. 16.—Theor.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle, and no straight line can be drawn from the extremity between that straight line and the circumference, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.

Con. Pst. 1. 12, I.

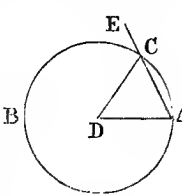
Dem. Def. 15, I.
5, I. The ∠s at the base of an isosc. △ are equal.
17, I. Any two ∠s of a △ are together less than two rt. ∠s.
Ax. 9. 19, I. The gr. ∠ of any △ shall be subtended by the gr. side.
Def. 2, III. A st. line is said to touch a ⊙, when it meets the ⊙, and being produced, does not cut it.
2, III. If any two · s be taken in the ⊙ce of a ⊙, the st. line which joins them shall fall within the ⊙.

E.1	Hyp. 1	Let ABC be a ⊙, fig. 3, D the cen., and AB the diam.;
2	,, 2	and the st. line AE be ⊥ to AB, at its extremity A;
3	Conc.1	then AE shall fall without the ⊙ ABC;
4	,, 2	& no st. line can be drawn from A, between AE and the ⊙cc, unless it cuts the ⊙.

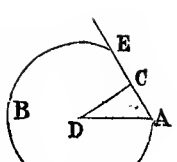
I. *The st. line* AE, *fig.* 3, *shall fall without the* ⊙ ABC.

Sup. 1 & 2.—If not, it must fall either *within* or *on* the ⊙ce.

	Sup. 1.	1st, Let AE fall *within* the ⊙.
C.	Pst. 1.	Join D and C, the · where AE cuts the ⊙ce.
D.1	Def.15,I. 5,I.	∵ DA = DC ∴ ∠ DAC = ∠ DCA;
2	H. 2.	but ∠ DAC is a rt. ∠, ∴ ∠ DCA is a rt. ∠.
3	D.1 &2, 17, I.	∴ in △ ADC, the ∠s DAC & DCA = 2 rt. ∠s; which is impossible
4	Conc.	∴ AE if ⊥ to DA does not fall *within* the ⊙.



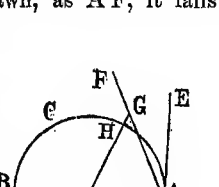
	Sup. 2.	2nd, If possible let A E fall *on* the ⊙ce.
C.	Pst. 1.	In ⊙ce take a · C and join D C.
D.1	D.1,2&3, 1st.	As before in △ A D C, ∠s D C A & D A C = 2 rt. ∠s.
2	17, I.	which is impossible;
3	Conc.	∴ A E cannot fall *on* the ⊙ce.



II. *No st. line can be drawn from* A, *between* A E *and the* ⊙*ce, unless it cuts the circle.*

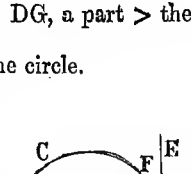
SUP. 1 and 2.—If such a line can be drawn, as A F, it falls either *without* or *on* the ⊙.

	Sup. 1.	1st. Let A E fall *without* the ⊙ A B C.
C.	12, I.	From D sup. D G ⊥ A F, and meeting ⊙ce in H.
D.1	Ax. 9. H. 2.	∵ ∠ D A G < ∠ D A E, but ∠ D A E a rt. ∠;
2		∴ ∠ D A G < a rt. ∠.
3	C.	But ∠ D G A is a rt. ∠, ∴ ∠ D G A > ∠ D A G;
4	19, I.	and ∴ D A > D G;
5	Def.15,I.Ax.9.	now D A = D H, ∴ D H > D G, a part > the whole; an impossibility;
6	Conc.	∴ A F cannot fall *without* the circle.



	Sup. 2.	2nd. If possible then, let A F fall *on* the ⊙ce.
C.	12, I.	Draw D G ⊥ A F, G being in the supposed ⊙ce.
D.1	D.1,2,3CaseII.	Then as before D A > D G;
2	Def. 15,I.	But D A = D G; which is impossible;
3	Conc.	∴ A F cannot fall *on* the ⊙ce.
4	Rec.	Hence, *The st. line drawn at rt. ∠s, &c.*

Q. E. D.



COR. I.—*If a st. line,* A E, *be drawn at rt. ∠s to any diam. of a* ⊙, A B, *from its extremity,* A, *it shall touch the* ⊙ *at the*

extremity: and a st. line touching the ⊙ at one • shall touch it at no other point.

D.1	Sup.&2,III.	For, if AE cut the ⊙, part would fall within the ⊙;
2	16, III. pt.I.	which has been proved to be impossible:
3	Def. 2, III.	∴ AE touches the ⊙ at A, the extremity of BA.
4	Remk.	And AE touches the ⊙ at no other point except A;
5	Sup.&2,III.	for if it could, as before, part would fall within the ⊙.
6	16,III.pt.I.	which has been proved to be impossible.

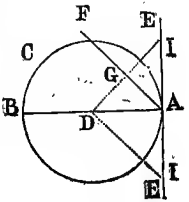
Cor. II.—*By* 28, I. *st. lines at rt. ∠s to the extremities of the same diam. are parallel.*

Cor. III.—*Tangents to a circle from the same point are equal.*

Sch.—1. Thus, a Tangent to a circle is a rt. line perpendicularly raised on the extremity of a diam., or of a radius.

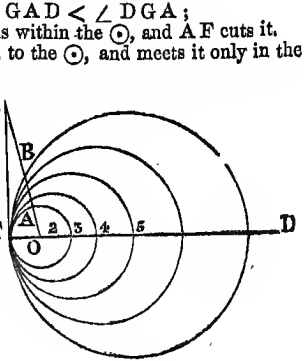
2. The 16th Prop. might have been proved *directly*; thus

C.	Pst. 1.	I. From D draw any line DI to meet AE;
D.1	H. 32, I.	∵ DAI is a rt. ∠, ∴ ∠ DIA < ∠ DAI;
2	19,I.Ax.A.	& ∵ DA < DI; ∴ the • I is out of the ⊙.
3	*Sim.*	So, we may prove the same of every • in AE except • A.
C.	Psts. 1,2,I.	II. Let AF make ∠ FAD acute; and DG ⊥ AF.
D.1	C. 32, I.	∵ DGA is a rt. ∠, ∴ ∠ GAD < ∠ DGA;
2	19,I.Ax.A.	and ∵ DG < DA, ∴ G is within the ⊙, and AF cuts it.
3	Conc.	∴ the line EAE is a tang. to the ⊙, and meets it only in the • A.



Use. and App.—I. The *infinite divisibility* of linear magnitude may be proved by this proposition.

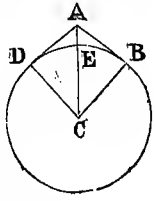
If in TD, or TD produced, points, O, 2, 3, 4, 5, be taken as centres of circles, to all of which TS shall be a com. tang.; then each circle as it cuts AS shall approach nearer to S, as B; but no circle shall ever pass through the point S, inasmuch as TS being a com. tang., the circles cannot touch TS except in the point T; for if they did, a curved line and a st. line would coincide.



II. Since by Prop. 16, III., the tang. AB is perp. to CB, the rad. of the circle, we can, by 47, I. compute the *distances and heights* of objects, on the earth's surface, when they are situated on the verge of the natural horizon; thus, 1st, if AB the distance of the verge of the horizon from the elevation AE be required,—$AB = \sqrt{CA^2}$ dist. of A from earth's centre; $- CB^2$ earth's rad.; where $CA^2 = (CE + EA)^2$.

2nd, If AE be the height of an object just visible from the verge of the horizon B or D; then $AE = CA - CE$ or CB; where $CA = \sqrt{AB^2 + BC^2}$;

Ex. 1. *Given* AE = 2 *miles, the height of the peak of Teneriffe, at what distance will its summit be seen?*



Here CB or CE = 4000 miles;
& CA = CE + AE = 4000 + 2 = 4002 miles,
Then $4002^2 - 4000^2 = AB^2$.
$\therefore$ 16016004 − 16000000 = 16004;
& $\sqrt{16004}$ = 126 miles = AB distance at which A is visible.

Ex. 2. *A meteor* A *was seen over a distance from* B *to* D *of* 500 *miles; required* EA, *its height above the earth.*

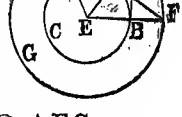
Here AE = AC − EC or = CB; and $\sqrt{CB^2 + AB^2} = AC$.
Or, $\sqrt{4000^2 + 250^2} = \sqrt{16062500} = 4007{\cdot}84$ miles;
and 4007·84 − 4000 = 7·84 miles, height of the meteor.

PROP. 17.—PROB.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

SOL. 1, III. Psts. 1, 3. 11, I.

DEM. Def. 15, I. 4, I. Ax. 1. Cor. 16, III.



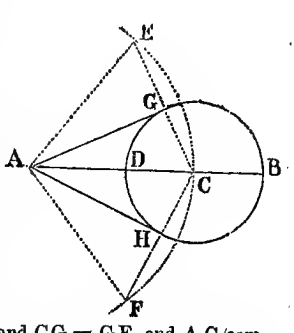
E.1	Dat.	I. Let A be a given • without the given ⊙ BCD;
2	Quæs.	from A to draw a tang. to the ⊙ BCD.
C.1	1, III. Pst. 1.	Find E cen. of ⊙ BCD, and join AE;
2	Pst. 3.	from E with EA desc., the ⊙ AFG;
3	11, I. Pst. 1.	at D draw the perp. DF; and join EBF and AB;
4	Sol.	then AB shall touch the ⊙ BCD in B.

D.1	C.1,2. Def.15,I.	∵ E is the com. cen. of ⊙s BDC, AFG; ∴ EA = EF, and ED = EB;
2	D. 1 and C.	and ∵ AE, EB = EF, ED, and ∠ E com. to △s AEB, FED;
3	4, I.	∴ DF = AB, △ FED = △ AEB, & ∠ EBA = ∠ EDF.
4	C. 3. Ax. 1.	but ∠ EDF is a rt. ∠, ∴ ∠ EBA is a rt. ∠;
5	Remk.	now EB is a line from the cen to the ⊙ce;
6	D.4.Cor.16,III.	∴ AB touches the ⊙ in B, and is drawn from A.
E.1	Dat.	II. Let D be the given · *in* the given ⊙ce BCD;
2	Quæs.	from D to draw a tang. to the ⊙ BCD.
C.1	1, III. Pst. 1.	Find the centre E, and join DE;
2	11, I.	from D draw DF perp. to DE;
3	Sol.	then DF is the tang. to the ⊙.
D.1	C. 2.	∵ DF is perp. at D to the rad. ED;
2	Cor. 16, III.	∴ DF touches the ⊙ BCD at the given · D.

Q. E. F.

Scн.—It is evident that from · A, out of the ⊙ BDG, two equal tangents, AG and AH, may be drawn.

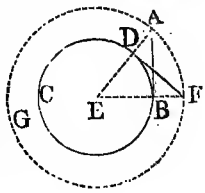
C.1	1, III. Pst. 1, 2.	Find C the cen. of ⊙ BDG, and draw ACB;
2	Pst. 2.	from A with AC desc. arc ECF;
3	Pst. 3.	and with rad. DB, from C, cut arc ECF in E and F;
4	Pst. 1.	join CE, CF, cutting the ⊙ BDG in G and H;
5	Sol.	the lines AG, AH, will be the two equal tangents.



D.1	Def. 15, I. 3, III.	For, ∵ AC = AE and CG = GE, and AG com.
2	8, I.	∴ △ AGE = △ AGC, and ∠ AGE = ∠ AGC;
3	Def.10,I.Cor.16,III.	∴ AG ⊥ CE and tang. to CG a rad.
4	*Sim.*	So AH is the second tang. to the ⊙ BDG.
5	Def. 15, I. 5, I.	Also ∵ CG = CH, AC com. and ∠ GCA = ∠ HCA;
6	4, I.	∴ tang. AG = tang. AH.

USE AND APP.—I. *Practically* a tangent will be drawn, from a point A *out* of a circle DBC by aid of a ruler; and from a point D *on* the circumference, by joining the point and the centre, E, and at the point, making, DF, a perpendicular to the radius.

II. Tangent lines, as AB, FD, determined by secants EA, EF, are of very frequent use, especially in trigonometry; tables of the proportional or relative lengths of chords, natural lines, tangents, and secants are constructed by aid of Prop. 47, Book I.



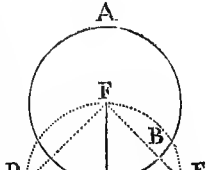
PROP. 18.—THEOR.

If a st. line touches a circle, the st. line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

CON. 1, III. Pst. 1. 12, I.

DEM. 17, I. 19, I. Def. 15, I. Ax. 9.

E.1	Hyp.	Let the st. line DE touch the ⊙ ABC in C;
2	1,III.Pst.1.	take the cen. F, and draw the st. line FC;
3	Conc.	then FC shall be perp. to DE.

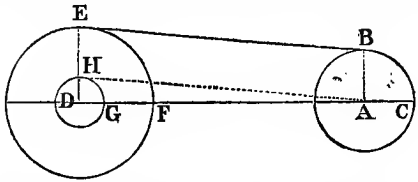


SUP.—*If not, from* F *draw* FBG *perp. to* DE.

D.1	C. 17, I.	∵ FGC is a rt ∠, ∴ GCF is an acute ∠;
2	19, I.	and ∵ to the gr. ∠ the gr. side is opp. ∴ FC > FG:
3	Def. 15, I. Ax. 9.	but FC = FB ∴ FB > FG, the less than the gr.; which is impossible;
4	Conc.	∴ FG is not perp. to DE.
5	*Sim.*	So, no line except FC is perp. to DE;
6	Conc.	∴ FC is perp. to DE.
7	Rec.	∴ *If a st. line touches, &c.* Q. E. D.

SCH.—Propositions 16 and 18 may be regarded as the *converse* of each other; in the one we prove, that a line perpendicular to the extremity of the radius is a tangent; in the other, that, if a line be a tangent to a circle, the radius from the point of contact will be perpendicular to it. Thus in Prop. 16, AE is proved to be perp. to AB, and in Prop. 18, FC perp. to CE: in the one case, the tang. is perp. to the rad.; in the other, the rad. is perp. to the tang.; two expressions for the same thing.

USE AND APP.—*To draw a tang. to each of two given* ⊙*s* ABC, DEF.



C.1	Pst.1. 3, I.	Join the centres A, D, and make FG = AC;
2	Pst. 3.	from D with rad. DG desc. ⊙ DGH;
3	17, III.	and from A draw AH, tang. to ⊙ DGH;
4	Pst. 1, 2.	join DH, and prod. it to E in ⊙ DFE;
5	31,I.Pst.1	from A draw AB ∥ DE, and join EB;
6	Sol:	then EB is tang. to each ⊙, ABC and DEF.
D.1	C 3,18,III	∵ AH is tang. in H to ⊙ DGH, ∴ ∠ DHA is a rt. ∠;
2	13, I. C.	and ∠ EHA is a rt. ∠; and ABEH is a ▱;
3	C.& 46,I.	and ∵ ∠ EHA is a rt. ∠ ∴ all the ∠s in ABEH are rt. ∠s.
4	D. 3	then ∵ DE from cen. D, & AB from cen. A, meet BE at rt. ∠s.
5	Cor 16,III.	∴ EB is tang. to both ⊙s, ABC and DEF. Q. E. F.

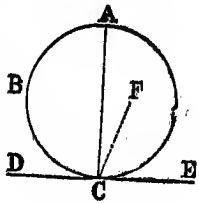
PROP. 19.—THEOR.

If a st. line touches a circle, and from the point of contact a st. line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

CON. Pst. 1.

DEM. 18, III. Ax. 11. All rt. ∠s. are equal to one another.—Ax. 9.

E.1	Hyp. 1	Let the st. line DE touch the ⊙ ABC in C;
2	,, 2	& from C let CA be at rt. ∠s to DE;
3	Conc.	then the cen. of the ⊙ shall be in CA.

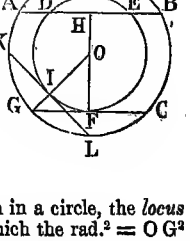


SUP.—*For if not, and it be possible, let* F *be the cen. and join* F C.

D.1	Sup.	then ∵ D E is a tang. and F C a rad. from the · of contact;
·2	18, III.	∴ F C is perp. to D E, and ∠ F C E a rt. ∠;
3	H. 2, Ax.11	but A C E is a rt. ∠, ∴ ∠ F C E = ∠ A C E;
4	Remk. Ax.9.	*i. e.* the part = the whole, which is impossible.
5	Conc.	∴ the · F is not the cen. of the ⊙ A B C.
6	*Sim.*	And thus no other · , except it be in C A, is the cen.;
7	Conc.	∴ the cen. of the ⊙ is in C A.
8	Rec.	Therefore, *If a st. line touches, &c.* Q. E. D.

SCH.—1. If in two concentric circles A B C, D E F, a chord A B of the greater meet the less, as in D and E, the parts A D and E B, intercepted between the two ⊙s are equal; and all chords of the greater ⊙, as G C, which touch the less, are bisected at the · s of contact, as F; and are equal.

C.	12, I.	From O the cen. draw O H ⊥ A B, and O F ⊥ G C.
D.1	3,III. Ax. 3.	I. ∵ A H = B H and D H = E H ∴ A D = B E.
2	C. 3, III.	II. ∵ ∠ O F C = ∠ O F G; ∴ G F = F C.
3	*Sim.*	Thus all chords of A B C tangs. to D E F, are bisd. at the · of contact.
4	14, III.	and III. such chords are equal.



2. If any number of equal chords, G C, K L, be drawn in a circle, the *locus* of their points of bisection, as F, I, &c., is a circle of which the rad.^{2} = O G^2 *minus* G F^2.

USE AND APP.—I. In Optics the properties of tangent lines are employed, among other purposes, for determining the part of a globe which may be enlightened by a luminous body, as by a lighthouse, a volcano, a meteor, or the sun.

II. By a tangent line representing the limit or extent of vision, the diameter of the earth may be ascertained, as in Prop. 6, Book II.

III. Tangent lines, too, serve to explain the Theory of the Phases of the Moon, and were employed by HIPPARCHUS, B.C. 160 to 145, to ascertain the distance of the sun.

IV. In Navigation the dip of the horizon corresponds with the tangent line from the point of observation; and were the tangent line to revolve round that point it would trace out the circle of the physical horizon. An approxi-

mate rule for ascertaining the distance of the verge of the natural horizon is to take the square root of the height of the spectator's eye in fect and multiply it by 1·3 ;—the product in feet will nearly give the distance, or rad. of the horizon, in miles ; thus from the mast of a ship 81 feet above the sea the horizon is 11·7 miles off; for $\sqrt{81} \times 1{\cdot}3 = 9 \times 1{\cdot}9 = 11{\cdot}7$; or,

How distant is the tangent point from the highest peak of Teneriffe, which has an elevation of about 11,946 feet ?

$\sqrt{11{,}946} \times 1{\cdot}3 = 109 \times 1{\cdot}3 = 141{\cdot}7$ miles.

V. In Dialling, also, or Gnomonics, tangents are employed for calculating the hour lines and ascertaining their exact position ; but Dialling is now comparatively of little importance.

Prop. 20.—Theor.

The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

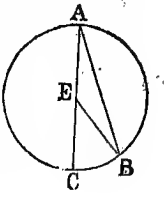
Con. Pst. 1 & 2.

Dem. Def. 15, I. 5, I.
32, I. In every △, if any side be produced, then the ext. ∠ = the two int. and opp. ∠s ; and the three int. ∠s = two rt. ∠s.

E.1	Hyp. 1	In a ⊙ ABC let E be the cen.; ∠ BEC the ∠ at the cen.
		and ∠ BAC the ∠ at the ⊙ce;
2	,, 2	and let ∠s BAC, BEC have for base the same arc BC;
3	Conc.	then the ∠ BEC = twice ∠ BAC.
C.	Pst. 1. 2.	Join AE and produce it to the ⊙ce F. (fig. 2.)

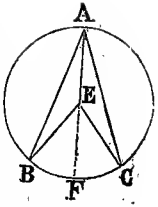
Case I. *Let* E *the cen. be* on AC, *one of the st. lines* AB, AC.

D.1	Def.15,I.5,I.	∵ EA = EB ∴ ∠ EAB = ∠ EBA; and ∠s EAB + EBA = 2 ∠ EAB;
2	C. 32. 1.	but ∵ in △ AEB, AE is prod. to C, ∴ ext. ∠ BEC = ∠s EAB + EBA
3	Conc.	∵ ∠ BEC = twice ∠ BAC.



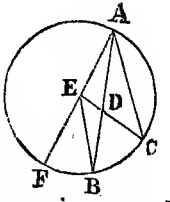
CASE II. *Let* E *the cen. be* within *the* ∠ BAC

D.1	Def.15,I.5,I.	∵ EA = EB, ∴ ∠ EAB = ∠ EBA,
2	Remk.	and ∠s EAB + EBA = twice ∠ EAB;
3	C. 32, I.	but ∠ BEF = ∠ EAB + ∠ EBA;
4	Conc.	∴ ∠ BEF = twice ∠ EAB:
5	*Sim.*	Thus also ∠ FEC = twice ∠ EAC;
6	Conc.	∴ the whole ∠ BEC = twice the whole ∠ BAC.



CASE III. *Lastly let* E *be* without *the* ∠ BAC.

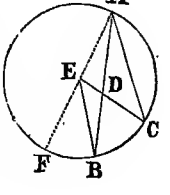
D.1	32, I.	then ∵ ∠ FEC = 2 ∠ FAC, & ∠ FEB = 2 ∠ FAB;
2		∴ rem. ∠ BEC = twice rem. ∠ BAC.
3	Rec.	Therefore in every case, *the angle, &c.* Q. E. D



COR.—*Any angle, as* ∠ CAB, *at the circumference is measured by half the arc* CB, *on which it stands.*

SCH.—I. The reasoning employed anticipates Prop. 5, Book V., and assumes, "that among four magnitudes, if the *first* = *twice* the second, and the *third* = *twice* the fourth, then the *first* + the *third* = *twice* (the second + the *fourth*); and also, that if *one magnitude* = *twice* another, and a *part* from the first = *twice* a part from the second; then the *rem.* of the first = *twice* the rem. of the second." This principle is neither sufficiently self-evident to be received as an axiom, neither has it been demonstrated; *another method of proof for Case III.* has therefore to be adopted.

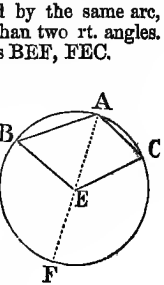
C.1	Pst. 1 and 2.	Join AE and prod. it to F; and let AB, EC inters. in D.
D.1	32, I.	∵ in △ BED ext. ∠ BDC = ∠ DEB + ∠ EBD;
2	Def.15, I. 5,I.	and ∵ EB = EA, ∴ ∠ EBD = ∠ EAD;
3	Conc.	∴ ∠ BDC = ∠ DEB + ∠ EAD.
4	32, I.	Again, ∵ in △ ADC ext. ∠ BDC = ∠ DAC + ∠ DCA;
5	Def.15, I. 5,I.	and ∵ EA = EC ∴ ∠ DCA = ∠ EAC = ∠ DAC + ∠ DAE;



6	D. 4 and 5.	$\therefore \angle BDC =$ twice $(\angle DAC + \angle DAE)$;
7	D. 3.	but $\angle BDC = \angle DEB + \angle EAD$;
8	Ax. 1.	$\therefore \angle DEB + \angle EAD =$ twice $(\angle DAC + \angle DAE)$;
9	*Sub.*	take away the com. $\angle EAD$;
10	Ax. 3.	$\therefore$ rem $\angle DEB =$ twice rem. $\angle DAC$;
		i. e. the $\angle$ at cen. = twice the $\angle$ at ⊙ce.

II. If Euclid's definition of an angle (Def. 9, Book I.) be strictly adhered to, the 20th Prop. is geometrically true only when the angle at the centre is less than a rt. angle; but if an angle may be regarded as any angular magnitude less than four right angles, the proposition is universally true. The relation between the angles at the centre and at the circumference, subtended by the same arc, includes the cases in which the angle at the centre is greater than two rt. angles. Take, for instance, the *re-entrant* angle BEC, made up of $\angle$s BEF, FEC.

C.	Pst. 1 and 2.	Join A E and prod. it to F, the re-entrant $\angle BEC = \angle$s BEF, FEC.
D.1	20, III.	$\because \angle BEF =$ twice $\angle BAF$, and $\angle FEC =$ twice $\angle CAF$;
2	Cone.	$\therefore$ re-entrant $\angle BEC =$ twice $\angle BAC$.



In this way the proposition is universally true.

III. The demonstrations often appear *plainer* when arranged exactly as a Simple Equation, and worked by the same rules; thus, in reference to the figures Pr. 20, III.

Case 2.	$\angle BEF = 2\angle BAF$, or	$12 = 6 + 6$
and	$\angle FEC = 2\angle FAC$,	$14 = 7 + 7$
$\therefore$ *Add.*	$\angle BEC = 2\angle BAC$,	$26 = 13 + 13$
Case 3.	$\angle FEC = 2\angle FAC$, or	$20 = 10 + 10$
and	$\angle FEB = 2\angle FAB$,	$12 = 6 + 6$
Sub.	$\angle BEC = 2\angle BAC$,	$8 = 4 + 4$

Thus, the sum or difference of the angles at the centre = twice the sum or difference of the angles at the circumference.

Use and App.—1. This proposition is applied in Trigonometry, of which examples will be found in the Use and Application of Prop. 21, Book III. 2. It was employed by Ptolemy to determine the eccentricity of the sun and the epicycle of the moon. 3. In ascertaining the earth's aphelion by three observations the angle at the centre of the orbit is taken double the angle at the circumference.

PROP. 21.—THEOR.

The angles in the same segment of a circle are equal to one another.

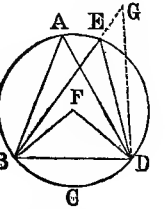
CON. 1, III. Pst. 1 and 2.

DEM. 20, III. AX. 7. AX. 2.

E.1	Hyp.	In ⊙ ABCD, let BAD, BED be ∠s in the same seg. BAED;
2	Conc.	then ∠ BAD = ∠ BED.

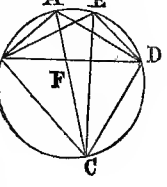
CASE I.—*Let the seg.* BAED *be greater than a semicircle.*

C.	1, III. Pst. 1.	Take F the cen. of ABCD and join BF, FD.
D.1	C.	∵ ∠ BFD is at F, the cen. and ∠ BAD at the ⊙ce,
2	„	and ∵ each ∠ has the same base, arc BCD;
3	20, III.	∴ ∠ BFD = twice ∠ BAD.
4	*Sim.*	For the same reason ∠ BFD = twice ∠ BED;
5	Ax. 7.	∴ ∠ BAD = ∠ BED.



CASE II.—*Let the seg.* BAED *be not gr. than a semicircle.*

C.	1, III. Pst. 1, 2.	Find F the cen., join AF, and prod. it to C, and join CE, CB, CD.
D.1	C.	∵ AC is a diam. ∴ seg. AEDC is a semic.
2	C.	also segs. BAEDC, DEABC each > a semic.
3	C. and Case I.	and ∵ BAEDC > a semic. ∴ ∠ BAC = ∠ BEC;
4	C. and Case I.	and ∵ DEABC > a semic. ∴ ∠ CAD = ∠ CED;
5	*Add.* Ax. 2.	Hence the whole ∠ BAD = whole ∠ BED.
6	Rec.	∴ *The angles in the same segment, &c.*



Q. E. D.

N.B. This Prop. may be very simply proved; for the angles of the segments are the halves of the same central angle standing on the same common base.

Cor.—*If*, as in fig. to Case I, *upon the base*, BD, *of a* $\triangle$, *there be described a segment*, BAED *of a circle, the vertex of the* $\triangle$ *shall fall* without, *as at* G, or within, *as at* F, *or* upon the arc, *as at* E, *according as the vertical angle is* $<$, $>$, *or* $=$ *the* $\angle$ A, *in the segment.*

For, if the vertex fall *without*, as at G, then, 16, I., $\angle$ BED $= \angle$ BAD and is $> \angle$ BGD; if *within*, as $\angle$ F, then, 21, I., $\angle$ F $> \angle$ BAD; and if *on*, as E, then 21, III., $\angle$ BED $= \angle$ BAD.

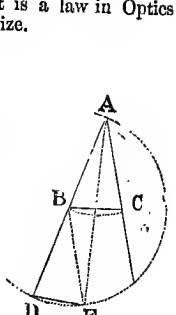
Sch. 1.—Any number of triangles whatever, upon the same base, and with their vertical angles equal, have the *locus* of their vertices in the arc of the same circle; hence, by constructing innumerable triangles fulfilling these two conditions, the vertices would form the segment of a circle.

2. From this Proposition it follows that the angle at the circumference has for its *measure* one half of the arc on which it stands.

Use and App.—I. This proposition may be employed for constructing a building in which all the spectators shall see any object under the same angle. A circular amphitheatre is of this kind, for let BD, fig. Case I., be the stage, and BAED the circle in which the seats are placed, then from every one of the seats the stage subtends the same angle of vision; and it is a law in Optics that a line seen under the same angle appears of the same size.

II. *To bisect an angle, as* $\angle$ DAC.

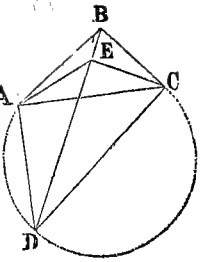
C.1	Pst. 3.	From A with AB desc. an arc BC;
2	3,I.Pst.3	take BD = BA, and with BD desc. the arc DFA;
3	3,I.Pst.3.	and from D with rad. = BC cut DFA in F;
4	Pst. 1.	and join BF, AF, BC, and DF,
5	Sol.	then the line AF bisects $\angle$ A.
D.1	C.	$\because \angle$ DBF is at the cen., and $\angle$ DAF at the $\odot$ce;
2	20, III.	$\therefore \angle$ DBF $= 2 \angle$ DAF;
3	8,I.Ax.1.	but $\angle$ BAC $= \angle$ DBF, $\therefore \angle$ BAC $= 2 \angle$ DAF.
4	Conc.	$\therefore \angle$ DAC is bisected by AF.



III. *To construct a figure representative of the distance of the place of observation from an object.*

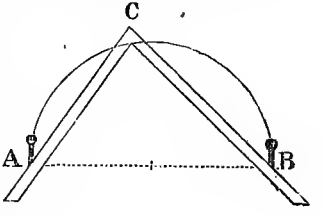
Ex.—Three objects, A, B, C, are distant, AB = 8 miles, BC = 7·2, and AC = 12 miles, from the place of observation D; the $\angle$ CDB = 25°, and $\angle$ BDA = 19°; required the distance from D to B.

From a scale of eq. parts, with the given distances, construct the $\triangle ABC$; at C make $\angle ACE = 19° = \angle BDA$, and at A, $\angle CAE = 25° = \angle CDB$; and through the three points A, E, and C, (Prop. 3. Use 4, III.) draw a circle ADCE; join BE, and produce it to intersect the circle in D; then BD is representative of the distance from D to B, and is equal to about 15 miles, taken from the scale. For $\angle$s ACE, ADE, being in the same segment, ADCE, are equal, by 21, III., and also $\angle$s CAE, CDE, are equal, being in the same segment, EADC. TATES' GEOMETRY, p. 56.



IV. *To draw the arc of any circle*, especially *on a large scale*, by means of two st. pieces of wood fastened so as to form a certain angle, ACB.

Fix into the ground, or on a floor, two pins, A and B, at any required distance, not greater than that of the extremities, A and B, of the angular frame ACB. Move the angular frame round, keeping the sides pressing close to the pins, and a tracing point or pencil at C will mark out the arc of a circle on the ground, or on a floor. The base, AB, remains the same, and the vertices of the triangles have their *locus* in the arc of the same circle.



This method of drawing, or tracing a circle, or the arc of a circle, without having its centre, may be employed for giving a spherical figure to metal cauldrons, or to optical glasses; also for making large Astrolabes, or for marking out the meridian lines, and the lines of latitude on large maps; indeed for every purpose which requires an arc of great magnitude.

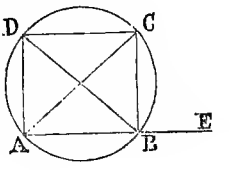
PROP. 22.—THEOR.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

CON. Pst. 1.

DEM. 32, I. 21, III. Ax. 2. Ax. 1.

E.1	Hyp.	Let ABCD be a qu. lat. in the ⊙ ABC.
2	Conc.1.	then $\angle$s ABC + ADC = 2 rt. $\angle$s.
3	,, 2.	& $\angle$s BAD + BCD = 2 rt. $\angle$s.
C.	Pst. 1.	Join AC, BD.



CASE I.—*The opp.* $\angle$s ABC + ADC = *2 rt.* $\angle$s.

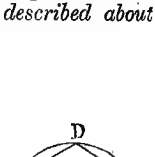
D.1	32, I.	In $\triangle$ CAB, the $\angle$s ABC + BCA + CAB = 2 rt. $\angle$s;
2	21, III.	but $\angle$ CAB = $\angle$ CDB, and $\angle$ ACB = $\angle$ ADB.
3	Ax. 2.	$\therefore$ the whole $\angle$ ADC = $\angle$ CAB, + $\angle$ BCA;
4	*Add.*	to each add $\angle$ ABC, and $\angle$s ADC + ABC = $\angle$s CAB + ABC + BCA;
5	D. 1.	but $\angle$s ABC + BCA + CAB = 2 rt. $\angle$s;
6	Ax. 1.	$\therefore$ $\angle$s ABC + ADC = 2 rt. $\angle$s.

CASE II.—*The other pair of opp.* $\angle$*s*, BAD, BCD, *also* = *2 rt.* $\angle$*s*.

D.1	32, I.	For in $\triangle$ BAD, $\angle$s BAD + ADB + DBA = 2 rt. $\angle$s.
2	D. 2, 4. Case I.	as before $\angle$s BAD + BCD = $\angle$s BAD + ADB + DBA;
3	Conc.	$\therefore$ $\angle$s BAD + BCD = 2 rt. $\angle$s.
4	Rec.	Therefore, *The opposite angles of any quadril., &c.* Q. E. D.

COR. I.—And conversely, *if the opposite angles of a quadrilateral be together equal to two rt. angles, a circle may be described about the quadrilateral.*

C.1	9, III. Use.	Through A, B, D, the vertices of three $\angle$s, draw a $\odot$ ABD; the $\odot$ce will also pass through the fourth vertex, C.
2	*Ass.* Pst. 1.	For take any other point E in the seg. and join AE, BE.
D.1	22, III., & H.	$\because$ $\angle$ E + $\angle$ D = 2 rt. $\angle$s, and $\angle$ C + $\angle$ D = 2 rt. $\angle$s.
2	*Sub.* Ax. 3.	$\therefore$ taking away com. $\angle$ D, $\angle$ E = $\angle$ C;
3	C.	and these angles have a com. base, AB;
4	Sch. 21, III.	$\therefore$ $\angle$s E and C, are both in the $\odot$ce.
5	Conc.	and $\therefore$ $\odot$ may be described about the qu.lat. ADBC.



COR. II.—*If any side of a quadril. in a circle*, as AB, in fig 1, *be produced* to E, *the ext.* $\angle$ EBC = *int. opp.* $\angle$ ADC; for they have a com. supplement $\angle$ CBA, or each of them together with the int. adj. $\angle$ = 2 rt. $\angle$s.

COR. III.—"*If two chords cut off similar segments from the same or different circles, the other segments will also be similar*, since the angles which they contain are supplemental to those in the former segments."

COR. IV.—"*If opp. angles of a quadril. be equal, they must be both rt. angles*, rt. angles being the only equal angles which are supplemental." LARDNER'S EUCLID, p. 110.

USE AND APP.—PTOLEMY availed himself of this proposition to construct the Tables of Chords; and in Trigonometry it may be applied to prove that the sides of an obtuse angled triangle have the same ratio to one another as the sines of their opposite angles; and hence, if any three be given, the fourth may be found.

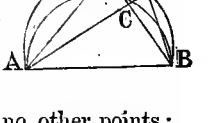
PROP. 23.—THEOR.

Upon the same straight line and upon the same side of it there cannot be two similar segments of circles, not coinciding with one another.

CON. Pst. 1, 2.

DEM. 10, III. Def. 11, III. 16, I.

E.	Sup.	If possible, on AB and on the same side of it let there be two sim. segs. ACB, ADB, not coinciding.
D.1	Sup.	∵ ⊙ ACB cuts ⊙ ADB in A and B;
2	10, III.	∴ these ⊙s ACB, ADB cut in no other points;
3	Conc.	∴ one of the segs. falls entirely within the other:



SUP.—*Let seg.* ACB *fall entirely within seg.* ADB;

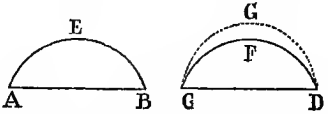
C.1	Pst. 1	In arc ACB take any • C, and join BC
2	Pst. 2	produce BC to D, and join CA, DA.
D.1	H.	∵ seg. ACB is sim. to seg. ADB;
2	Def.11,III.	and sim. segs. of ⊙s contain equal ∠s;
3	Conc.	∴ ∠ ACB = ∠ ADB, *i. e.* the ext. ∠ = the int. ∠;
4	16, I.	but this is impossible;
5	Rec.	Therefore, *there cannot be two similar, &c.* Q. E. D.

SCH.—This proposition, relating to two similar segments of a circle, is the same in principle with the 7th of Book I., which says, that, "on the same base and on the same side of it, there cannot be two triangles which have their sides terminated in one extremity of the base equal, and likewise those terminated in the other extremity equal, not coinciding with one another:" and as the only purpose for which 7, I., was employed was to prove 8, I.; so the purpose to which 23, III. is applied is the demonstration of 24, III. We throw such propositions away as soon as we have used them, yet they are needful links in the chain of geometrical argument.

PROP. 24.—THEOR.

Similar segments of circles upon equal straight lines are equal to one another.

DEM. 23, III. AX. 8. Magnitudes which coincide are equal.



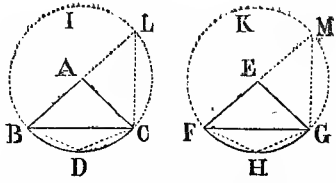
E.1	Hyp.	Let AEB, CFD, be sim. segs. of ⊙s on equal st. lines, AB = CD;
2	Conc.	then seg. AEB = seg. CFD.
D.1	*Super.*	Suppose seg. AEB placed on seg. CFD, so that A be on C, AB on CD, and ares AEB, CFD, on the same side of CD;

2	H.	then $\because$ AB = CD, $\therefore$ B shall coincide with D.
3	D. 2 and 1.	and AB coinciding with CD, and the segs. being on the same side of CD,
4	23, III.	$\therefore$ the seg. AEB must coincide with seg. CFD.
5	Sup.	If not, the arc AEB would take another direction, as CGD;
6	Remk.	thus on CD, and on the same side of it, there would be two similar segs. of ⊙s not coinciding, CGD and CFD;
7	23, III.	which is impossible.
8	Conc. Ax.8.	$\therefore$ seg. AEB coincides with seg. CFD; and seg. AEB = seg. CFD.
9	Rec.	$\therefore$ *Similar segments of circles, &c.* Q. E. D.

Cor. I.—*Similar segments having equal chords have also equal arcs;* to be established by the same principle of superposition.

Cor. II.—*Similar segments having equal chords are parts of equal circles*; for circles which agree in more than two points agree in every point.

Cor. III.—*If the radii,* AB, EF, *and angles,* BAC, FEG, *of sectors* BACD, FEGH, *are equal, the sectors themselves are equal.*

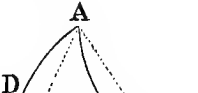


C.1	Pst. 1.	Draw the chords BC and FG;
2	Pst. 2, 1,	Produce BA to L, FE to M,
3	Pst. 1.	and join LC, MG, BD, DC, FH, HG.
D.1	4, I.	Then $\because$ $\triangle$ BAC = $\triangle$ FEG;
2	20, III.	and $\because$ $\angle$ BAC = $\angle$ FEG, and $\angle$ BLC = $\angle$ FMG;

D.3	22, III.	also ∵ ∠ BDC = ∠ FHG,
4	Def. 10, III.	∴ seg. BDC is sim. to seg. FHG.
5	D.1.24, III.	Again ∵ BC, = FG ∴ seg. BDC = seg. FHG;
6	*Add.*	To each seg. add equal △s BAC, FEG;
7	Ax. 2.	∴ BDC + BAC = FHG + FEG,
		i. e. sect. BACD = sect. FEGH.

USE AND APP.—*Curve lined figures, as* ADB, CEA, *are often reduced to rectilineal figures by this proposition.*

For, if two like segments, ADB and AEC of circles are described on the equal sides, AB, AC, of an equilateral △ ABC, it is evident that by *transposing* the seg. AEC on ADB, the △ ABC = the curve lined fig. ADB, CEA.



PROP. 25.—PROB.

A segment of a circle being given, to describe the circle of which it is the segment.

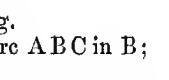
CON. 10, I. 11, I. Pst. 1. 23, I. Pst. 2.

DEM. 6, I. If a △ have two of its ∠s equal, then the sides opp. the equal ∠s shall be equal.
Ax I. 9, III. Def. 10, I. 4, I.

E.1	Dat.	Let ABC be the seg. of a circle,
2	Quæs.	to desc. the ⊙ of which it is the seg.
C.1	10,I.11,I.	Bis. AC by the perp. DB cutting arc ABC in B;
2	Pst. 1.	and join AB;
3	Remk.	there will be *two* cases according as the ∠s BAD, ABD are = or ≠.

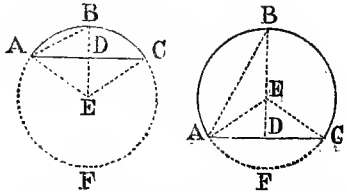
CASE I. *Let* ∠ BAD = ∠ ABD.

D.1	H. 6, I.	∵ ∠ DBA = ∠ DAB.
		∴ DA = DB;
2	C.& Ax.1	and ∵ DC = DA,
		∴ DA = DB = DC;



D.3	C & D, 2.	And ∵ from D to the ⊙ce, DA, DB, DC are equal;
4	9, III.	∴ D is the cen. of the ⊙ of which ACB is an arc.
5	Sol.	Hence, if from D with rad. DA or DB or DC a ⊙ be desc., it will be that of which ABC in an arc.
6	Remk.	And ∵ the cen. is in AC, ∴ the seg. ABC is a semicircle.

CASE II. *Let* ∠ BAD ≠ ∠ ABD.

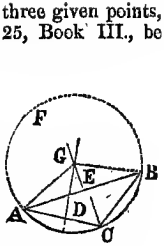


C.1	23, I.	At A in AB make ∠ BAE = ∠ ABD;
2	Psts. 2 & 1.	prod., if necessary, BD to meet AE in E, and join EC.
D.1	C. 1. 6, I.	∵ ∠ ABE = ∠ BAE, ∴ BE = AE;
2	C. Def. 10, I.	and ∵ AD = DC, DE com. and ∠ ADE = ∠ CDE,
3	4, I.	∴ in △s ADE, CDE, base AE = base EC;
4	D. 1, Ax. 1.	but AE = EB, ∴ BE also = EC;
5	D. 3, 4. 9, III.	and ∵ AE = EB = EC, ∴ E is the cen. of the ⊙.
6	Sol.	Hence, if from E with rad. EA, EB, or EC, a ⊙ be described, it will be that of which ABC is an arc.
7	Remk.	If ∠ ABD > ∠ BAD, the cen. E falls *without* the seg., which therefore is less than a semicircle;
8	Remk.	but, if ∠ ABD < ∠ BAD, the cen. E falls *within* the seg., which therefore is greater than a semicircle.
9	Rec.	∴ *A segment of a circle being given, &c.* Q. E. F.

SCH.—This problem might be proposed in another way,—as, *to inscribe a triangle in a circle*; or, *to make a circle pass through three given points, provided they are not in a st. line.* The mode of doing this has been pointed out in Use and App. II., III., and IV., of Prop. 3, Book III., and also in the App.

of Prop. 9, Book III. If, for making a circle pass through three given points, A, B, C, not in a st. line, the method pursued in Prop. 25, Book III., be followed, the process will be—

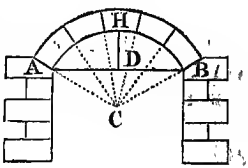
C.1	Pst. 1.	Join the three given · s A, B, C;
2	10, I. 11, I.	bis. A C in D by a perp. from D, and A B in E by a perp. from E;
3	Sol.	the · G where the perpendiculars inters. is the cen. of the required ⊙.
4	Pst. 1.	Join G A, G C, G B.
D.1	C. 4, I.	∵ A D = D C, D G com. and ∠s at D equal, ∴ G A = G C.
2	*Sim.*	for a like reason G B = G C,
3	Ax. 1. 9, III.	∴ G A = G B = G C; and with either as rad. a ⊙ may be drawn through A, B, C.



Cor.—In like manner the remainder of the ⊙ce of a segment of a circle may be completed.

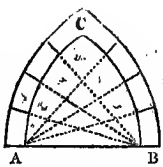
Use and App.—The proposition is of frequent use in all cases when a circle, or an arc, has to be drawn through any three points. Thus:

1. For constructing an arch, of which A B the span and D H the perp. height to the centre, are given:



2. For drawing the plan of a Gothic arch, in which the span A B, and the radii of the arcs intersecting in C are the same:

When A B, the span of the arch, or distance of two of the three points, A, B, is great, the method may be adopted, which is given in Use and App. IV., Prop. 21, Book III., p. 49.



3. For cutting stone, wood, or metal, so that a circle shall pass through three given points:

4. For finding the apogee of the moon, and the eccentricity of the earth's orbit.

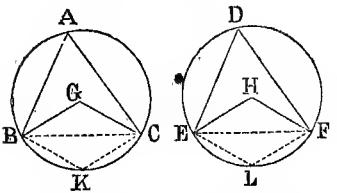
Prop. 26.—Theor.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or circumferences.

Con. Pst. 1.

Dem. Def. 1, III. 4, I. Def. 11, III. 24, III. Ax. 8.

E.1	Hyp. 1.	Let ⊙ ABC = ⊙ DEF;
2	„ 2.	and ∠ BGC = ∠ EHF at their centres; and ∠ BAC = ∠ FDE at their ⊙ces;
3	Conc.	then arc BKC = arc ELF.
C.	Pst. 1.	Join BC, and EF.



D.1	H. Def. 1, III.	∵ ⊙ ABC = ⊙ DEF, ∴ BG = GC = EH = HF each to each;
2	H. 4, I.	and ∠ G = ∠ H, ∴ BC = EF.
3	H. Def. 11, III.	And ∵ ∠ at A = ∠ at D, ∴ seg. BAC is sim. to seg. EDF;
4	D. 2. 24, III.	but ∵ BC = EF, ∴ seg. BAC = seg. EDF;
5	H. Ax. 3.	Now ⊙ ABC = ⊙ DEF, ∴ rem. seg. BKC = rem. seg. ELF;
6	Conc.	and arc BKC = arc ELF.
7	Rec.	∴ *In equal circles, equal angles; &c.* Q. E. D.

Cor. 1.—Since by Cor. 4, Prop. 22, bk. III, if the opp. ∠s of a qu. lat. in a circle are equal, those ∠s are rt. ∠s, it follows from Prop. 26, *if the opp. ∠s be equal their opp. diagonal must be a diameter*, and, *the segment a semicircle.*

Cor. 2.—In the same or equal circles one central or circumferential angle is less than, equal to, or greater than another, as the arc of the one is less than, equal to, or greater than, the arc of the other.

Cor. 3.—The diameters which intersect at rt. angles divide the circumference into four equal arcs, or the circle into four equal parts.

Cor 4.—When the sum of the central angles equals four right angles, the sum of their arcs equals the whole circumference.

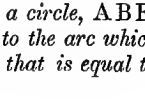
Cor. 5.—When the sum of the angles at the circumference equals two rt. angles, the sum of their arcs also equals the whole circumference.

Cor. 6.—Similar arcs of equal circles are equal.

Cor. 7.—*Parallel chords*, AE, CD, *of a circle intercept equal arcs*, AC ED.

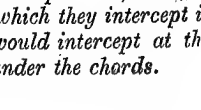
Cor. 8.—*If two chords*, AB, CD *intersect* within *a circle*, ABE, *the sum of the intercepted arcs*, AC + DB *is equal to the arc which an angle would intercept at the circumference*, BAE, *that is equal to the angle*, BFD, *under the chords.*

C.	31, I.	Draw AE ∥ CD.
D.1	C.29, I.26, III.	∵ AE ∥ CD ∴ ∠ EAF = ∠ AFC, and ∴ arc ED = arc AC;
2	*Add.*	To arc ED add arc. DB, ∴ arc EB = arc AC + arc DB;
3	D. I.	but ∠ BAE *i. e.* FAE = ∠ AFC;
4	15, I. Ax. 1.	and ∠ AFC = ∠ BFD,
5	Conc.	∴ ∠ BAE = ∠ BFD. Q. E. D.



Cor 9.—*If two chords*, AB, CD, *intersect at a point* F, without *a circle the difference of the arcs*, BD ~ AC, *which they intercept is equal to the arc*, BE, *which an angle* BAE, *would intercept at the circumference that is equal to the angle* BFD, *under the chords.*

C.	31, I.	Draw AE ∥ CD or FD.
D.1	as in Cor. 8.	then arc AC = arc DE
2	*Sub.* Ax. 3.	from arc BD take arc DE, ∴ arc BE = arc BD − arc AC.
3	29, I.	and ∠ EAB = ∠ BFD.



Lardner's *Euclid p.* 113.

Use and App. 1.—Since, by Cor. 8, of the preceding Prop. ∠ BFD = ∠ BAE, and by Cor. 20, III, an ∠ at the ⊙ce, BAE, on an arc EB is measured by half the arc, BE, on which it stands, namely by $\frac{BE}{2}$ this property

is available *for finding the true central angle of an imperfectly constructed theodolite*, or of any similar instrument, in which the revolving limbs AF, FB, in the last figure but one, are not at the centre of the circle; thus, if the arc AC, as shown by one limb FA, is 50° 20′ and the arc BD, as shown by the other limb FB, is 48° 12′ *the true central angle* $= \frac{50° 20' + 48° 12'}{2} = \frac{98° 32'}{2} = 49° 16'$.

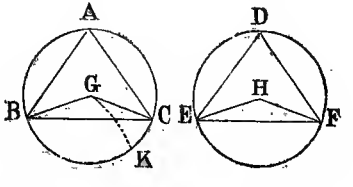
2. We may also apply Cor. 9 to determine the ∠ DFB, when the arcs AC and BD are given; for if arc AC = 48°, and arc BD = 100°, then ∠ DFB = 100° − 48° = 52°.

PROP. 27.—THEOR.

In equal circles the angles which stand upon equal arcs are equal to one another, whether they be at the centres or the circumferences.

CON. 23, I.

DEM. 20, III. Ax. 7. 26, III. Ax. 1. Ax. 9.



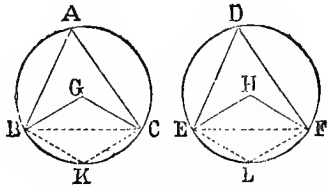
E.1	Hyp. 1.	Let ABC, DEF, be eq. ⊙s, G and H being the centres;
2	Hyp. 2.	and let ∠s BGC, EHF, at G and H, and ∠s BAC, EDF, at the ⊙ces, stand upon the eq. arcs BC, EF;
3	Conc.	then ∠ BGC = ∠ EHF and ∠ BAC = ∠ EDF.

SUP.—If ∠ BGC = ∠ EHF, then (20, III., and Ax. 7) ∠ BAC = ∠ EDF; but if not, one must be the greater.

C.1	Sup.	Let ∠ BGC > ∠ EHF.
2	23, I.	At G in BG make ∠ BGK = ∠ EHF.

D.1	C.26, III.	Then ∵ ∠ BGK = ∠ EHF, ∴ arc BK = arc FE;
2	H.2.Ax.1	but arc EF = arc BC, ∴ arc BK = arc BC;
3	Ax. 9.	*i. e.* the less = the gr., which is impossible;
4	Remk.	∴ ∠ BGC not ≠ ∠ EHF, *i. e.* ∠ BGC = ∠ EHF;
5	20, III.	but ∠ at A = ½ ∠ BGC, and ∠ at D = ½ ∠ EHF;
6	Ax. 7.	∴ ∠ at A = ∠ at D,
7	Rec.	∴ *In equal circles, the angles, &c.* Q. E. D.

Cor. I.—*In the same or in equal circles,* ABC, DEF, *the sectors* BGC, EHF, *which stand upon equal arcs,* BC, EF, *are equal,* and conversely.

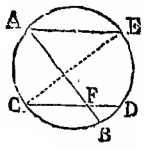


C.	Pst. 1.	Join BC, EF; and from ⋅s K, L, draw KB, KC, LE, LF.
D.1	Def.15,I.27,III.	∵ lines BG, GC = EH, HF, and ∠ BGC = ∠ EHF.
2	4, I.	∴ base BC = base EF, and △ BGC = △ EHF.
3	*Sub.*	From eq. ⊙s ABC, DEF, take equal arcs BC, EF;
4	Ax. 3.	∴ rem. arc BAC = rem. arc EDF;
5	Def. 11, III.	∴ ∠ BKC = ∠ ELF, and seg. BKC is sim. to seg. ELF;
6	D. 2 & 24, III.	but base BC = base EF, ∴ seg. BKC = seg. ELF.
7	*Add.*	To the eq. △s BGC, EHF, add the equal segs. BKC, ELF;
8	Ax. 2.	∴ the sector BGCK = the sector EHFL.

N.B. The *Converse* may be left for the student to demonstrate.

COR. II.—As in Cor. 7, P. 26. III., *if the chords* AE, CD, *of a circle are parallel, they intercept equal arcs*, and vice versâ.

C.	Pst. 1.	Join EC.
D.1	29, I. 26, III.	$\because \angle AEC = \angle ECD$, $\therefore$ arc ED = arc AC.
2	D. 1. 27, III.	Again $\because$ arc AC = arc ED, $\therefore \angle ECD = \angle AEC$;
3	27, I.	$\therefore$ AE ∥ CD.



SCH.—1. Propositions 26 and 27 are *converse* propositions; and what is true of equal circles is true of equal arcs in the same circle.

2. As in Cor. 2, Prop. 15, I., all the angles formed by any number of lines diverging from a common centre are together equal to four rt. angles, so the sum of the angles at the centre of a circle subtended by arcs, which together make up the whole circumference, is equal to four rt. angles. Also the sum of the angles at the circumference subtended by those same arcs is equal to two rt. angles.

3. And, since eq. arcs of eq. circles subtend eq. angles, such eq. arcs contain similar segments.

USE AND APP.—1. By Cor. II. of this Prop., a parallel through a given point E to a given st. line CD may readily be drawn; for join EC, and make $\angle CEA = \angle DCE$, and AE is parallel to CD.

2. The principle on which *the area of a sector* is ascertained may be developed from Cor. I. of this proposition, for the area of the triangle BGC, added to the area of the segment BKC gives the area of the sector BGEK.

Or, when the rad. and $\angle BGC$ are given, by principles hereafter to be proved, the Area of the Sector $= \frac{\text{Area of } \odot \times \angle BGC}{360°}$.

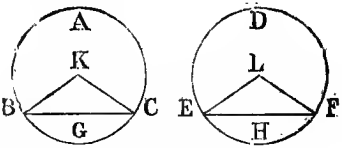
PROP. 28.—THEOR.

In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less.

CON. I, III. Pst. 1.

DEM. Def. 1, III. 8, I. 26, III. Ax. 3

E.1	Hyp. 1.	Let ABC and DEF be eq. ⊙s, and BC, EF eq. st. lines in them;
2	,, 2.	and let BC, EF cut off two gr. arcs BAC, EDF, and two less BGC, EHF;
3	Conc.	then the gr. arc BAC = the gr. EDF; and the less BGC = the less EHF.
C.	1, III. Pst. 1.	Take K, L, centres of the ⊙s, and join KB, KC, LE, LF.



D.1	H. Def. 1, III.	∵ ⊙ ABC = ⊙ DEF, ∴ KB, KC = LE, LF, each to each;
2	H. 8, I.	and BC = EF, ∴ ∠ BKC = ∠ ELF;
3	26, III.	∴ the arc BGC = the arc EHF;
4	H. Ax. 3.	but ⊙ ABC = ⊙ DEF; ∴ rem. arc BAC = rem. arc EDF.
5	Rec.	Therefore, *in equal circles, equal st. lines, &c.* Q. E. D.

Sch. As in other instances, the principle of the proposition extends to equal st. lines in the same circle, in which also such equal st. lines cut off equal *arcs.*

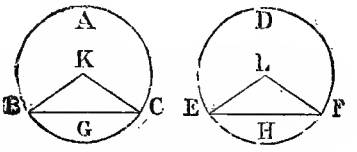
Prop. 29.—Theor.

In equal circles equal arcs are subtended by equal straight lines.

Con. 1, III. Pst. 1.

Dem. 27, III. Def. 1, III. 4, I.

E.1	Hyp.	Let ⊙ ABC = ⊙ DEF, and arc BGC = arc EHF,
2	Conc.	then on joining BC, and EF, chord BC = chord EF,
C.	1, III. Pst. 1.	Find K, L centres of the ⊙s, and join KB, KC, LE, LF.



D.1	H. 27, III.	∵ arc BGC = arc EHF, ∴ ∠ BKC = ∠ ELF;
3	H.Def.1,III.	and ∵ ⊙ ABC = ⊙ DEF, ∴ BK, KC = EL, LF each to each;
3	D. 1 and 4, I.	and ∠ BKC = ∠ ELF, ∴ chord BC = chord EF;
4	Rec.	Therefore, *in equal circles equal arcs &c.* Q. E. D.

Cor. 1.—By the same kind of demonstration it may be shown that, *in the same or in equal circles, equal sectors stand upon equal arcs; and conversely.*

Cor. II.—From Prop. 26, 27 and 29, *straight lines which intercept equal arcs are parallel;* and *parallel st. lines intercept equal arcs;* for the alternate angles are equal.

Use and App.—1. We may declare generally that whatever has been proved with respect to equal circles is also true when applied to the same circle.

2. In Spherical Trigonometry, Props. 26, 27, 28, and 29 are of continual use. By means of Props. 27 and 28, Theodosius demonstrated that the arcs of the circles of the Italian and Babylonian hours, comprehended between two parallels, are equal; and, in the same way, it may be proved that the arcs of circles of the astronomical hours, comprehended between the two parallels to the equator, are equal.

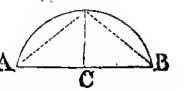
Prop. 30.—Prob.

To bisect a given arc of a circle, i. e., to divide it into two equal parts.

Con. 10, I. 11, I. Pst. 1.

Dem. Def. 10, I. 4, I. 28, III. Cor. 1, III.

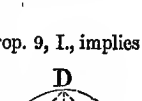
E.1	Datum.	Let ADB be the given arc;
2	Quæs.	it is required to bisect it.
C.1	Pst.1 & 10.11,I.	Join D B, and bis. A B in C by perp. C D;
2	Sol.	then the arc A D B is bis. in D, *i. e.* arc A D = arc B D.
3	Pst. 1.	Join A D, and D B.
D.1	C.1. Def. 10, I.	∵ AC = BC, CD com. and ∠ ACD = ∠ BCD,
2	4, I.	∴ base A D = base B D.
3	28, III.	But eq. st. lines cut off eq. arcs, the gr. = the gr., and the less = the less;
4	Cor. 1, III.	and ∵ D C passes through the cen., arcs A D, D B, each < a semicircle.
5	Conc.	∴ arc AD = arc D B, and ADB is bis. in D.



Q. E. F.

Con.—Hence, by successive bisections, as in Sch. 1 and 3, Prop. 9, I., *a given arc may be divided into any number of equal parts that are the powers of two*, as 4, 8, 16, 32, &c.

Sch.—1. The bisection of a given rectil. angle, A D B, Prop. 9, I., implies a bisection of A D B, an arc of a circle; but, as by Plane Geometry, a rectil. angle, except in the case of a right angle, D C B, cannot be divided into 3, 5. 6, &c., equal parts, so an arc of a circle, except in the case of a quadrant, D B, which is the circular measure of a right angle, D C B, cannot be cut into 3, 5, 6, &c., equal parts.

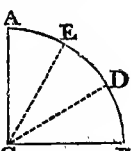


N.B. If, as is mentioned below, other plane curves, besides the circle, had been admitted by Euclid, any angle could also be divided into 3, 5, 6, &c., equal parts.

2. *To trisect. a quadrant,* AB, *i. e.* to divide a rt. ∠, ACB, into three equal parts.

C. 1	1, I.	On CB const. an equil. △ BEC,
2	9, I.	and bis. its ∠ BCE by CD,
3	Sol.	the rt. ∠ ACB will be trisected, *i. e.* ∠BCD = ∠DCE = ∠ ECA.
D. 1	Cor. 32, I. C. 2.	∵ ∠ BCE = ⅔ of a rt. ∠, and is bis. by CD;
2	Conc.	∴ ∠ BCD = ∠ DCE.
3	H. & D. 1.	Also ∵ ACB is a rt. ∠, and BCE = ⅔ of a rt. ∠;
4	Ax. 3.	∴ ∠ ACE = ⅓ of a rt. ∠;
5	Conc.	∴ ∠ ACE = ∠ ECD = ∠ DCB, *i. e.* the quadrant is trisected.

Or,—From A and B, with the rad. of the circle, describe arcs cutting the quadrant in D and E; join EC, DC, and the quadrant is trisected.

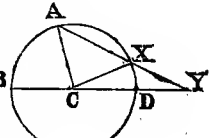


3. By successive bisections of the one-third of a rt. ∠, the 1-6th, 1-12th, 1-24th, &c., of a rt. ∠, or of a quadrant is obtained.

4. The division of a quadrant into five equal parts depends on P. 10, IV.

5. Since EUCLID confines himself to straight lines and circles, *the trisection of an angle,* or. *of an arc cannot be effected by his Geometry*; if, however, other curves, formed by the sections of the Cone, were admitted among the curves of our Plane Geometry, the problem could readily be solved. For, "if with two-thirds of any given line, A, as a major axis, an hyperbola be described whose asymptotes," or incoincident lines, "make an angle of 120°; and if with A as a base, and a point on the branch of the hyperbola adjacent to the single third of A as a vertex, a triangle be described, the larger of the angles adjacent to A will always be double of the smaller. Consequently, one of the external angles will be triple of one of its internal and opposite angles; so that by describing on a straight line A a segment of a circle containing the supplement of any given angle less than 180°, that circle will cut the branch of the hyperbola in a point which, being joined with the further extremity of A, will give an angle equal to the given angle."—PENNY CYCL., XXV., p. 260.

6. What is required for the trisection of an arc or of an angle is the solution of the following problem; "from a given point, as A, in the circumference of a ⊙ ABD, to draw a st. line, AXY, such that the part XY, between the circumference and a given diameter BD produced, shall be equal to the radius CA."



D. 1	H. 6, I.	∵ CX = XY, ∴ ∠ XCY = ∠ XYC;
2	32, I.	but ∠ AXC or CAX = 2∠ XCD, and ∠ ACB = ∠ XCD + ∠ AXC;
3		∴ ∠ ACB = 3 ∠ XCD, ∴ arc AB = 3 arc XD.
4	Conc.	Thus ∠ XCD = ⅓ of ∠ ACB, and ∴ arc XD = ⅓ arc AB.

7. One of the *Trochoidal* curves, known by the name of the *trisectrix*, is peculiarly possessed of the property of dividing any arc into three equal parts.

USE AND APP.—By this problem, the semicircle is divided into quadrants, and the quadrant into arcs of 45°, 22½°, &c.; and the Mariner's Compass, as in Use 4, 9, I., into 32 equal parts called Rhumbs; but the division into single degrees cannot be performed by Euclid's Geometry.

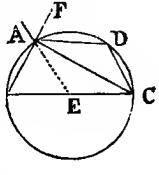
PROP. 31.—THEOR.

In a circle, the angle in a semicircle is a right angle, but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

CON. 10, I. Psts. 1 & 2.

DEM. Def. 15, I. 5, I. AXS. 1, 2. Def. 10, I. 22, III.
32, I. If a side of a △ be produced, the ext. ∠ = the two int. and opp. ∠s; and the three int. ∠s of every △ are together equal to two rt. ∠s.
17, I. Any two angles of a △ shall together be less than two rt. angles.

E.1	Hyp. 1.	Let A B C be a ⊙, B C its diam. and E its cen.
2	,, 2.	and from C let C A divide the ⊙ into segs. A B C, A D C, of which seg. A B C is > a semic. and A D C < a semic.
3	Conc. 1.	Then ∠ B A C in the semic. is a rt. ∠;
4	,, 2.	∠ A B C in seg. A B C is < a rt. ∠;
5	,, 3.	and ∠ A D C in seg. A D C is > a rt. ∠.
C.1	10, I.	Bisect diam. B C in E; E is the cen.
2	Pst. 1 & 2	join E A, B A; and produce B A to F;
3	Pst. 1.	in arc A D C take any · D, and join A D, D C.



CASE I.—*The* $\angle$ BAC *in the semicircle shall be a rt.* $\angle$.

D.1	Def. 15, I. 5, I.	Now, $\because$ EB = EA = EC; $\therefore$ $\angle$ EBA = $\angle$ EAB, and $\angle$ EAC = $\angle$ ECA;
2	*Add.* Ax. 2.	add the eqs. and the whole $\angle$ BAC = the two $\angle$s ABC, ACB;
3	32, I.	but in $\triangle$ ABC, the ext. $\angle$ FAC = the two int. $\angle$s ABC, ACB;
4	Ax. 1. Def. 10, I.	$\therefore$ $\angle$ BAC = $\angle$ FAC, and $\therefore$ each is a rt. $\angle$.
5	Conc.	$\therefore$ the $\angle$ BAC in the semic. is a rt. $\angle$.

CASE II.—*The* $\angle$ ABC, *in a seg.* ABC *gr. than a semic., shall be less than a rt.* $\angle$.

D.1	17, I.	And $\because$ in $\triangle$ ABC the $\angle$s ABC, BAC are $<$ 2 rt. $\angle$s.
2	Case I.	and that $\angle$ BAC is a rt. $\angle$;
3	Conc.	$\therefore$ $\angle$ ABC must be $<$ a rt. $\angle$.
4	Rec.	$\therefore$ the $\angle$, in a seg. $>$ a semic., is less than a rt. $\angle$.

CASE III.—*The* $\angle$ ADC, *in a seg.* ADC $<$ *a semic., shall be gr. than a rt.* $\angle$.

D.1	C. 22, III.	$\because$ ABCD is a quadrilateral in a $\odot$; $\therefore$ $\angle$s ABC, ADC = 2 rt. $\angle$s;
2	Case II.	and $\because$ $\angle$ ABC is $<$ a rt. $\angle$;
3	Conc.	$\therefore$ the other $\angle$ ADC is $>$ a rt. $\angle$.
4	Rec.	$\therefore$ *In a circle the angle, &c.*

COR.—*If one angle of a triangle be eq. to the other two, it is a rt. angle.*

E.1	Hyp.	For in $\triangle$ ABC, let $\angle$ BAC = $\angle$s ABC + ACB;
2	Conc.	then $\angle$ BAC is a rt. $\angle$.
C.	Pst.	Produce BA to F.
D.1	32, I.	$\because$ $\angle$ FAC = $\angle$s ABC + ACB; $\therefore$ $\angle$ FAC = $\angle$ BAC,
2	Def. 10, I.	$\therefore$ $\angle$ BAC is a rt. $\angle$.

SCH.—The converse of Prop. 31, is, "the segment which contains an acute angle is greater than a semicircle, and that which contains an obtuse angle is less than a semicircle."

2. The Demonstration which LARDNER gives of the 31st Prop. is remarkable for its elegance and brevity; it is founded on Prop. 20, Book III.;—

∵ the central ∠ on a semicircle = two rt. ∠s;
∴ the circumferential ∠ = one rt. ∠.

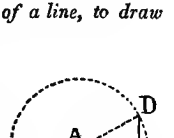
Again, ∵ the cen. ∠ on an arc less than a semic. is less than two rt. ∠s;
∴ the circumferential ∠ is < a rt. ∠.

And ∵ the cen. ∠ on an arc gr. than a semic. is gr. than two rt. ∠s;
∴ the circumferential ∠ is > a rt. ∠.

USE AND APP.—I. From the property, Case I., P. 31, III., that the angle in a semicircle is a rt. angle, the following Problems are derived:

PROB. 1. *From a point* B, *in a line, or at the extremity of a line, to draw a perpendicular.*

From any • A, out of the line, with the distance AB, describe a semic. meeting CA produced in D, and join BD; then BD is the perp.



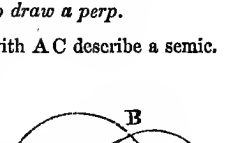
N.B. PELITARIUS, a mathematician, often quoted by BILLINGSLEY, gives this Cor. to Pr. 31.—"If in a circle be inscribed a rectangle triangle, the side opposite unto the right angle shall be the diameter of the circle."

PROB. 2.—*From a point* D, without CB *a line, to draw a perp.*

Join D, C, and bisect DC in A; and from A with AC describe a semic. and join AB; DB is the perp. required.

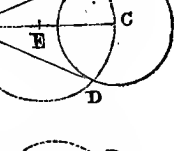
PROB. 3.—*From a point* A, without *a circle, to draw a tangent.*

Join A and the cen. C; bis. AC in E; and from E with EA desc. a circle; and draw AB and AD; then AB and AD are both tangs. from A to the ⊙ BCD.



The Demonstration may be left to the Student.

Or, join A, C; 10, I., bis. AC in D, and with DA desc. a semicircle ABC; where the semic. cuts the ⊙ B, is the tangent point; and AB the tangent from A.

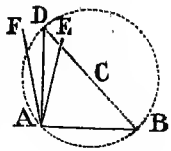


II. By means of a square *the centre of a circle may be easily found;*

For, if B the angular point of the square, CBD (as in the fig. to Prob. 1. above) touch any point in the ⊙ce; and if also the sides of the square, BD, BC, fall upon two other points, C, D of the ⊙ce; then the line CD is a diam. and its middle point the centre of the circle.

III. In this proposition workmen possess a *very simple way of trying if their squares be exact;*

For on the hypotenuse D B desc. a semicircle D A B, and apply the angular point A of the square on the ⊙ce of the circle, and one of the sides of the sq. A B, so that the point B of the square may fall upon B the extremity of the diam ; then if D, the extremity of the other side of the square, fall upon D the other extremity of the diam. B D, the square is correct ; but if that other side, as A E, falls within the circle, the ∠ E A B is < a rt. ∠ ; if without, as A F, the ∠ F A B is > a rt. ∠.



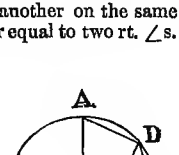
PROP. 32.—THEOR.

If a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle ; the angles which this line makes with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

CON. 11, I. Pst. 1.

DEM. 19, III. 31, III. 32, I. AXS. 1, 2 & 3. 22, III.
13, I. The adj. ∠s which one st. line makes with another on the same side of it shall either be two rt. ∠s, or be together equal to two rt. ∠s.

E.1	Hyp. 1.	Let E F touch ⊙ A B C D in B;
2	,, 2.	and from B let B D cut the ⊙;
3	Conc. 1.	then ∠ F B D = ∠ B A D in the altr. seg. D A B.
4	Con. 2.	and ∠ D B E = ∠ B C D in the altr. seg. D C B.
C.1	11, I.	From B draw B A ⊥ E F, cutting the ⊙ in A;
2	Pst. 1.	in arc. D B take any • C, and join A D, D C, C B.



CASE I.—*The* ∠ F B D = ∠ B A D *in the altr. seg.* D A B.

D.1	H. and C. 1.	∵ E F touches the ⊙ in B ; and B A ⊥ from B;
2	19, III. 31, III.	∴ the cen. of the ⊙ is in B A, and ∠ A D B is a rt. ∠ ;

D.3	32, II.	and ∴ also ∠s BAD, ABD, = a rt· ∠ ;
4	C. Ax. 1.	but ABF is a rt. ∠, ∴ ∠ ABF = ∠s BAD, ABD ;
5	*Sub.*	take away ∠ ABD ;
6	Ax. 3.	∴ rem. ∠ DBF = rem. ∠ BAD, in the altr. seg of the ⊙.

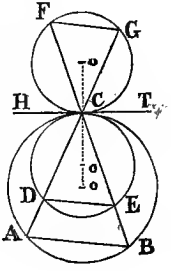
CASE II.---*The* ∠ DBE = ∠ BCD *in the altr. seg.* DCB.

D.1	C.	∵ fig. ABCD is a quadrilateral in a ⊙ ;
2	22, III.	∴ the opp. ∠s BAD + BCD = 2 rt. ∠s ;
3	13, I.	but ∠s DBF + DBE = 2 rt. ∠s ;
4	Ax. 1. 5, I.	∴ ∠s DBF + DBE = ∠s BAD + BCD, and ∠ DBF = ∠ BAD ;
5	*Sub.*	take away the equals DBF and BAD ;
6	Ax. 3.	∴ rem. ∠ DBE = rem. ∠ BCD in the altr. seg. of the ⊙.
7	Rec.	∴ *If a st. line touches a circle, &c.* Q. E. D.

COR. I.—Also, conversely, "*if from the end,* B, *of a line,* DB, *cutting the circle, there be drawn a st. line,* EF, *such that the* ∠*s* FBD, EBD, *which it makes with the cutting line, are equal to the* ∠*s* BAD, BCD, *in the altr. seg. of the* ⊙, *that st. line must touch the circle.*"

COR. II.—*If two or more circles,* ABC, DEC, FGC, *touch each other,* either externally *or* internally, *and through the point of contact,* C, *two st. lines,* AG, BF, *be drawn, meeting their* ⊙*ces, the chords* AB, DE, FG, *of the intercepted arcs will be parallel.*

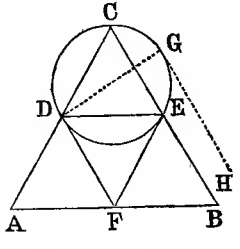
C.	17, III.	Through C, draw HT a tangent.
D.1	11,12,III.	∵ a line joining the centres passes through C ;
2	18, III.	and ∵ HT makes rt. ∠s with that line ;
3	16, III.	∴ HT is a tang. to the ⊙s ABC, DEC.
4	32, III.	But ∠ HCF = ∠ CGF in altr. seg. of ⊙ FCG ;
5	15, I. Ax. 1	and ∠ HCF = ∠ TCB ; ∴ ∠ CGF = ∠ TCB ;



D.6	32, III.	also $\angle$ TCB = $\angle$ CAB in the altr. seg. of $\odot$ ABC;
7	Ax. 1.	$\therefore$ $\angle$ CGF = $\angle$ CAB, and they are alt. $\angle$s;
8	27, I.	$\therefore$ FG and AB are parallel.
9	*Sim.*	So it may be shown that FG $\parallel$ DE.
10	Rec.	Therefore, *if two or more circles, &c.* Q. E. D.

Cor. III.—Also, *if two or more circles touch each other, either* internally *or* externally, *at a common · of contact*, C, *any line, as* AG, *passing through the · of contact, will cut off sim. segments from each; i. e.*, the segs. ACA, DCD, and GCG, are similar.

Cor. IV.—*In an equil* $\triangle$, ABC, *if the sides be bisected in* D, E, F, *and st. lines be drawn joining the points of bisection, of those lines, two* FD, FE, *will be tangents to the circle* DEC, *which passes through* D, E, *the ends of the other line, and* C *the angular point opp. to that line.*



D.1	C.	$\because$ DEF is an eq. lat. $\triangle$;
	Sch.4,32,I.	$\therefore$ $\angle$ FDE = $\frac{2}{3}$ of a rt. $\angle$;
2	C.	also $\because$ ABC is an eq. lat. $\triangle$;
		$\therefore$ $\angle$ ACB = $\frac{2}{3}$ of a rt. $\angle$;
3	Ax. 1.	$\therefore$ $\angle$ C = $\angle$ FDE;
4	C.	but $\angle$ DCE is in the altr. seg. of $\odot$ CDE;
5	Cor.32,III.	$\therefore$ DF, and also EF, touch the $\odot$ in D & E.

Cor. V.—On the same principle, *tangents*, as FD, FE, *through the extremities*, DE, *of the same chord*, DE, *make the angles on the same side* of DE *equal; i. e.* $\angle$ FDE = $\angle$ FED.

Cor. VI.—Also, *if tangents, as* DF, GH, *are parallel, the line* DG *joining the points of contact*, D, G, *is a diameter*; for $\angle$ FDG = $\angle$ HGD, and 29, I., each is a rt. $\angle$; and 19, III., the line DG, from the points of contact, D and G, passes through the centre; $\therefore$ Def. 17, I., DG is a diameter.

Sch.—The 32nd Prop. is sometimes enunciated thus—"If a tangent be drawn to a circle, and from the point of contact a line be drawn cutting off an arc, the angle, between the tangent and the line cutting the circle, will be equal to an angle at the circumference of the circle standing on the arc cut off."

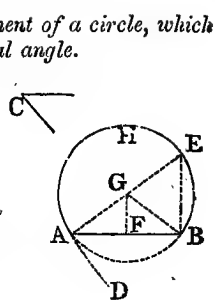
Use and App.—The Proposition is preliminary to the proof of Prop. 33, and is required for the demonstration of all Theorems dependent on the equality of the angles formed by a tangent and secant, and the angles in the alternate segments of the circle.

Prop. 33.—Prob.

Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Sol. 10, I. 23, I. 11, I. Pst. 1 & 3.

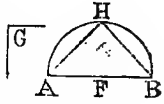
Dem. 31, III. Def. 10, I. Def. 15, I. 4. I. Cor. 16, III. 32, III. Ax. 1.



E.1	Data.	Given A B a st. line, and C a rectil. ∠;
2	Quæs.	on AB to desc. a seg. of a ⊙ with an ∠ = ∠ C.

Case I.—*Let the given angle* C *be a rt.* ∠.

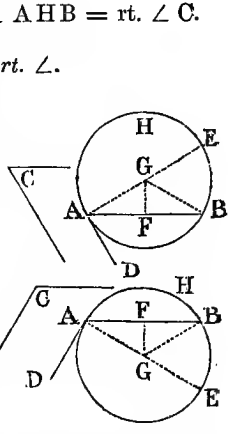
C.1, 10, I. Pst. 3.		Bis. AB in F; and from F, with FA, desc. the semic. AHB;
2	*Sum.* Pst. 1.	Take a · H in the arc, and join HA, HB;
3	Sol.	then AHB is the seg. required.



D. C. 31, III. ∵ AHB is a semic.; ∴ ∠ AHB = rt. ∠ C.

Case II.—*But if the given* ∠ C *be not a rt.* ∠.

C.1	23, I.	At A in AB make ∠ BAD = ∠ C;
2	11, I.	from A draw AE ⊥ AD;
3	10.11, I. Pst. 1.	bis. AB in F by perp. FG; and join GB;
4	Sol.	then the seg. AHB is the seg. required.
D.1	C. 3. Def. 10, I.	∵ AF = FB, FG com., and ∠ AFG = ∠ BFG;
2	4, I. Def. 15, I.	∴ AG = GB, and ⊙ from G, with GA, passes through B.
3	*Sum.*	Let this ⊙ be AHB; in seg. AHB the ∠ = ∠ C.



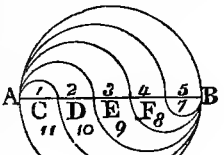
D.4	C.2.Cor.16,III.	∵ at A, AD is ⊥ to AE a diam.,
		∴ AD is a tang.;
5	C.	and ∵ AB from the · of contact, A, cuts the ⊙,
6	32, III.	∴ ∠ DAB = ∠ in the altr. seg. AHB;
7	C. 1. Ax. 1.	but ∠ DAB = ∠C; ∴ ∠C = ∠ in the seg. AHB.
8	Rec.	∴ *upon the given line is described, &c.*

Q. E. F.

Sch.—Though not belonging to this Proposition, yet from relating to the division of a circle into segments bounded by curved lines, it is *convenient* here to introduce the following Problem :—

To divide a given circle, of which the diameter is AB, *into any number of equal parts, of which the perimeters also are equal.*

By Use and App. 2, P. 34, I., divide the diam. AB into the required number of eq. parts, as C, D, E, F, B; then on one side of the diam., beginning from the extremity A, desc. the semicircles numbered 1, 2, 3, 4, 5, &c., of which the diameters are AC, AD, AE, AF, &c.; and on the other side, beginning from B, the semicircles 7, 8, 9, 10, 11, &c., of which the diameters are BF, BE, BD, BC, &c. ; then of the segments bounded by curved lines, 1 + 11 = 5 + 7, 2 + 10 = 4 + 8; and all the parts thus taken in pairs = 3 + 9, and are equal, one curved segment to the other, both in area and perimeter.—Leslie's Geometry.



By Def. 1. III., the semicircle, of which AB and BA are diameters, are equal ; and ∵ diam. AC = diam. BF, diam. AD = diam. BE, and diam. AE = diam. BD, and diam. AF = diam. BC ; ∴ semicircles 1 and 7 are equal, 2 and 8, 3 and 9, 4 and 10, 5 and 11.

If from semic. on AB, equal to semic. on BA, we take semic. on AF and semic. on BC, the remaining space No. 5 = rem. space No. 11; and space 1 = space 7; ∴ the whole area 5 + 7 = the whole area 1 + 11 ; and in like manner for the areas of all the other curved lined segments.

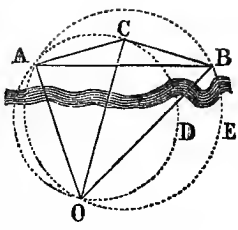
Also the perimeter of semic. on AB = that on BA; of semic. 1 = that of semic 7; of semic. 5=that of 11, &c. ; ∴ the perim. of fig. 5 + 7 = the perim. of 11 + 1 ; and so on for the perims. of the other curved lined segments.

Use and App.—The cases are very numerous in which the 33rd Prop. may be employed.

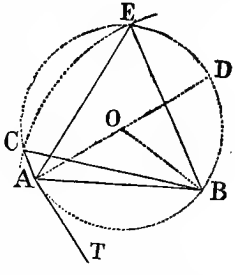
I.—In Coast Surveying, for noting soundings and bearings, this Problem is of great importance and utility; when from a map, or by some other means, the distances of three objects are known, and the observer wishes to ascertain his own position relatively to them. By a sextant the angles at the place of observation between the respective distances are taken ; and from these angles and distances the construction or calculation is made. Thus—

Given the distances of three land-marks, A, B, C, *from each other; required their distances from* O, *the place of observation.*

With a sextant take ∠s AOC, COB, AOB;
through A and B draw a ⊙ of which the ∠ = ACB;
and through A and C draw a circle of which the ∠ = AOC;
then, unless the observer be in the ⊙ce of the ⊙ passing through A, C, and B, it is evident the ⊙s ABE and ACD cut in the point O, where the observer stands; ∴ the lines OA, OC, OB, are *representative* of the actual distances from the observer to the three given places.



II. *On a given line* AB, *measuring* 17 *miles to describe a seg. of a circle which shall contain an* ∠ *of any fixed number of degrees as* 55°, *being the angle between the line* AB *and* C, *a station* 19 *miles from* B; *and to ascertain the distance of* C *from* A.



C.1	3, I.	From a scale of eq. pts. set off AB = 17;
2	33, III.	desc. a seg. of a ⊙ CDB with ∠ ACB = 55°;
3	Pst. 3.	from B with rad. = 19 cut the seg. in C and E;
4	Pst. 1	join CA, EA, and EB;
5	Sol.	then AC or AE = the distance required;
6		applied to the scale AC = 3·5 and AE = 19.

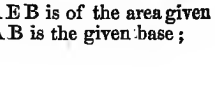
N.B. The distance = 19, cuts the seg. in two parts, C and E, and thus the problem has two solutions.

III. *Given the base* AB *and vertical* ∠ = ∠ TAB, *to find the* locus *of the vertex.*

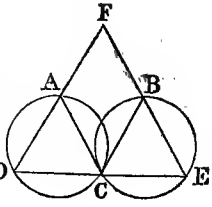
C.	33, III.	On AB desc. a seg. with ∠ = ∠ TAB.
D.1	21, III.	∵ all the ∠s in seg. ACDB are equal;
2	Conc.	∴ the *locus* of the vertex is in that seg.

IV. *Given the vert.* ∠, *the base* AB, *and the area to construct a triangle.*

C. 1	(41, I.)	On AB const. rect. ADCB = *twice* the area;
2	33, III.	and on AB *a* seg. AEB with ∠ = the given ∠;
3	Pst. 1. Sol.	join EA, EB; and AEB is the req. △.
D. 1	41, I.	∵ △ AEB = ½ rect. AC; ∴ △ AEB is of the area given;
2	C.	and ∠ AEB = the vert. ∠, and AB is the given base;
3	D. 1, 2.	∴ △ AEB is the △ required.

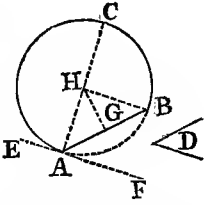


V. *Through three given points,* A, B, C, *to draw three st. lines, so as to make an equilateral triangle.*



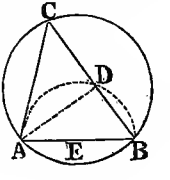
C. 1 | 33, III. | Join A C, B C, and on each desc. a seg. with an $\angle = \frac{2}{3}$ of a rt. $\angle$, as seg. A D C, B E C;
2 | Pst. 1. | through C draw any line E D, cutting the $\odot$s in D, E;
3 | Psts. 1, 2. | join D A, E B; and prod. them to meet in F;
4 | Sol. | then $\triangle$ D E F is the eq. lat. $\triangle$ required.
D. 1 | C. | $\because$ in seg. A D C $\angle$ D $= \frac{2}{3}$ of a rt. $\angle$; and in seg. B E C $\angle$ E $= \frac{2}{3}$ of a rt. $\angle$;
2 | Ax. 1. | $\therefore \angle$ D $= \angle$ E.
3 | 32, I. D. 1. | And $\because \angle$s D + E + F = 2 rt. $\angle$s; and $\angle$s D + E $= \frac{4}{3}$ rt. $\angle$s;
4 | Ax. 3. C. | $\therefore \angle$ F $= \frac{2}{3}$ of a rt. $\angle$; and $\therefore$ $\triangle$ D E F is eq. angular;
5 | Cor. 6, I. | and $\therefore$ also it is equilateral. Q. E. F.

VI. *Given the* $\angle$ D *equal to the vertical* $\angle$ *of a triangle, and the base* A B, *to find the* locus *of the vertex.*



At A, by 23, I., make the $\angle$ B A F = $\angle$ D; and 11, I., $\angle$ F A H = a rt. $\angle$; bisect, 10 and 11, I., A B in G by the perp. G H; and from H, with rad. H A or H B, desc. a $\odot$ A B C; the *locus* of the vertex will be at any point in the arc of the segment A C B.

VII. *Given* A B *the base,* $\angle$ A C D *the vertical* $\angle$, *and* A D *the perp. from the extremity of the base* A *on the opp. side* B C, *to construct the trangle.*



On A B, by 33, III., make a segment containing the given $\angle$ A C D; bisect A B, 10, I., in E; and with rad. E A or E B desc. the semicircle A D B; and from A inflect the perp. A D upon the semicircle in the point D; B D A is a rt. $\angle$, 31, III.; and B D produced to C, and C A joined, give the triangle required.

The Demonstrations may serve as Exercises.

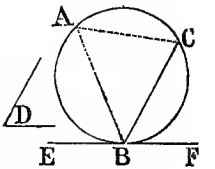
Prop. 34.—Prob.

From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

Con. 17, III. 23, I.

Dem. 32, III. Ax. 1.

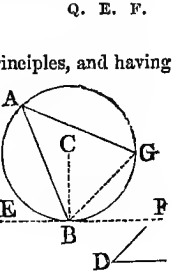
E. 1	Data.	Given the ⊙ A B C and the rectil. ∠ D;
2	Quæs.	to cut from the ⊙ a seg. with an ∠ = D.
C. 1	17, III.	Draw E F a tang. in B to the ⊙ A B C;
2	23, I.	at B in B F make ∠ F B C = ∠ D;
3	Sol.	then seg. B A C contains an ∠ = ∠ D.
D. 1	C. 1.	∵ the st. line E F touches the ⊙ A B C,
2	C. 2.	and ∵ B C is drawn from B the • of contact;
3	32, III.	∴ ∠ F B C = the ∠ in the altr. seg. B A C;
4	C. 2.	but ∠ F B C = ∠ D; ∴ in the altr. seg. the ∠ = ∠ D.
5	Conc.	∵ from the ⊙ A B C, the seg. B A C is cut off, containing an ∠ = the given ∠ D. Q. E. F.



Sch.—The following method, on exactly the same principles, and having the same demonstration, gives the *full* construction from Problems 11 and 23, Book I.

From a given ⊙, A B G *to cut off a segment which shall contain an* ∠ *equal to a given rectil.* ∠ D.

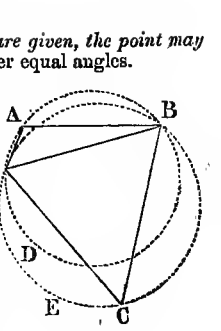
Draw any Rad. C B; and at B, 11, I., draw E F at rt. ∠s with C B; at B, the tangent-point of the rad., make the ∠ F B G = ∠ D; the segment B A G contains an ∠ B A G = ∠ G B F = ∠ D.



Use and App.—I. By the last two Problems, if three observations be taken, the eccentricity of the annual orbit of the earth and its aphelion may be found.

II. Also in Optics, *if two unequal lines*, AB, BC, *are given, the point may be ascertained where they will appear equal*, that is, under equal angles.

C. 1	2 & 3, I.	Place AB, BC so as to be conterm. in B;
2	33, III.	on AB, and BC, within the ∠ ABC, construct ⊙s, each containing eq. ∠s in their respective segs. ADB, BCE;
3	Pst. 1.	from F, the · of inters. of the ⊙s, draw FA, FB, FC;
4	Conc.	∠ AFB = ∠ BFC, and the lines AB and BC appear equal from · F.
D. 1	C.	∵ the ∠ in seg. ADB = a certain ∠;
2	C.	and ∵ the ∠ in seg. BCE = the same ∠;
3	Ax. 1.	∴ ∠ AFB = ∠ BFC, and AB appears = to BC. Q. E. F.



PROP. 35.—THEOR. *(Very Important.)*

If two st. lines cut one another within a circle, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

CON. 10, I. 12, I. 1, III. Pst. 1.

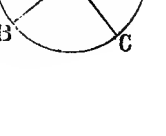
DEM. Def. 15, I. 3, III. 47, I. Axs. 1, 2, 3, I.
36, I. Parlms. upon eq. bases and between the same ‖s are equal.
5, II. If a straight line be divided into two equal parts, and also into two uneq. parts, the rect. contained by the uneq. parts, together with the square of the line between the · s of section, is equal to the square of half the line.

E. 1	Hyp.	Within ⊙ ABCD let AC and BD cut in E;
2	Conc.	then AE · EC = BE · ED.

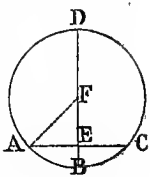
There are *four* Cases of this proposition, according as the intersecting lines pass through the centre of the ⊙ or not.

CASE I. — *Let both lines*, AC, BD, *pass through the centre* E.

D. 1	Def. 15, I.	∵ AE = EC = BE = ED;
2	36, I.	∴ AE · EC = BE · ED.

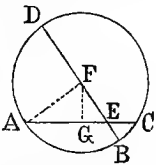


CASE II.—*Let* BD, *passing through the cen., cut* AC *not passing through the cen. at rt. ∠s in the point* E.



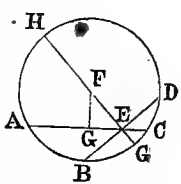
C.	10, I.	Bis. BD in F, the centre; join FA.
D.1	C. & 3, III.	∵ BD through F cuts AC not passing through F, and at rt. ∠s in E; ∴ AE = EC;
2	C.	and ∵ BD is div. equally in F, and uneq. in E;
3	5, II.	∴ BE · ED + EF² = FB², *i. e.* = FA²;
4	47, I.	but AE² + EF² = FA²;
5	Ax. 1.	∴ BE · ED + EF² = AE² + EF²;
6	*Sub.*	take away the com. sq. EF,
7	Ax. 3.	and rem. BE · ED = rem. AE²; *i. e.* AE · EC.

CASE III.—*Let* BD *passing through the centre cut* AC, *not passing through the centre, but* not *at rt. ∠s in the point* E.



C.1	10, I.	Bis. BD as before, & join AF.
2	12, I.	and from F, draw FG, perp. to AC.
D.1	C. 2. 3, III.	∵ FG through cen. F cuts AC ⊥; ∴ AG = GC;
2	C.	and ∵ AC is divided eq. in G and uneq. in E;
3	5, II.	∴ AE · EC + GE² = AG² or GC²;
4	*Add.* Ax. 2.	Add GF², ∴ AE · EC + GE² + GF² = AG² + GF²;
5	47, I.	but GE² + GF² = EF², & AG² + GF² = AF²;
6	Ax.1.Def.15,I.	∴ AE · EC + EF² = AF², *i. e.* = FB²;
7	5, II.	but FB² = BE · ED + EF²;
8	Ax. 1.	∴ AE · EC + EF² = BE · ED + EF²;
9	*Sub.* Ax. 3.	take away EF²; and AE · EC = BE · ED.

CASE IV.—*Let neither* AC *nor* BD *pass through the centre.*



C.1	1,III.Pst.1&2.	Find the cen. F; join EF; and prod. it to meet the ⊙ in H and G.
D.1	Case III.	∵ AE · EC = GE · EH; and BE · ED = GE · EH;
2	Ax. 1.	∴ AE · EC = BE · ED.
3	Rec.	∴ *If two st. lines cut one another, &c.* Q. E. D.

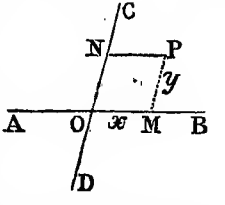
COR.—Conversely, *if the rectangles be equal, contained by two lines intersecting within a circle, the extremities of the lines*, A, C, B, D, *shall be in the circumference of the same circle.*

SCH.—1. The Proposition is sometimes enunciated thus—"If two chords of a circle cut one another, the rectangles under their segments terminating in the point of section shall be equal."

2. The rectangles are each equal to $Rad.^2 - EF.^2$.

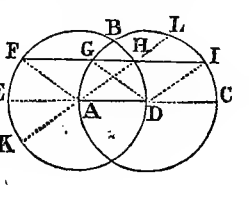
3. When, as in Case I., the fig. B A D is a semicircle, and A E, *the ordinate*, is perp. to B D, *the abscissa*, then the square of the ordinate is equal to the product of the abscissæ, *i. e.* $AE^2 = BE \cdot ED$.

Note.—The terms *ordinate* and *abscissa* may require some explanation When any two st. lines, A B, C D, in the same plane, meet in a common point O, that point may be considered as the *origin* of the lines from which they diverge, and the lines themselves as *axes*. To know the position of any point, as P, in the plane of the axes A B, C D, we must know—1st, between which of the angles P is, whether between $\angle$s A O C, A O D, B O C, or $\angle$ B O D; and, 2nd, how far P is from each axis, the distances being measured on parallels to the axes, *i. e.* on the sides of the parallelogram O M P N. Either of the parallels O N or O M is named the *abscissa*, and the other parallels, P N or P M, the *ordinates*, the *abscissa* being the line cut off, the *ordinate* the line which determines the point of section; the *two* are named *co-ordinates*, because together they order or determine the position of the point. Thus, with respect to the point P, if O M be the abscissa, P M is the ordinate. All points in P M have the same abscissa, all in P N the same ordinate. It is usual to denote the length of the abscissa by the letter x, and that of the ordinate by the letter y.



USE AND APP.—I. If of two equal circles, A B C, D B E, the centres A and D be each on the circumference of the other, and a com. chord F G H I be drawn parallel to A D, the line joining the centres, then the lines A F, A H, D G, D I, F G, and H I, joining the points F, G, H, I, where the com. chord cuts the circles, and A, D, the extremities of the line joining the centres, form parallelograms; and if A H be produced to meet the circumferences in K and L, G H = H L and F I = K L.

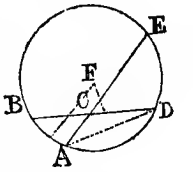
C.1	Pst. 2.	Prod. A D to the ⊙ces in E and C;
2	Pst. 1, 2.	join A H, and prod to K and L;
3	Pst. 1.	join A F, D G, and D I;
4	Conc. 1.	then figs. A D I H and A D G F are ▱s;
5	Conc. 2.	and G H = H L, and F I = K L.



D.1	H.Cor.2.27,III.	$\because$ FI $\parallel$ EC ; $\therefore$ arc FE = arc DH = arc IC ;
2	Cor. 2. 27, III.	and $\therefore$ also $\angle$ IDC = $\angle$ HAD ;
3	28, I. Def. A.	now AH $\parallel$ DI ; $\therefore$ fig. ADHI is a $\square$;
4	*Sim.*	and in like manner fig. ADGF is a $\square$.
5	34, I.	Hence HI = AD = FG ;
6	35, III.	but GH · HI = AH · HL ; and HI = AH
7	Ax. 3 & 2.	$\therefore$ GH = HL ; and FI = 2 AD + GH ;
8		also 2 AD + GH = KH + HL = KL,
9	Conc.	$\therefore$ FI = KL. Q. E. D.

2. By this Prop. we arrive at a practical way of *finding a line which is the fourth proportional to three given lines*, or *the third proportional to two given lines.*

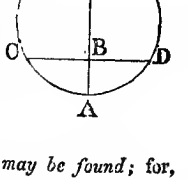
Ex. 1. Let there be *three* lines, AC = 2 eq. pts., BC = 3, and CD = 4; required a fourth line in proportion to the other three.



C.1	Pst. 2.	From any · B draw an indef. line ;
2	3, I.	on it set the 2nd and 3rd terms, BC = 3, and CD = 4 ;
3	3, I. Pst. 2.	from C set CA = 2 ; and prod. AC indef.;
4	Use 9, III.	through the three · s, A, B, D, desc. a ⊙ cutting AC produced in E ;
5	Sol.	the distance CE is the 4th proportional ; and measured, CE = 6 eq. pts.
D.1	C.	$\because$ within a ⊙ the lines BD and AE cut each other.
2	35, III.	$\therefore$ BC · CD = AC · CE, and CE is the 4th proportional.

If of the four segments any three be given, the fourth may be found; for $\frac{BC \cdot CD}{AC} = CE$; $\frac{BC \cdot CD}{CE} = AC$; $\frac{AC \cdot CE}{BC} = CD$; and $\frac{AC \cdot CE}{CD} = CB$.

Ex. 2. Next, let there be *two* lines, AB = 3 eq. pts. and BC = 6; required a third line in proportion to the other two.



C.1	11, 1.	Set the two lines so as to form a rt. $\angle$ at B ;
2	Pst. 2 & 3 I.	prod. AB, CB indef.; and make BD = BC;
3	Use 9, III.	through the three · s, C, A, D desc. a ⊙ cutting AB in E;
4	Sol.	the dist. BE is the 3rd proportional ; and measured, BE = 12 eq. pts.
D.1	C.	$\because$ CD and AE, within a ⊙, cut each other in B ;
2	35, III.	$\therefore$ BC · BD, or BC^2 = AB · BE, and BE is the 3rd proporortional.

If of the three segments any two be given, the third may be found; for, $\sqrt{AB \cdot BE} = BC$; $\frac{BC^2}{AB} = BE$; and $\frac{BC^2}{BE} = BA$.

PROP. 36.—THEOR. *(Important.)*

If from any point without *a circle two st. lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.*

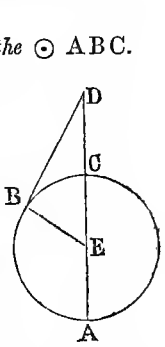
CON. 10, I. 1, III. 12, I.

DEM. 18, III. 47, I. AXS. 1, 2, 3, I. 3, III. 17, III.
6, II. If a st. line be bisd. and produced to any point; the rect. contained by the whole line thus produced, and the pt. of it produced, together with the square of half the line bisd., is equal to the square of the st. line which is made up of the half and the part produced.

E. 1	Hyp. 1	Let D be any · without ⊙ ABC;
2	„ 2	and DCA, DB two st. lines from · D, of which DCA cuts and DB touches the ⊙;
3	Conc.	then $AD \cdot DC = DB^2$

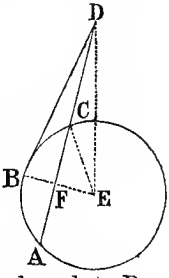
REMK.—Of this Proposition there are two Cases, according as, DCA, the cutting line, passes through the centre of the circle, or not.

CASE I.—*Let* DCA *pass through the centre of the* ⊙ ABC.



C. 1	10, I.	Bis. AC in E, and E is the cen. of ⊙ ABC;
2	Pst. 1.	join BE.
D. 1	H. & C. 2	∵ DB is a tang., and BE a line from E to B;
2	18, III.	∴ ∠ EBD is a rt. ∠.
3	C. 1.	And ∵ AC is bisd. in E and prod. to D;
4	6, II.	$\therefore AD \cdot DC + EC^2 = ED^2$;
5	Def. 15, I.	but EC = EB; ∴ $AD \cdot DC + EB^2 = ED^2$;
6	D. 2 & 47. I.	and EBD being a rt. ∠, ∴ $ED^2 = EB^2 + BD^2$;
7	Ax. 1.	∴ $AD \cdot DC + EB^2 = EB^2 + BD^2$;
8	*Sub.* Ax. 3.	take away EB^2; and $AD \cdot DC = DB^2$, *i.e.* = the square on the tang.

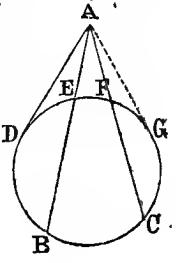
CASE II.—*Let* DCA not *pass through the centre of the* ⊙ ABC.



C.1	1, III. & 12, I.	Find the cen. E, and draw EF perp. to AC;
2	Pst. 1.	and join EB, EC, ED.
D.1	H. & 18, III.	∵ as before, ∠ EBD is a rt. ∠;
2	C. 1.	and ∵ EF, through the cen., cuts at rt. ∠s AC, not through the cen.;
3	3, III.	∴ EF bisects AC, *i. e.* AF = FC:
4	D. 3.	And ∵ AC is bisd. in F, and prod. to D;
5	6, III.	∴ AD · DC + FC^2 = FD^2;
6	*Add.*	to each equal add FE^2;
7	Ax. 2.	∴ AD · DC + FC^2 + FE^2 = FD^2 + FE^2:
8	D. 1. & 47, I.	but EFD being a rt. ∠, ∴ ED^2 = DF^2 + FE^2;
9	D. 1. & 47, I.	and EC^2 = CF^2 + FE^2;
10	D. 6. 7. 8. Ax. 1.	∴ AD · DC + EC^2 = ED^2;
11	Def. 15. I.	but CE = EB, ∴ AD · DC + EB^2 = ED^2;
12	H. & 47, I.	and EBD being a rt. ∠, ∴ AD · DC + EB^2 = EB^2 + BD^2;
13	*Sub.* Ax. 3.	take away EB^2, ∴ AD · DC = BD^2; *i. e.* = the square on tang.
14	Rec.	∴ *If from any point* without *a circle, &c.*

Q. E. D.

COR I.—*If from any point* A, without *a circle,* BDC, *there be drawn two st. lines,* AB, AC, *cutting it in* E *and* F; *then the rectangles contained by the whole lines and the parts of them without the circle shall be equal to one another; i. e.* BA · AE = CA · AF.

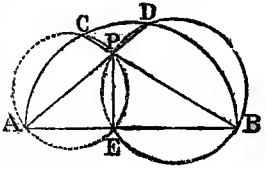


C.1	17, III.	From AD draw a tang. to the ⊙.
D.1	36, III.	∴ BA · AE = AD^2, and CA · AF = AD^2;
2	Ax. 1.	∴ BA · AE = CA · AF.

COR. II.—*If from the same point,* A, *two tangents,* AD, AG, *be drawn to the same circle, they are equal;* for the squares on them are each equal to the same rectangle.

SCH.—Prop. 36 may also be enunciated thus; "If any chord of a circle be produced to cut a tangent to the same circle, the square of the tangent shall be equal to the rectangle under the segments of the chord so produced."

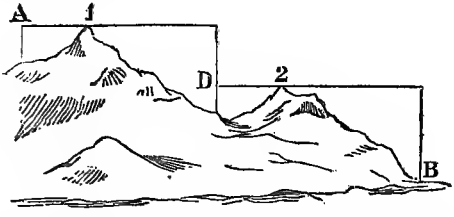
USE AND APP.—I. In any semicircle, ABDC, if from A, B, the extremities of the diameter, chords be drawn, AD, BC, intersecting within the semicircle, the sum of the rectangles formed by any two of such chords, AD, BC, intersecting within it, and by the sections of the chords, AP, BP, between the extremities of the diameter AB and the intersecting point P, shall equal the square on the diameter, AB.



C. 1	12, I.	From P draw PE perp. to AB;
2	Use 9, III.	and draw ⊙s through A, P, E, and B, P, E.
D. 1	Cor. 36, III.	Now in ⊙ APE, $AB \cdot BE = CB \cdot BP$;
2	Cor. 36, III.	and in ⊙ BPE, $BA \cdot AE = DA \cdot AP$;
3	2, II.	Also $AB^2 = AB \cdot BE + BA \cdot AE$;
4	Conc.	$\therefore AB^2 = CB \cdot BP + DA \cdot AP$.

II. From this Prop. is deduced the *Art of taking a true Level* on the earth's surface.

1°. In the more considerable problems of Levelling, it is usual to employ a telescopic sight, and staves with sliding vanes, the distances and observations being entered in a field-book. Thus

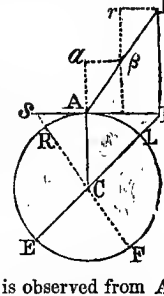


	BACK SIGHTS.			FORE SIGHTS.		
Station.	Distance.	Height.	Corrected.	Distance.	Height.	Corrected.
1	1420 links	1ft. 5in.	1ft. 4½in.	2448 links.	6ft. 8in.	6ft. 7in.
2	2030 „	6 2	6 1¼	2870 „	7 9	7 7¾
			7 5¾			14 2¾

Then 14ft. 2¾in. *minus* 7ft. 5¾in = 6ft. 9in. the *height* of A above B.

2.° *In measuring an Ascent*, AD, the perpendicular and horizontal lines may be found by means of a spirit level and a staff;

For $a\beta + r\,D = AB$; and $A\,a + \beta r = BD$.



3°. But if the horizontal distance should exceed 100 yards, it will be necessary *to correct it for the curvature;* the horizontal line AB being a tangent, whilst the line of the true level, that of the earth's curvature, AL, is at every point equally distant from the earth's centre. The *deviation* of the horizontal from the true level = BL; and, by 36, III., $BE \cdot BL = AB^2$, $\therefore BL = \frac{AB^2}{BE}$.

N.B. The Greek letters of the diagram α, γ, have been changed to *a*, *r*.

Example.—A fountain, B, one mile distant from A, is observed from A, to be on the same horizontal level with the point A; how much is B above A? *i. e.* how much is B farther from the earth's centre C, than the points A or L? CL or CA being 3956 miles.

Here BC — LC = BL;

For, by 47, I., $BC = \sqrt{3956^2 + 1^2} = \sqrt{15649937} = 3956{\cdot}0012639$,

Then, BL = 3956·00012639 *minus* 3956 = ·00012639 of a mile = 8·00808 inches.

4°. By following the same method it will be found that the *deviation* of the horizontal from the true level,—

For 1 mile = 8 inches; 2 miles = 32 inches; 3 miles = 6 feet; and for 4 miles = 10·6 feet, &c.

Or,—Two-thirds of the square of the horizontal level in miles gives the deviation in feet.

5°. The distance within which an object may be seen at sea, or the distance of the *boundary* of the horizon from the spectator is ascertained on the same principle; for the rad. of the horizon $AB = \sqrt{BE \times BL}$.

Ex.—The Peak of Teneriffe, BL, is 2·5 miles above the sea level; what will be the radius of its horizon, or the greatest distance from which it is visible?

Here, BE = 7912 + 2·5 = 7914·5 miles

$\therefore AB = \sqrt{7914{\cdot}5 \times 2{\cdot}5} = \sqrt{19786{\cdot}25} = 140{\cdot}66$ miles.

6°. From an elevation, as the top-mast of a ship, an object will be seen at a greater distance; in this case the *sum of the horizons* of the object and of the elevation will give the whole horizon; *i. e.* AB + AS = SB.

Ex.—The top-mast of a ship, RS, is 100 feet above the surface of the sea; at what distance to a spectator standing there will the summit of Teneriffe be visible?

Here SF = 7912 + ·01893 = 7912·01893 miles.

$\therefore AS = \sqrt{7912{\cdot}01893 \times {\cdot}01893} = \sqrt{149{\cdot}7745183449} = 12{\cdot}23824$ miles.

As above AB = 140·66, $\therefore$ SB = 140·66 + 12·23824 = 152·89824 miles rad. of the whole horizon.

7°. The *earth's diameter may be ascertained* from knowing the horizon AB, and the height, BL, of an object; for $LE = BE - BL$, and $BE = \frac{AB^2}{BL}$.

Ex.—From the summit of Teneriffe the radius, AB, of the horizon is 140·66 miles, and the height of BL above the sea level is 2·5 miles; reqnired the earth's diameter.

Here $BE = \frac{140{\cdot}66 \times 140{\cdot}66}{2 \cdot 5} = \frac{19785{\cdot}2356}{2 \cdot 5} = 7914{\cdot}098$;

And $LE = 7914{\cdot}098$ *minus* $2{\cdot}5 = 7911{\cdot}598$ miles, earth's diameter.

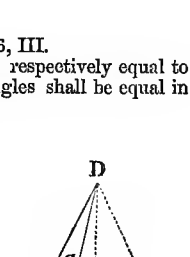
PROP. 37.—THEOR.

If from a point without *a circle there be drawn two lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.*

CON. 17, III. 1, III. Pst. 1.

DEM. 18, III. 36, III. AX. 1. Def. 15, I. Cor. 16, III. 8, I. If two △s have the three sides of the one respectively equal to the three sides of the other; then these triangles shall be equal in every respect.

E.1	Hyp. 1.	Let D be any · without the ⊙ ABC;
2	„ 2.	and from D two st. lines be drawn, of which DCA cuts the ⊙, and DB meets it, so that $AD \cdot DC = DB^2$;
3	Conc.	then DB shall be a tang. to ⊙ ABC.
C.1	17, III.	Draw DE a tang. in E to ⊙ ABC;
2	1, III.	find F the cen. of the ⊙ ABC;
3	Pst. 1.	and join FB, ED and FE.

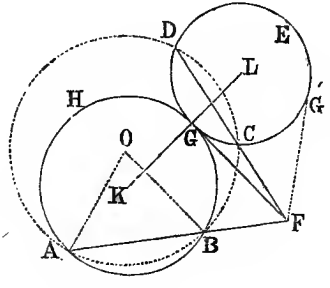


D.1	C. 1 & 18, III.	∵ DE is a tang.; ∴ ∠ FED is a rt ∠:
2	H. 2 & C. 1.	and ∵ DCA cuts, and DE touches the ⊙;
3	36, III.	∴ AD · DC = DE^2;
4	H. & Ax. 1.	but AD · DC = DB^2,
		∴ DE^2 = DB^2, and DE = DB.
5	Def. 15,I. D. 4.	Also ∵ FE = FB, FD com. and DE = DB;
6	8, I.	∴ in △s DEF, DBF, ∠ DEF = ∠ DBF;
7	D. 1. Ax. 1.	but DEF is a rt. ∠; ∴ ∠ DBF is a rt. ∠;
8	C.	thus BD is ⊥ to the rad. FB, at its extr. B;
9	Cor. 16, III.	∴ DB touches the ⊙ at B.
10	Rec.	. *If from any point* without *a circle, &c.*
		Q. E. D.

Cor.—*Tangents, as* DB, DE, *from the same point,* D, *without a circle are equal.*

D.1	36, III.	∵ DB^2 = AD · DC and DE^2 also = AD · DC;
2	Ax. 1.	∴ DB^2 = DE^2, and DB = DE.

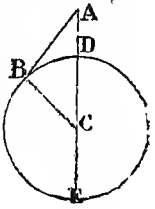
Use and App.—I. By the 36th and 37th Propositions the Problem is solved, "*through two given points,* A,B, *to describe a circle touching a given circle* CDE."



C. 1	*Assum.*	In ⊙ CDE assume any point D of the ⊙ce;
2	Use 9, III.	and through, A, B, D desc. a ⊙ cutting ⊙ CDE in D, C;
3	Pst. 1. 2.	join DC, and prod. AB, DC to meet in F;
4	17, III.	from F draw FG, FG′ tangs. to ⊙ CDE;
5	25, III.	and desc. a ⊙ through · s A, B, G;
6	Sol.	then the ⊙ ABH will touch ⊙ CDE in G.

D. 1	C. 4 & 36, III.	∵ FG is tang. to ⊙ CDE; ∴ $FG^2 = FD \cdot FC$;
2	H.	but A, B, C, D are all in the ⊙ce of ⊙ ABCD;
3	Cor. 36, III.	∴ $FD \cdot FC = FA \cdot FB$.
4	Ax. 1.	∴ also $FA \cdot FB = FG^2$; and ∴ FG is tang. of ⊙ ABH;
5	Remk.	Now FG is a tang. common to both ⊙s;
6	Conc.	∴ the ⊙s ABH, CDE touch in the · G.

II. The last three Propositions, 35, 36, and 37, are amongst the most important in Plane Geometry. It was by their aid that Maurolico, of Messina, in the sixteenth century, calculated the diameter of the earth; for by the method which has been shown, having ascertained AD the vertical height of a mountain; the ∠ BAC made by the vertical line, and AB the line from the summit A to B the boundary of sight; he found the length of AB, by Trigonometry;



then ∵ $AB^2 = AE \cdot AD$, ∴ $AE = \frac{AB^2}{AD}$; and

AE − AD = ED the diameter of the earth.

REMARKS.

1. In classifying the Propositions of the Third Book, it will be useful to consider them under five general heads;—

1°. The propositions which relate to the Centre of a circle; 1—15 and 20.
2°. To the Tangents of a circle; 16—19.
3°. To the Segments of circles; 21—25.
4°. To the Angles in circles, or in their segments; 26—34.
5°. To the Equality of the rectangles contained by the segments of lines intersecting each other within, or without the circle; 35-37.

2. Of the 37 Propositions which are found in this book only six are Problems, namely:—

Pr. 1. To find the Centre of a circle.
Pr. 17. To draw a Tangent to and from a circle.
Pr. 25. To complete the circle of which a Segment is given.
Pr. 30. To bisect a given Circumference.

Pr. 33. On a given line to draw the segment of a circle containing an angle of a given magnitude; and

Pr. 34. From a circle to cut off a segment which shall contain an angle of a given magnitude.

3. From the Properties of Plane Figures demonstrated in this and in the preceding books, various other problems, however, may be deduced; as

1. To draw a circle through three given points not in a st. line; Use 9, and 25, III.
2. To describe an oval on any given major axis, or a spiral with the radius of the *eye* given; Use 11, III.
3. To join two given points by a serpentine line, or *cima recta*; Use 12, III.
4. To draw a tangent to each of two given circles; Use 18, III.
5. To draw the arc of a chord without knowing the centre of the circle; Use 21, III.
6. To reduce curve-lined figures to rectilineal of equal areas; Use 24, III.
7. Through a given point to draw a line parallel to another line; Use 27, III.
8. To trisect a quadrant; Use 30, III.
9. From a point *in* a line, or *from* the extremity of a line to raise a perpendicular; and from a point *without* a line to drop a perpendicular; Use 31, III.
10. Given the base and vertical $\angle$ of a $\triangle$ to find the *locus* of the vertex; given the vertical angle, the base, and the area of a $\triangle$ to construct the triangle; and given any three points not in a st. line to describe through them an equilateral triangle; Use 33, III.
11. To determine the point where two unequal lines will appear equal, *i. e.* under the same angle; Use 34, III.
12. To find a line proportional to two given lines; and also a line proportional to three given lines; Use 35, III.

4. Were it required in a work like the present, problems might be introduced which show how circles may be drawn which are tangents to two or three given circles, or to two st. lines and a circle; or to two circles and a st. line, &c.; but for these reference may be made to "Geometry, Plane, Solid, and Spherical," Book III., § 8.

5. Several of the principles on which the Levelling and Surveying of Land, and Geographical and Astronomical Observations depend, have been given in the Third Book,—such as the *Methods* :—

1. Of computing the distances and heights of objects when situated on the verge of the natural horizon ; Use 16, III.
2. Of determining the part of a globe which may he enlightened by a luminous body, as by a meteor, volcano, lighthouse, &c. ; of explaining the theory of the phases of the moon ; of ascertaining the distance of the sun ; and of obtaining the dip of the horizon ; Use 19, III.
3. Of constructing a figure representative of the distance of the place of observation from an object ; and of tracing the arc of a circle for giving a spherical figure to optical glasses ; Use 21, III.
4. Of finding the true centre of an imperfectly constructed theodolite, or similar instrument ; Use 26, III.
5. In Coast Surveying, for noting soundings, bearings, &c.; Use 33, III.
6. For taking a true level on the earth's surface ; Use 36, III.; and
7. For calculating the earth's diameter ; Use 37, III.

These are but Examples of the *many useful purposes* to which geometrical science may be applied ; and they may serve to redeem geometry from the prejudiced objection that it is a system of theoretical reasoning without practical results. The practical results are really most important, and in the actual business and occupations of life are an everyday's demand.

GRADATIONS IN EUCLID.

BOOK IV.

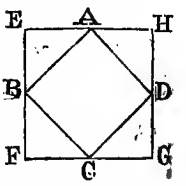
CONTAINING THE METHODS OF CONSTRUCTING REGULAR STRAIGHT-LINED FIGURES IN AND ABOUT A CIRCLE, AND CIRCLES IN AND ABOUT REGULAR STRAIGHT-LINED FIGURES.

Excepting Prop. A, Theor, the fourth book of Euclid's Plane Geometry consists entirely of Problems; it is in fact the Application of the third book to the purposes of inscribing and circumscribing triangles and other regular straight-lined figures in and about circles, or circles in and about such regular figures. The former books supplied the means of drawing regular plane figures of 3, 4, 5, and 15 sides; and by continued bisections of making them of 6, 12, 24, &c., or 8, 16, 32, &c.; or 10, 20, 30, &c., sides. The employment of those means constitutes the object of the book on which we are now entering.

In Trigonometry, Astronomy, and the various departments of Civil and Military Engineering, the fourth book is found of essential service; we also deduce from it the method of obtaining, with sufficient exactness, the quadrature of the circle, and of proving that circles are to one another in the proportion of the squares of their diameters.

Definitions.

1. A rectilineal figure is said to be inscribed in another rectilineal figure when the angular points of the inscribed figure touch the sides of the figure in which it is inscribed, each upon each.



Thus, the fig. ABCD is inscribed in the fig. EFGH.

2. In like manner, a figure is said to be *described* about another figure when the sides of the circumscribed figure touch the angular points of the figure about which it is described, each upon each;

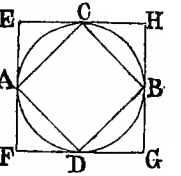
Thus the fig. EFGH is circumscribed about the fig. ABCD.

It is noteworthy that Euclid gives no example of one rectilineal figure being inscribed in another rectilineal figure, or circumscribed about it.

3. A rectilineal figure is said to be *inscribed in a circle* when each angular point of the inscribed figure touches the circumference of the circle;

Thus, the qu. lat. ACBD is inscribed in the circle ADBC.

4. A rectilineal figure is said to be described about a circle when each side of the circumscribed figure touches the circumference of the circle;



Thus, the qu. lat. EFGH is described about the circle ABCD.

5. In like manner, a circle is said to be *inscribed* in a rectilineal figure when the circumference of the circle touches each side of the figure;

Thus, the circle ABCD is inscribed in the quadrilateral EFGH.

6. A circle is said to be *described* about a rectilineal figure when the circumference of a circle touches each corner of the figure about which it is described;

Thus, the circle ABCD is described about the figure ADBC.

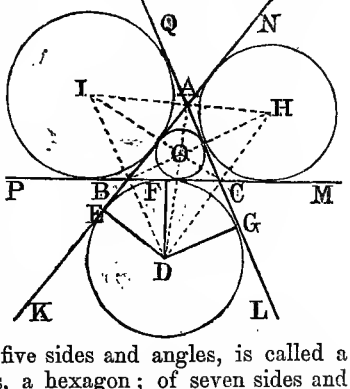
7. A straight line is said to be fitted exactly into a circle, or to be applied in it, when the extremities of it are on the circumference of the circle;

Thus, the lines AC and AD are applied to the circle ABCD.

DEFINITIONS ADDITIONAL TO THOSE OF EUCLID.

8. A circle is said to be *exscribed* to a triangle when, having for centre the point of intersection of any two straight lines that bisect the exterior angles of the triangle, the circle touches a side of that triangle.

Thus, ⊙ EFG from cen. D, (where the lines CD, BD, intersect, which divide the ext. ∠s CBK, BCL, each into two equal parts), and touching in F the side BC of the △ ABC, *is exscribed* to that triangle; as also are the ⊙s from the centres H and I.



9. "Any rectilineal figure, of five sides and angles, is called a pentagon; of six sides and angles, a hexagon; of seven sides and angles, a heptagon; of eight sides and angles, an octagon; of nine sides and angles, a nonagon, of ten sides and angles, a decagon; of twelve sides and angles, a duodecagon; of fifteen sides and angles, a quindecagon," &c.

10. "These figures are included under the general name of *polygons*; and are called *equilateral* when their sides are equal; and *equiangular* when their angles are equal. Also, when both their sides and angles are equal they are called *regular polygons*." POTTS' EUCLID, p. 124.

N.B.—The force of the propositions in Simpson's Edition is often lessened by not rendering the Greek original into English corresponding, as far as differences of idiom will admit, more closely with Euclid's text. To avoid this, Galbraith and Haughton's rendering of the general enunciation is often followed, though they have not been so thoroughly exact as is desirable.

PROPOSITIONS.

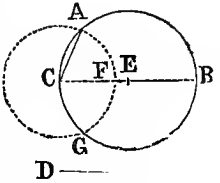
Prop. 1.—Prob.

Into a given circle to fit exactly a right line equal to a given right line, which is not greater than the diameter of the circle.

Sol. 1, III. To find the centre of a given circle.
3, I. From the greater of two given lines to cut off a part equal to the less.
Pst. 3 and 1. A circle may be described from any centre at any distance from that centre. A st. line may be drawn from any one point to any other point.

Dem. 15, I. A circle is a plane figure contained by one line, which is called the circumference, and is such that all st. lines drawn from a certain point within the figure to the circumference, are equal to one another.
Ax. 1. Things which are equal to the same thing, are equal to one another.
Def. 7, IV. A st. line is said to be fitted exactly into a circle, or to be applied in it, when the extremities of it are on the circumference of the circle.

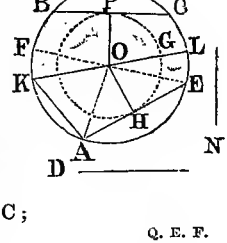
E.1	Dat. 1.	Let ABC be the given ⊙;
2	„ 2.	and D the st. line $>$ CB diam. of ⊙ ABC;
3	Quæs.	in ABC to place a st. line = D.
C.1	1, III.	Find E the cen. of ABC, and draw any diam., BC, through it;
2	Sup.	if BC = D, the required thing is done;
3	H.	but if not, and BC is $>$ D;
4	3, I.	from CB cut off CF = D;
5	Pst. 3 & 1.	and from C, with CF, desc. ⊙ GFA, and join CA;
6	Sol.	then CA is the line required.



D.1	C.5. Def. 15, I.	∵ C is cen. of ⊙ GFA ∴ CA = CF;
2	C. 4. Ax. 1.	but CF = D, ∴ D = CA.
3	Def. 7, IV.	∴ in ⊙ ABC, a st. line has been placed, CA = the given st. line D. Q. E. F.

USE AND APP.—I. *Within a given* ⊙, ABC, *to place a line of a given length*, D, *not greater than the diam. of the given* ⊙, *which line shall pass through* A, *a given point in the* ⊙*ce.*

C.1	1, IV.	In the ⊙ ABC place BC = D the given line;
2	1, III. 12, I.	find O cen. of ⊙ ABC, and draw OP perp. to BC;
3	Pst. 3.	with OP, from cen. O, draw ⊙ PGH;
4	17, III. Pst. 2.	from A draw AH tang. to ⊙ PGH, and prod. AH to cut ⊙ ABC in E; and join OH;
5	Sol.	then AE = BC = D.
D.1	Def. 15, I. & 14, III.	∵ OH = OP; ∴ AE = BC;
2	C. 1. Ax. 1.	but BC = D; ∴ AE = D. Q. E. F.



II. *To draw that diam. of a* ⊙ *which shall pass at a given distance*, N, *from a given point* A

C.1	Pst. 3. & 1, IV.	With OA = N, desc. a ⊙ ABC, and in it place AK = N;
2	Sol.	then KO produced to L is the diam. required.
D.1	Def. 16, I. & C.	∵ KL is a diam., and AK = AO = N;
2	Conc.	∴ KL, a diam., passes at the given distance from A.

PROP. 2.—PROB.

In a given circle to inscribe a triangle equiangular to a given triangle.

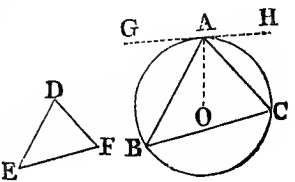
CON. 17, III. To draw a st. line from a given point, either *without* or *in* the circumference, which shall touch a given circle.

23, I. At a given point in a given line to make a rectil. ∠ equal to a given rectil. ∠. Pst. 1.

DEM. 32, III. If a st. line touches a ⊙, and from the point of contact a st. line be drawn cutting the circle, the ∠s which this line makes with the line touching the ⊙, shall be equal to the ∠s which are in the altr. segs. of the ⊙. Ax. 1.

Cor. 3, 32, I. If two △s have two ∠s of the one respectively equal to two ∠s of the other, then the third ∠ of the one shall be equal to the third ∠ of the other.

Def. 3, IV. A rectil. figure is said to be inscribed in a circle when each angular point of the inscribed figure touches the ⊙ce of the circle.

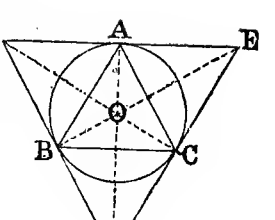


E.1	Data.	Let A B C be the given ⊙, and D E F the given △ ;
2	Quæs.	in ⊙ A B C to inse. a △ eq. ang. to △ D E F.
C.1	17, III.	To any point, A, in the ⊙ce, draw a tang. G A H;
2	23, I.	at A, in A H, make ∠ H A C = ∠ D E F;
3	23, I.	and at A, in A H, make ∠ G A B = ∠ D F E;
4	Pst. 1.	join B C;
5	Sol.	then △ A B C is the △ required.
D.1	C. 1, 2,	∵ H A G is a tang., and A C from A cuts the ⊙;
2	32, III.	∴ ∠ H A C = ∠ A B C in the alternate seg.;
3	C. 2. Ax. 1.	but ∠ H A C = ∠ D E F; ∴ ∠ A B C = ∠ D E F;
4	*Sim.*	In like manner ∠ A C B = ∠ D F E;
5	Cor. 3. 32, I.	∴ rem. ∠ B A C = rem. ∠ E D F;
6	Def. 3, IV.	Hence △ A B C is eq. ang. with △ E D F, and is inscribed in the ⊙. Q. E. F.

Sch.—The *Analysis* of a problem is a very useful exercise, and, that the learner may become accustomed to the method, some examples will be given. Thus, of Prop. 2, setting out with the admission that the △ A B C has its angles respectively equal to the ∠s D, E, F, the Analysis will be—

Through the point A draw A G H, a tangent to the ⊙ ;
then, ∵ ∠ C A H = ∠ A B C = ∠ E, ∴ the line A C is given in position ; and, being cut by the ⊙ce, the point C is given.
In the same way it will appear that the point B also is given ;
and ∵ the three points, A, B, C, are given, ∴ their junction forms △ A B C, inscribed in the circle.

Use and App.—*An eq. lat.* △ A B C, *being inscribed in a circle, and through the angular points* A, B, C, *tangents,* D E, E F, F D, *being drawn, these tangents will also form an eq. lat.* △, D E F, *the area of which is four times that of the inscribed eq. lat.* △.



C. 1	1, III.	Find O the cen. of the given circle,
2	Pst. 1.	and join OA, OB, OC, OD, OE, OF.
D. 1	18, III.	∵ ∠s OBD, OCD, are rt. ∠s,
2	47, I.	∴ $OD^2 = OB^2 + BD^2 = OC^2 + CD^2$;
3	Def. 15, I.	but OB = OC, ∴ $BD^2 = CD^2$ and BD = DC;
4	D. 3. and H.	and ∵ OB = OC, BD = DC, and ∠ OBD = ∠ OCD;
5	4, I.	∴ △ OBD = △ OCD, ∠ BOD = ∠ COD, and ∠ BDO = ∠ CDO; *i. e.* ∠s BOC and BDC are each bisected by DO;
6	26, I.	also DO bisects the line BC;
7	3, III.	∴ DO bis. BC at rt. ∠s, and passes through the vertex A.
8	*Sim.*	So, EO bis. AC, and passes to the vertex B; and FO bis. AB, and passes to the vertex C.
9	Cor. 5, I. 32, I.	Now ∠ OBC = ⅓ of a rt. ∠; ∴ ∠ DBC = ⅔ of a rt. ∠;
10	*Sim.*	So ∠s DCB, BDC, each = ⅔ of a rt. ∠,
11	Cor. 6, I.	∴ △ BDC is eq. lat. and = △ ABC;
12	*Sim.*	So △s ACE, ABF are each eq. lat. and = △ ABC;
13	Conc.	∴ △ DEF = 4 △ ABC.
14	D. 11, 12.	Also DE = 2 DC = 2 BC;
15		EF = 2 AE = 2 AC = 2 BC;
16		and FD = 2 FB = 2 AB = 2 BC;
17	Conc.	∴ △ DEF is equilateral. Q. E. F.

PROP. 3.—PROB.

About a given circle to circumscribe a triangle equiangular to a given triangle.

SOL. Pst. 2. 1, III. 23, I. 17, III.

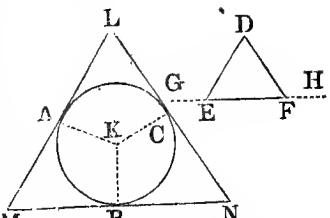
DEM. 18, III. If a st. line touches a ⊙, the st. line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Cor. 1. 32, I. All the interior ∠s of any rectil. fig., together with four rt. ∠s, are equal to twice as many rt. ∠s as the figure has sides.

Ax. 3, I. If equals be taken from equals the remainders are equal.

13, I. The ∠s which one st. line makes with another upon one side of it are either rt. ∠s, or together equal to two rt. ∠s. Ax. 1.

Def. 4, IV. A rectil. fig. is said to be described about a ⊙, when each side of the circumscribed fig. touches the ⊙cc of the ⊙.



E.1	Data.	Let A B C be the given ⊙, & D E F the given △ ;
2	Quæs.	about the ⊙ A B C to desc. a △ eq. ang. to the △ D E F.
C.1	Pst. 2.	Produce E F both ways to G, H;
2	1, III. Pst. 1	find K the cen. of ⊙ A B C, and draw K B;
3	23, I.	at K in B K make ∠ B K A = ∠ D E G, and ∠ B K C = ∠ D F H;
4	17, III.	and through A, B, C, draw L M, M N, N L, tangs. to ⊙ A B C;
5	Sol.	then △ L M N shall be the △ required.
D.1	C. 4.	∵ L M, M N, N L, are tangs to ⊙ A B C;
2	C. 2, 3.	and K A, K B, K C, lines from the cen. to A, B, C;
3	18, III.	∴ the ∠s at A, B, and C are all rt. ∠s.
4	Cor. 1, 32, I.	And ∵ the 4 ∠s of the qu. lat. A M B K = 4 rt. ∠s;
5	D 2.	and two of the four, K A M, K B M, are rt. ∠s;
6	Ax. 3.	∴ the other two ∠s, A K B, A M B = 2 rt. ∠s;
7	13, I.	but ∠s D E G, D E F = 2 rt. ∠s;
8	Ax. 1.	∴ ∠s A K B, A M B, = ∠s D E G, D E F;
9	C. 3. Ax. 3.	and ∠ A K B = ∠ D E G, ∴ ∠ A M B = ∠ D E F.
10	*Sim.*	In like manner ∠ L N M = ∠ D F E;
11	32, I. Ax. 3.	∴ rem. ∠ M L N = rem. ∠ E D F;
12	Def. 4. IV.	∴ △ L M N is eq. ang. with △ D E F; and is described about the ⊙ A B C. Q. E. F.

Sch.—*Analysis:* We suppose the problem to have been solved, the △ L M N being described about the given ⊙ A B C, so that ∠ L = ∠ D, ∠ M = ∠ E, and ∠ N = ∠ F.

Join K the cen. of the ⊙ to the tang. points A, B, C.
In the qu. lat. B K C N, the four ∠s = four rt. ∠s ;
and ∵ ∠s K B N, K C N, are 2 rt. ∠s ; ∴ ∠s B K C, B N C = 2 rt. ∠s.
But ∠ N being given, its supplement ∠ B K C is also given ;
consequently, K B and K C, the two radii, are given in position.
Thus, it may be shown that the ∠ A K B is given, and the line K A given in position.
The inters. of K A, K B, K C, with the ⊙cc, or the points A, B, C, are given;
∴ the tangs. M N, N L, L M, are given in position.
Thus the △ L M N is given.

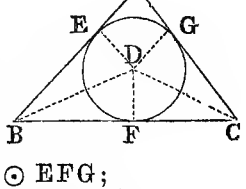
PROP. 4.—PROB.

To inscribe a circle in a given triangle.

CON. 9, I. To bisect a given rectil. $\angle$, *i.e.* to divide it into two equal parts
12, I. To draw a perp. to a given st. line of unlimited length from a given point without it. Pst. 3.

DEM. Ax. 11. All rt. $\angle$s are equal to one another.
26, I. If two $\triangle$s have two $\angle$s of the one equal to two $\angle$s of the other, each to each, and one side equal to one side; then shall the other sides be equal, each to each, and also the third $\angle$ of the one to the third $\angle$ of the other. Ax. 1.
9, III. If a point be taken within a circle, from which there fall more than two equal st. lines to the $\odot$ce, that point is the cen. of the $\odot$.
Cor. 1, 16, III. If a st. line be drawn at rt. $\angle$s to any diam. of a $\odot$, from its extremity, it shall touch the $\odot$ at the extremity; and a st. line touching the $\odot$ at one point shall touch it at no other point.
Def. 5, IV. A $\odot$ is said to be inscribed in a rectil. fig. when the $\odot$ce of the $\odot$ touches each side of the figure.

E.1	Datum.	Let ABC be the given $\triangle$;
2	Quæs.	in it to inscribe a $\odot$.
C.1	9, I.	Bis. $\angle$s ABC, ACB, by BD, CD, meeting in D;
2	12, I.	from D draw DE, DF, DG, perps. to AB, BC, CA.
3	Pst. 3.	with DF as rad. draw a $\odot$ EFG;
4	Sol.	then the $\odot$ EFG is the $\odot$ required.
D.1	C. 1, 2.	$\because$ $\angle$EBD = $\angle$ FBD, $\angle$DEF = $\angle$ DFB, and BD com.;
2	26, I.	$\therefore$ DE = DF;
3	*Sim.* Ax. 1.	So DG = DF; $\therefore$ DE = DG = DF;
4	9, III.	$\therefore$ the points E, F, G, are in the $\odot$ce of the $\odot$;
5	C. 2.	And $\because$ the $\angle$s at E, F, G, are rt. $\angle$s;
6	Cor. 1. 16, III.	$\therefore$ AB, BC, CA touch the $\odot$ EFG,
7	Def. 5. IV.	and $\therefore$ $\odot$ EFG is inscr. in $\triangle$ ABC.



Q. E. F.

SCH.—1. In other words, the 4th Prop. is—*To describe a* $\odot$ *which shall touch three given st. lines not parallels.*

2. In the *Analysis* of this problem we assume that the $\odot$ EFG has been inscribed in the $\triangle$ ABC.

The cen. of the ⊙ being D, join DA, DB, DC.
Now ∵ DE = DF, DB, com. to △s BFD, BED, and the ∠s at E and F are equal. being rt. ∠s;
∴ △ BFD = △ BED, ∠ DBF = ∠ DBE, and thus BD is given in position.

By a similar argument it may he shown that DA and DC are given in position :—

∴ their point of inters., D, is also given.

USE AND APP.—I. Connected with this problem is the following theorem: *If three ∠s, A, B, C, of a △ ABC be bisected by st. lines, BD, CD, AD, these lines will intersect in the same point.*

C.	12, I.	From D, the inters. of BD and CD, draw DE, DF, DG, perps. to AB, BC, and CA.
D.1	C. and H.	∵ ∠s DFC, DCF = ∠s DGC, DCG, and DC is com.;
2	26, I.	∴ FD = GD.
3	*Sim.* Ax. 1.	So ED = GD; ∴ ED = FD;
4	Conc.	∴ the lines BD, CD, AD, have a com. point of inters. D.

II. An expression for the *Area of a Triangle*, and for *the Radius of the inscribed circle* may be deduced from this theorem.

First.—For the *Area of the Triangle*, the sides AB = *a*, BC = *b*, and CA = *c*, and the radius, DF = *r*, of the inscribed ⊙ being given;
The △ ABC = △s ABD + BDC + CDA;
the area of ABD = $a \times \frac{r}{2}$; of BDC = $b \times \frac{r}{2}$ and of CDA = $c \times \frac{r}{2}$;
∴ the area of △ ABC = $(a + b + c) \frac{r}{2}$.

Second.—For the *Radius of the inscribed circle*, the area and sides being given;
$$r = \frac{2 \text{ area of } \triangle}{a + b + c}.$$

Ex. 1. The sides of a △ are 52, 56, and 60 yards, and the rad. of the inscribed ⊙ 16 yards, required the area.

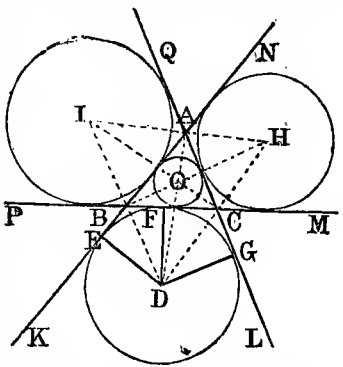
Here $(52 + 56 + 60) \times \frac{16}{2} = 168 \times 8 = 1{,}344$ square yards.

Ex. 2. The area of a △ is 216 square feet, and the sides 18, 24, and 30 feet, required the radius of the inscribed circle.

Here, $\frac{2 \times 216}{18 + 24 + 30} = \frac{432}{72} = 6$ feet.

III. The Corollaries to Prop. 4, given by Galbraith and Haughton, in their "Manual of Euclid," Bk. IV., pp. 6, 7, open up an interesting subject of inquiry, namely—*The Properties of Circles* exscribed *to a Triangle.* From those Corollaries we select the following, referring to Def. 8, IV. :—

1. The bisectors of any internal ∠, as DA of ∠ BAC, and of the remaining two ext. ∠s, as DB and DC of ∠s CBK, BCL, intersect in the same point, D, which is the cen. of the ⊙ EFG, touching the side, BC, opposite the given ∠ BAC and the productions, BK, CL, of the two sides AB, AC, containing it.



2. The radius, as DF, of the exscribed ⊙, EFG, may be found in numbers, by dividing the area of the △ ABC, by half the difference between the sum of the two sides and the base; because the whole triangle is the difference between the sum of the two triangles ACD and ABD, whose bases are the sides AB, AC, and common altitude the radius of the exscribed circle, and the △ BCD, whose base is the third side BC, and altitude the same radius DF.

3. From 13, II., a Formula is deduced for calculating the *Area of a Triangle* in the terms of the sides; $2s$ being the perimeter or sum of the sides, and s the semi-perimeter, and a, b, c, the sides of the triangle :—

$$\text{Area of } \triangle = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

4. By aid of this Formula and Cor. 2, given above, we can readily express *the radii of the three exscribed circles*; r denoting the radius of the inscribed circle, and r', r'', r''', the radii of the circles exscribed to the sides opposite A, B, C, respectively;

$$\text{Then, } r' = \sqrt{\frac{s\cdot s-b\cdot s-c}{s-a}},\quad r'' = \sqrt{\frac{s\cdot s-a\cdot s-c}{s-b}},\quad r''' = \sqrt{\frac{s\cdot s-a\cdot s-b}{s-c}};$$

$$\text{and by Use 2, P. 4, IV. } r = \frac{2\ area}{a+b+c}, \text{ or } r = \sqrt{\frac{s-a\cdot s-b\cdot s-c}{s}}.$$

5. By taking the product of the four radii, $r\,r'\,r''\,r'''$, the following remarkable expression is obtained for the area of a triangle :—

$$\text{Area of } \triangle = \sqrt{r\,r'\,r''\,r'''}.$$

Ex. 1. Find the radii of the inscribed and exscribed circles in the triangle of which the sides are $a = 100$, $b = 86$, and $c = 72$.

$$\text{Here } s = \frac{100+86+72}{2} = \frac{258}{2} = 129,\ s-a = 29,\ s-b = 43,\ \text{and } s-c = 57.$$

$$\text{And } r = \frac{2\ Area}{a+b+c} \text{ or } r = \sqrt{\frac{s-a\cdot s-b\cdot s-c}{s}}$$

Now area $= \sqrt{129 \times 29 \times 43 \times 57} = \sqrt{9169191} = 3028{\cdot}06$; and

$r = \frac{6056{\cdot}12}{258} = 23{\cdot}473$, or $r = \sqrt{\frac{29 \times 43 \times 57}{129}} = \sqrt{551} = 23{\cdot}473$

$r' = \sqrt{\frac{129 \times 43 \times 57}{29}} = \sqrt{\frac{316179}{29}} = \sqrt{10902{\cdot}7} = 104{\cdot}416$

$r'' = \sqrt{\frac{129 \times 29 \times 57}{43}} = \sqrt{\frac{213237}{43}} = \sqrt{4959} = 70{\cdot}420$

$r''' = \sqrt{\frac{129 \times 29 \times 43}{57}} = \sqrt{\frac{160863}{57}} = \sqrt{2822{\cdot}1579} = 53{\cdot}124$

Ex. 2. From the two formulas given above, find the area of the foregoing triangle :—

1°. Area $= \sqrt{129 \times 29 \times 43 \times 57} = \sqrt{9169191} = 3028{\cdot}06$.

2°. Area $= \sqrt{23{\cdot}473 \times 104{\cdot}416 \times 70{\cdot}420 \times 53{\cdot}124} = \sqrt{9168919{\cdot}3231896}$
$= 3028{\cdot}0223$.

6. In a rt. angled $\triangle$ the *diam.* of the *inscribed* $\odot$ is equal to the difference between the sum of the sides and the hypotenuse ; and the *diam.* of the $\odot$ *exscribed* to the hypotenuse is equal to the perimeter of the triangle.

N.B. For other properties of the inscribed, circumscribed, and exscribed circles of a given triangle, consult the Appendix to Galbraith and Haughton's *Manual of Euclid*, Bk. IV., p.p. 20-25.

Prop. 5.—Prob.

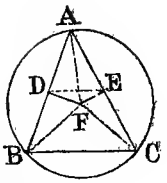
To circumscribe a circle about a given triangle.

Con. 10, I. To bisect a given finite st. line. Pst. 1 and 3.
11, I. To draw a st. line at rt. $\angle$s to a given st. line from a given point in the same.

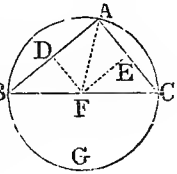
Dem. Ax. 9. The whole is greater than its part.
Ax. 12. If a st. line meets two st. lines, so as to make the two int. $\angle$s on the same side of it taken together less than two rt. $\angle$s, these two st. lines, being continually produced, shall at length meet upon that side on which the $\angle$s are less than two rt. $\angle$s.
Def. 10, I. When a st. line standing on another st. line makes the adj. $\angle$s equal to each other, each of these $\angle$s is called a *rt.* $\angle$; and the st. line which stands on the other is called a *perp.* to it.
4, 1. If two $\triangle$s have two sides and the incl. $\angle$ of one equal to two sides and the incl. $\angle$ of the other, the $\triangle$s are equal in every respect.
Ax. 1.
Def. 6, IV. A circle is said to be *desc.* about a rectil. fig. when the $\odot$ce of the $\odot$ touches each corner of the fig. about which it is described.
31, III. In a $\odot$ the $\angle$ in a semi $\odot$ is a rt. $\angle$; but the $\angle$ in a seg $>$ a semi $\odot$ is less than a rt. $\angle$; and the $\angle$ in a seg. $<$ semi. $\odot$, is greater than a rt. $\angle$.

E. 1	Datum.	Let A B C be the given △;
2	Quæs.	To desc. a ⊙ about the △ A B C.
C. 1	10, I.	Bisect A B, A C in D, E;
2	11, I. Post. 1.	and from D, E, draw D F, E F ⊥s to A B, A C and join D E;
3	C. 2.	then ∵ ∠s A D F, A E F are rt. ∠s,
4	Ax. 9	∴ ∠s E D F, D E F are < two rt. ∠s;
5	Ax. 12.	and ∴ D F, E F being prod. shall meet;
6	Sup. Pst. 1.	let them meet in F, and join F A;
7	Sup.	If F be not on B C, join also F B and F C;
8	Pst. 3.	with F A or F B desc. a ⊙;
9	Sol.	the ⊙ A B C is c. scr. about the △ A B C.
D. 1	C.1.2. Def.10,I	∵ A D = D B, D F com., and ∠ A D F = ∠ B D F;
2	4, I.	∴ △ A D F = △ B D F, and A F = F B.
3	*Sim.* Ax. 1.	So C F = F A, ∴ B F = F C;
4	D. 2, 3.	thus F A = F B = F C;
5	Rem.	∴ a ⊙ from F, with rad. F A or F B, or F C, will pass through the extrs. of the other two;
6	Def. 6, IV.	∵ the ⊙ passes through the angr. • s A, B, C;
7	Sol.	and ∴ the △ A B C is c. sc. by the ⊙ A B C.

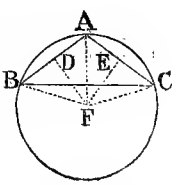
Q. E. F.



Cor. 1. When F, the centre of the ⊙, falls *within* the △, each angle is an acute ∠, (31, III.); when *on* a side of the △, the ∠ opp. that side is a rt. ∠, (31, III.); and when *without* the △, the ∠ opp. the side nearest the centre is an obt. ∠, (31, III.); and *conversely*.



If the given △ be acute angled, the cen. of the ⊙ falls *within* it; if rt. angled, the centre is *on* the side opp. the rt. ∠; and if obtuse angled, the centre falls *without* the triangle, beyond the side opp. the obtuse angle.



2. The perps. bisecting the sides of a △ meet at the centre of the circumscribing circle.

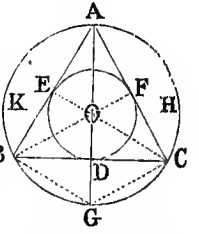
3. Perpendiculars from each ∠ on the opp. side intersect in the same point.

Sch. 1. This proposition is identical; 1° with that (in 3, III. and 9, III.) of describing a circle through three given points, A, B, C, not in the same st. line; 2° through two given points, A, B, and touching a given st. line; or 3° through a given point A, and touching two given st. lines.

2. It is also identical with describing a circle; 1° through two given points and touching a given circle; 2° through a given point and touching two given circles; or 3° touching three given circles.

3. And it may be further extended to solve the problems of describing a circle; 1° through a given point, touching a given st. line, and also touching a given circle; 2° touching two given st. lines and also a given circle; or 3° touching a given st. line, and two given circles. See *Geom. Plane, Solid, and Spherical*, pp. 114, 118.

Use and App.—"*If one circle* EDF *is inscribed in an equil.* △ ABC *and another* KAH *circumscribed about it, the circles* EDF, KAH *are* concentric, *and the diam* OG *of one is* double *the diam.* OD, *of the other.*

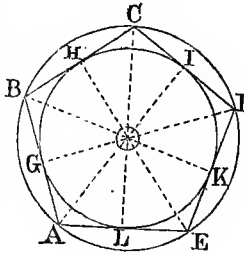


C. 1	9, I.	Bis. the ∠s A, B, C by the st. lines AG, BF, CE;
2	Pst. 1.	and join BG, GC.
D. 1	H. and C. 1.	∵ BA = CA, AG com., and ∠ BAD = ∠ CAD;
2	4, 1. Dcf. 10, I.	∴ BD = DC, and ∠s at D rt. ∠s.
3	*Sim.*	So AE = EB, and AF = FC, and ∠s at E, F, are rt. ∠s.
4	Cor. 1, III.	∴ AG, CE, BF, pass through O the cen. of ⊙ ABC;
5	1, III.	and O, the cen. of AG a diam., is the cen. of ⊙ ABC.
6	H. and C.	Again ∵ EB = BD, OB com., and ∠ EBO = ∠ DBO;
7	4, I	∴ OE = OD;
8	*Sim.* Ax. 1.	So OF = OD; ∴ OE = OF = OD;
9	Cor. 1, III.	∴ O is also the cen. of the ⊙ EDF.
10	Cor. 31. III.	Next, ∵ AKG and AHG are sem. cs.;
11	Ax. 11,	∴ ∠ ABG = ∠ ACG;
12	H.	and ∵ △ ABC is eq. ang.; ∴ each ∠ = ⅔ of a rt. ∠;
13		∴ also ∠ DBG = ⅓ rt. ∠ = ∠ DCG.
14	D. 2.	Now ∠ BDO = ∠ CDG, ∠ DBO = ∠ DCG, and BD = DC:
15	26, I.	∴ OD = DG, or 2 OD = OG = OA.
16	Conc.	∴ the diam. of ⊙ AKGH = 2 diam. of ⊙ EDF. Q. E. D.

PROP. A.—THEOR.

A circle may be described about any regular polygon, or inscribed within it; and conversely.

E. 1	Hyp.	Let the reg. polygon be ABCDE;
2	Con. 1.	the ⊙ ABD may be circumscribed;
3	„ 2.	or ⊙ GLI may be inscribed.
C. 1	Use 9, III.	Through A, B, C desc. a ⊙, O being the centre;
2	10, I,	bis. the sides AB, BC, &c., in G, H, &c.;
3	Pst.	and join OA, OB, OC, &c.; OG, OH, OI, &c.
D. 1	C. 1, 2, & 3, III.	∵ CI = ID, IO com. and ∠ CIO = ∠ DIO;
2	4, I.	∴ OD = OC; and ⊙ passes through D;
3	*Sim.* Def. 6, IV.	∴ ⊙ also passes through E; and ∴ a ⊙ is descr. about ABCDE.
4	C. 1, 2.	Next, ∵ AG = AL, OA com. and ∠ OAG = ∠ OAL;
5	4, I. and 3, III.	∴ OG = OL, and the ⊙ touches AB and AE.
6	*Sim.*	So the ⊙ touches H, I, and K.
7	Def. 5, IV.	∴ ⊙ GLI is inscribed in the polygon.
8	Rec.	∴ *A circle may be described, &c.* Q. E. D.



USE AND APP. 1.—In a reg. polygon the angles at the centre, opposite to the sides, are all equal, and together make up four rt. angles; therefore *in constructing a polygon,* $\frac{360°}{\text{No. of sides}}$ = each ∠ at the centre; and the *chord* of that ∠ = *a side;* and the *circumference* being divided *by the chord,* the points of the polygon will be obtained.

2. *To inscribe a polygon in a given circle;* divide the circle into as many equal parts as there are sides of the polygon, and join the points of the circumference by chords.

3. *To circumscribe a polygon*; draw tangents at the points where the circumference is divided into equal parts and produce them until they meet.

4. *The area of a regular polygon*; as ABCDE, is equal to the product of the perimeter into half the radius, $\frac{OL}{2}$, of the inscribed circle;

For area of $\triangle$ OAE $= AE \times \frac{OL}{2}$; and there are as many triangles identical with $\triangle$ OAE as there are sides of the polygon; $\therefore$ taking n to represent the number of sides, *area of polygon* $= n \times AE \times \frac{1}{2}$ OL.

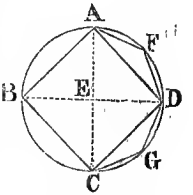
5. By an extension of the principle, as shown in 41, I. Use 4, the *area of a circle* = the product of the rad. and of half the ⊙ce.

Prop 6.—Prob.

To inscribe a square in a given circle.

Con. 1, III. 11, I. Pst. 3.

Dem. Def. 15, I. Ax. 11. 4, I. 31, III. Def. 3, IV.
Def. 30, I. A square has all its sides equal and all its angles rt. angles.



E.1	Datum	Let ABCD be the given circle;
2	Quæs.	to inscribe a square in ABCD.
C.1	1, III. 11, I.	Draw the diams. AC, BD at rt. ∠s to each other;
2	Pst. 3.	and join AB, BC, CD, DA;
3	Sol.	then the fig. ABCD is the □ required.
D.1	Def. 15, I.	Here EA = EB = EC = ED;
2	D.1C.1Ax.11	∵ BE = ED, EA com, and ∠s at E equal;
3	4, I. and *Sim.*	∴ BA = DA; and BA = DA = BC = CD;
4	Conc.	∴ the quadril. ABCD is equilateral.
5	C. 1. 31, III.	Again, ∵ BD is a diam., ∴ ∠ BAD is a rt. ∠;
6	*Sim.*	also each of the ∠s ABC, BCD, CDA is a rt. ∠;
7	Conc.	∴ the qu. lat. ABCD is equiangular
8	D.4.7.Def. 30	∴ the qu. lat. is a square;
9	Def. 3, IV.	and inscribed in the ⊙ ABCD. Q. E. F.

Con.—The square on the radius, EC^2, of an inscribed ⊙, is half the square inscribed in the circle, and one-fourth the square on its diam., AC^2; *i. e.* $EC^2 = \frac{ABCD}{2} = \frac{AC^2}{4}$.

Sch.—In other words, the inscribed square $ABCD = 2EC^2 = \frac{1}{2}AC^2$; for the $\angle$s at E being rt. $\angle$s, we have, by 47, I, $BC^2 = BE^2 + CE^2 = 2r^2$; $\therefore BC = r\sqrt{2}$; And $AC^2 = 4EC^2 = 2BC^2$.

Ex. 1.—The radius being 8, required the area of the inscribed square.

$$2 \times 8 \times 8 = 128.$$

Ex. 2.—When the area of the insc. square is 288, what is the diam. of the circle?

$$\sqrt{(2 \times 288)} = \sqrt{576} = 24.$$

Use and App.—By bisecting the arcs AD, DC, CB, BA, and joining the points F, G, &c., an *octagon* will be formed; and by continuing the bisections, other regular polygons of 16, 32, 64, &c., sides.

Prop. 7.—Prob.

To circumscribe a square about a given circle.

Con. 1, III. 11, I. Pst. 1. 17, III.

Dem. 18, III. Def. 15-17, I. Def. 30. Def. 4, IV.

28, I. If a st. line falling upon two other st. lines makes the ext. $\angle$ = the int. and opp. $\angle$ upon the same side of the line; or makes the int. $\angle$s upon the same side together equal to two rt. $\angle$s; the two st. lines shall be parallel.

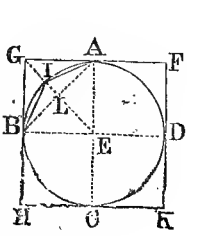
30, I. Straight lines parallel to the same straight line are parallel to each other.

Def. A. I. A parallelogram is a four-sided fig., of which the opp. sides are parallel.

34, I. The opp. sides and $\angle$s of $\square$s are equal to one another, and the diam. bisects them; *i. e.* divides them into two equal parts.

Cor. 2. 46, I. Every $\square$ having one rt. $\angle$ has all its $\angle$s rt. $\angle$s.

E.1	Datum.	Given a $\odot$ ABCD;
2	Quæs.	to descr. a square about it.
C.1	1, III.	Find E the cen. of the $\odot$;
2	11, I.	at E draw two diams. AC, BD, at rt. $\angle$s to each other;
3	17, III.	and through A, B, C, D, tangents forming fig. GHKF;
4	Sol.	then GHKF is the $\square$ required.



D.1	C. 3 and 2.	∵ GF is a tang. at A, and AE a st. line from A to cen. E;
2	18, III.	∴ the ∠s at A are rt. ∠s;
3	*Sim.*	So the ∠s at B, C, D, are rt. ∠s.
4	C. 2 & 3. 28, I.	And ∵ the ∠s AEB, EBG, are rt. ∠s, ∴ GH ∥ AC;
5	*Sim.* 30, I.	also FK ∥ AC, and GH ∥ FK;
6	*Sim.*	and GF, FK, each ∥ BD;
7	Def. A.I.	∴ the figures GK, GC, AK, FB, BK, are □s;
8	34, I.	and ∴ GF = HK and GH = FK;
9	Def.15-17. 34,I.	and ∵ AC = BD; AC = GH = FK; and BD = GF = HK;
10	Conc.	∴ GH = FK = GF=HK, and FGHK is equilateral.
11	D. 7. C. 2.	Again, ∵ GBEA is a □, and ∠ AEB a rt. ∠;
12	34, I.	∴ ∠ AGB is a rt. ∠;
13	Cor. 2. 46, I.	and ∴ ∠s at H, K, and F are rt ∠s;
14	Conc.	∴ FGHK is rectangular;
15	Def.10,14.Def.30	∴ the fig. FGHK is a square,
16	Def. 4, IV.	and it is circumscribed about the ⊙ ABC.

Q. E. F.

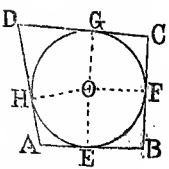
Cor.—In the same circle the circumscribed square GHFK is double of the inscribed square; *i. e.* BD^2 or $AC^2 = 2AB^2 = 4AE^2$.

Use and App. 1.—By bisecting the arcs and drawing chords a *regular octagon* may be *inscribed*, AI, IB, &c.; and by drawing tangents through the angular points of the inscribed octagon, a *reg. octagon* may be *circumscribed.*

2.—A reg. octagon, AIB, &c., inscribed in a circle, ABCD, is equal to the rectangle under the sides, AB, BD, of the inscribed and circumscribing squares.

D.1	6, 7, IV.	For BD and AE are at rt. ∠s; and BD ∥ GF ∥ HK side of circumscribing □;
2	28, III.	also AI = IB, and arc AI = arc IB; and EI ⊥ AB bisects it in I;
3	41, I.	∴ 2△ EAI = IE · AL; and 2 △ EBI = IE · BL.
4	*Add.*	∴ 2 the qu. lat. BEAI = IE · (AL + BL) = IE · AB;
5		Or 4 BEAI = 2 BE · BA = BD · BA;
6	Remk.	Now the octagon is 4BEAI; and BD = side of circumscrg. □, and AB = side of inscribed squarc.

3.—If a quadril. ABCD be circumscribed about a circle EFGH, any two of its opp. sides AB+CD, or AD+BC = half its perimeter, *i. e.* $\frac{AB+BC+CD+AD.}{2}$



D.1	Cor. 37, III.	∵ tangs. AE = AH, BE = BF, CF = CG, and DG = DH;
2	Ax. 1.	∴ AE + BE + CF + DG = AH + BF + CG + DH;
3		*i. e.* AB + CD = AD + BC, half the perimeter.

PROP. 8.—PROB.

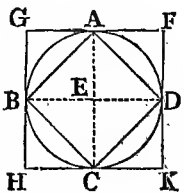
To inscribe a circle in a given square.

CON. 10, I. 31, I. Through a given point to draw a parallel to a given st. line.

DEM. Def. 30, I. Cor. 2. 46, I. 34, I. Def. 5, IV.
Ax. 7. Things which are halves of the same are equal to one another.
29, I. If a line fall upon two par. st. lines, it makes the alt. ∠s equal to one another; and the ext. ∠ = int. and opp. ∠ upon the same side; and likewise the two int. ∠s upon the same side together = two rt. ∠s.

Cor. 1. 16, III. If a st. line be drawn at rt. ∠s to any diam. of a ⊙ from its extremity, it shall touch the ⊙ at the extremity; and a st. line touching the ⊙ at one point shall touch it at no other point.

E.1	Datum.	Let GHKF be the given square,
2	Quæs.	to inscr. a circle in that □.
C.1	10, I.	Bis. the sides GH, GF in B and A;
2	31, I.	through A and B draw AC ∥ GH or FK, and BD ∥ GF or HK;
3	Pst. 3.	with rad. EA, from E, draw a ⊙;
4	Sol.	the ⊙ ABCD is inscr. in the sq. GHKF.



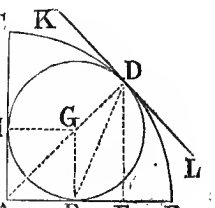
D.1	C. 2.	Each of the figs. GD, DH, GC, CF, GE, EK, HE, EF is a □ ;
2	Def. 30.	and ∵ each contains an ∠ of the sq. GHKF;
3	Cor. 2. 46, I.	∴ each of those figures is rectangular;
4	34, I.	and ∴ of each, the opp. sides are equal.
5	Def. 30. E 1.	Now ∵ GF = GH, and GA = ½GF, and GB = ½GH;
6	Ax. 7. D. 3.	∴ GA = GB, and BE = EA;
7	*Sim.*	Thus EA = ED = EC = EB;
8	Conc.	∴ the ⊙ from E, with rad. EA, passes through the · s B, C, D.
9	29, I.	And ∵ the ∠s at A, B, C, D, are rt. ∠s;
10	C.	and the st. lines GF, FK, KH, HG are at the ends of diams. BD, AC;
11	Cor. 16, III.	∴ each of those st. lines is a tang to the ⊙ ;
12	Def. 5, IV.	∴ the ⊙ ABCD is inscribed in the □ GHKF.

Q. E. F.

N.B.—The diagram will illustrate the Cor. to Pr. 7, IV.

Sch.—Euclid confines himself in this book to the inscription and circumscription of circles and regular rt. lined figures,—but circles may be inscribed in segments and sectors ; for example,

To inscribe a circle in a given quadrant ABC.



C.1	9, I. 31, I.	With AD bis. ∠ CAB and draw DE ∥ AC;
2	9, I. 31, I.	with DF bis. ∠ ADE, and draw FG ∥ AC and meeting AD in G ;
3	31, I. 11, I.	also draw GH ∥ AB and KL ⊥ AD in D ;
4	Sol.	the cen. of the required ⊙ is G; and GD or GF its rad.
D.1	C. 1. H.	∵ ∠ DAB = ½ ∠ CAB a rt. ∠, and GFA is a rt. ∠ ;
2	6, I.	∴ ∠ AGF = ½ a rt. ∠, and AF = FG ;
3	34, I. Ax. 1.	but AF = HG ; ∴ FG = HG.
4	C. 2. 29, I.	Again ∵ ∠ GDF = ∠ EDF, and ∠ EDF = ∠ DFG;
5	Ax. 1. 6, I. D. 3.	∴ ∠ GDF = ∠ DFG, and GF = GD = HG ;
6	9, III.	∴ the ⊙ HFD passes through the points, D, F, H.
7	Cor. 46, I.	Also ∵ the ∠s at H and F are rt. ∠s ;
8	16, III.	∴ the ⊙ touches AC and AB in H and F ;
9	11, III.	and ∴ AG, joining the centres A and G, passes through D;
10	C. 3 and D. 6.	and ∵ LD or KD, a perp. to AD at D, is a tang. to arc CDB and to the ⊙ DFH ;
11	11, III.	∴ the ⊙ DFH touches the arc CDB.

Q. E. F.

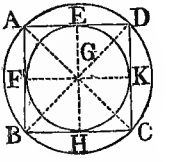
PROP. 9.—PROB.

To circumscribe a circle about a given square.

CON. Pst. 1 and 3.

DEM. Def. 30, I. Axs. 11 & 7. Def. 6, IV. 8, I. If the △s have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the ∠ which is contained by the two sides of the one, shall be equal to the ∠ contained by the two sides equal to them of the other.

6, I. If two ∠s of a △ be equal to one another, the sides opp. to the equal ∠s shall be equal to one another.



E.1	Datum.	Let ABCD be the given □;
2	Quæs.	to circumsc. a ⊙ about ABCD.
C.1	Pst. 1. 3.	Join AC, BD cutting in G; and with GA desc. a ⊙;
2	Sol.	*that* ⊙ ABCD is the ⊙ required.
D.1	Def. 30, I.	∵ in △s ABC, ADC, DA = AB, CB = CD, and AC com.;
2	8, 1.	∴ ∠ DAC = ∠ BAC, *i. e.* ∠ BAD is bisd. by AC;
3	*Sim.*	So ∠s ABC, BCD and CDA are bisected by BD and AC.
4	Def. 30, I.	Now ∵ ∠ DAB = ∠ ABC;
5	D. 3.	and ∠ GAB = ½∠ DAB, and ∠ GBA = ½∠ ABC:
6	Ax. 7 & 6. I.	∴ ∠ GAB = ∠ GBA, and GA = GB.
7	*Sim.*	So GA = GB, and GC = GB = GD;
8	Ax. 1.	∴ GA = GB = GC = GD.
9	9, III.	Hence a ⊙ from G, with rad. GA, will pass through B, C, and D.
10	D. 8.	And ∵ the ⊙ ABC passes through the angular • s of the sq. ABCD,
11	Def. 6, IV.	∴ the ⊙ ABC is c. scrd. about the □ABCD.

Q. E. F.

N.B. In the diagram a circle is also inscribed in the square ABCD.

Prop. 10.—Prob.

To construct an isosceles triangle, having each of the angles at the base double of the third, or vertical angle.

Con. Pst. 3. 1. IV. Pst. 1. 5, IV. 11, II. To divide a given line into two parts, so that the rect. contained by the whole and one of the parts shall equal the sq. of the other part.

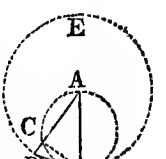
Dem. Ax. 1. 32, III. 6, I. 37, III. If from a · *without* a ⊙ there be drawn two lines, one of a which cuts the ⊙, and the other meets it; if the rect. contained by the whole line which cuts the ⊙, and the part of it without the circle, be equal to the sq. of the line which meets it, the line which meets shall touch the ⊙.

Ax. 2. If equals be added to equals the wholes are equal.

32, I. If a side of a △ be produced, the ext. ∠ = the two int. and opp. ∠s, and the three int. ∠s of every △ are together equal to two rt. ∠s.

Def. 25, I. An Isosceles △ is that which has two equal sides or legs.

5, I. The ∠s at the base of an isosc. △ are equal to each other; and if the eq. sides be produced, the ∠s on the other side of the base shall be equal.



E.1	Data.	Given the ∠s ABD, ADB each = 2∠ BAD;
2	Quæs.	to construct an isosc. △ ABD.
C.1	11, II.	Take any line AB and divide it so that AB · BC = AC²;
2	Pst. 3.	from A with rad. AB desc. ⊙ BDE;
3	1, IV. Pst. 1.	in ⊙ BDE place BD = AC ⊁ diam. of ⊙ BDE, and join AD;
4	Sol.	then in △ ABD, ∠ ABD = ∠ BDA = 2 ∠ BAD.
5	Pst. 1. 5., IV.	Join DC, and about △ ADC desc. ⊙ ACD.
D.1	C. 1. Ax. 1.	∵ AB · BC = AC², and AC = BD; ∴ AB · BC = BD²;
2	C.	and ∵ from B, a · out of ⊙ ACD, are drawn BCA, BD, one cutting the ⊙ in C, the other meeting it in D;
3	D. 1. 37, III.	and ∵ AB · BC = BD², ∴ BD touches the ⊙ ACD in D.

D.4	D. 3. & C.	Again ∵ BD touches the ⊙ ACD, and DC a st. line from the · of contact D, cuts that ⊙;
5	32, III.	∴ ∠ BDC = ∠ DAC in the alt. seg.;
6	*Add.* Ax. 3.	Add ∠ CDA to each;
		∴ ∠ BDA = ∠ CDA + ∠ DAC;
7	32, I. Ax. 1.	But ∵ ext. ∠ BCD = ∠s CDA, DAC,
		∴ ∠ BDA = ∠ BCD;
8	C. 5, I. Ax. 1.	and ∵ AD = AB, and ∠ BDA = ∠ CBD;
		∴ ∠ CBD or DBA = ∠ BCD;
9	Ax. 1.	∴ ∠ BDA = ∠ DBA = ∠ BCD.
10	D. 7. 9. 6, I.	Again ∵ ∠ DBC = ∠ BCD;
		∴ BD = DC;
11	C.3. Ax.1.5,I.	but BD = CA, ∴ AC = CD and ∠ CDA = ∠ DAC;
12	Ax. 2.	∴ ∠s CDA + DAC = 2 ∠ DAC;
13	D. 7. 32, I.	but ∠ BCD = ∠ CDA + ∠ DAC;
		∴ ∠ BCD = 2∠ DAC.
14	D. 9.	Now ∠ BCD = ∠ BDA = ∠ DBA;
15		∴ ∠ BDA = DBA = 2∠ DAB.
16	Rec.	∴ the △ ABD is the isosc. △ required.

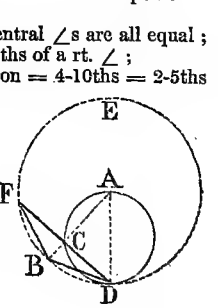
Q. E. F.

USE AND APP.—The following are some of the various problems which bear a close relation to the 10th :—

1°. *The side* AC *inscribed in the smaller* ⊙ ACD *equals the side of a regular* pentagon *in that circle, and also equals the side of a regular* decagon *in the larger* ⊙ BDE.

Prel. 1	2 Cor. 15, I.	∵ the ∠s formed by lines from a central point = 4 rt. ∠s;
2		and ∵ in a regular polygon the central ∠s are all equal;
3		∴ the central ∠ of a pent. = 4-5ths of a rt. ∠;
4		and ∴ the central ∠ of a decagon = 4-10ths = 2-5ths of a rt. ∠.

First—For the *decagon* in ⊙ DEF.,



D.1	10,IV.Prel.4.	∵ AC = BD; and ∠ A = 2-5ths of a rt. ∠;
2	Conc.	∴ AC or BD = the side of a reg. decagon in ⊙ BEF.

Second,—For the *pentagon* in ⊙ A C D.

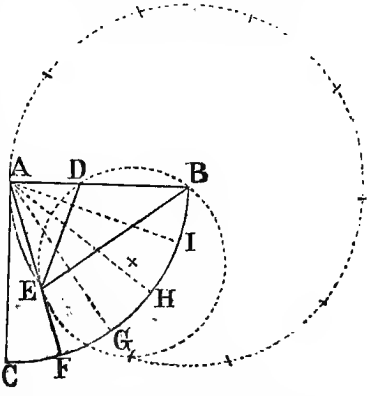
D.1	10, IV., & C.	∵ C D = B D = A C; and A C subtends ∠ A D C at the ⊙ce of a ⊙ A C D.
2	20, III.	and ∵ also ∠ A D C = 2-5ths of a rt. ∠ = ½ the ∠ at the centre of ⊙ A C D;
3	Ax. 6.	∴ the ∠ at the cen. of ⊙ A C D = 4-5ths of a rt. ∠;
4	Conc.	and ∴ A C = side of a reg. pent. in the ⊙ A C D.

2°. *On the side* D C *being produced to meet the circle* B D E *in* F, *and* F B *being joined, the* ∠ A B F = *three times* ∠ B F D.

D.1	C. and 32, I.	∵ ∠ B A D = ∠ A D C, and ∵ ∠ B C D = ∠s B A D + A D C;
2	Ax. 1.	∴ ∠ B C D = 2 ∠ B A D.
3	20, III.	Now, ∠ B A D = 2 ∠ B F D; and ext. ∠ B C D = 4 ∠ B F D = ∠s A B F + B F D;
4	*Sum.*	from each side take away ∠ B F D;
5	Ax. 3.	∴ ∠ A B F = 3 ∠ B F D.

3°. *To quinquisect*, i.e., *divide a quadrant* B A C *into five equal parts.*

C.1	10, IV.	Construct an isosc. △ B A E, having ∠ B A E = ∠ B E A = 2 ∠ A B E;
2	9, I. Sch.	prod. A E to F, and div. ∠ F A B into 4 equal parts;
3	Sol.	then ∠ C A F = ∠ F A G = ∠ G A H = ∠ H A I = ∠ I A B = 1-5th ∠ B A C.
D.1	C. & Def.	∵ of a decagon the central ∠ A B E = 2-5ths of a rt. ∠;
2	C.	and ∵ ∠ B A F = 2 ∠ A B E = 4-5ths of a rt. ∠;
3	Datum.	but ∠ B A C is a rt. ∠;
4	Ax.	∴ ∠ C A F = 1-5th of a rt. ∠, and ∠ F A B = 4-5ths of a rt. ∠.
5	C. 2.	and ∠ F A B is div. into four equal parts;
6	Conc.	∴ the quadrant B A C has been divided into five equal parts. Q. E. F.



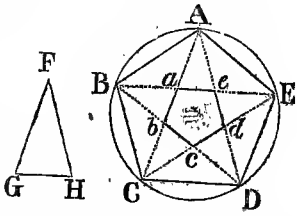
It is evident that, by the process of continual bisection, the quadrant may now be divided into 10, 20, 40, 80, &c., equal parts.

PROP. 11.—PROB.

To inscribe an equilateral and equiangular pentagon in a given circle.

CON. 10, IV. 2, IV. 9, I. Pst. 1.

DEM. 26, III. In eq. circles, eq. angles stand upon eq. arcs, whether they be at the centres or circumferences.
29, III. In eq. circles eq. arcs are subtended by eq. st. lines. AX. 2.
27, III. In eq. circles, the angles which stand upon eq. arcs are equal to one another, whether they be at the centres or the ⊙ces. Def. 3, IV.



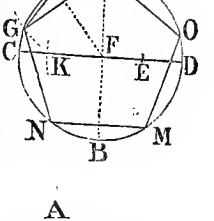
E.1	Dat.	Let ABCDE be the given ⊙;
2	Quæs.	to inscr. therein a reg. pent.
C.1	10, IV.	Desc. an isosc. △ FGH, with ∠ G = ∠ H = 2 ∠ F;
2	2, IV.	in ⊙ ABCDE insc. △ ACD equ. ang. to △ FGH; so that ∠ CAG = ∠ F, and ∠s ACD, CDA each = ∠ G = ∠ H;
3	9, I,	Bis. ∠s ACD, ADC by CE, CB cutting the ⊙ce in B and E;
4	Pst. 1.	join AB, BC, DE and EA;
5	Sol.	then the fig. ABCDE is the pent. required.
D.1	C. 2, & 1.	∵ △ ACD is eq. ang. to △ FGH, and ∠ G = ∠ H = 2 ∠ F;
2	Ax. 1,	∴ ∠ ACD = ∠ ADC = 2 ∠ CAD.
3	C. 3.	and ∵ the eq. ∠s ACD, ADC are bisected by CE, DB;
4	Ax. 7.	∴ ∠ DAC = ∠ ACE = ∠ ECD = ∠ CDB = ∠ BDA;
5	26, III.	but eq. ∠s at the ⊙ce stand on eq. arcs;
6	Conc.	∴ the arcs, AB, BC, CD, DE, EA are all equal;
7	29, III.	and eq. arcs have equal chords;
8	Conc.	∴ the chords AB, BC, CD, DE, EA are all equal;
9	Conc. 1.	∴ the pentagon is equilateral.
10	*Add.*	Again, to each of the eq. arcs AE, DE, add arc BCD;

11	Ax. 2.	∴ the whole arc ABCD = the whole arc EDCB;
12	C.&D.10&11	but ∠ AED stands on arc ABCD, and ∠ BAE on arc EDCB;
13	27, III	∴ ∠ BAE = ∠ AED;
14	*Sim.*	So ∠s ABC, BCD, CDE, each = ∠ BAE, or ∠ AED;
15	Conc. 2.	∴ the pent. is equiangular.
16	D. 9.	And it has been shown to be equilateral;
17	C.	and ∵ the angular · s A, B, C, D, E are in the ⊙ce,
18	Def. 3, IV.	∴ the pent. is inscribed in the circle.

Q. E. F.

Scu. I.—In a reg. pentagon we may remark; 1st., that each diagonal, as A C, is parallel to the side, as E D, to which it is not conterminous; 2nd., that triangles, B C *b*, E D *e*, A E *d*, &c., are isosc. △s, equiangular with △ C A D, &c., and having the ∠ at the base = *twice* the vert. angle; 3rd., that the fig. A B *c* E is a lozenge, also B C D *e*, &c.; and 4th., that fig. *a b c d e* is a reg. pentagon. Lardner's *Euclid*, p. 130.

II. *Practically*, a pentagon is inscribed in a circle, by drawing two perpendicular diameters, A B, C D, and bisecting the rad. F D in E; from E with E A desc. A K, and from A with A K desc. K G; then A G is the side of the pentagon; and if the arc A G, be bisected in H, the chord A H is a side of a decagon. See Euclid 10, xiii.

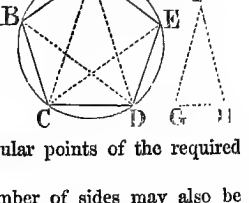


III. It is generally *by aid of isosc.* △s, with the ∠s at the base *equimultiples* of the vert. ∠, that reg. rectil. figures which have an *odd* number of sides are inscribed in circles. Thus, *for the pentagon*;

By 10, iv., desc. an isosc. △ F G H, having each of the ∠s, G and H at the base double of ∠ F, the ∠ at the vertex.

2, iv., in the given ⊙ insc. a △ A C D eq. ang. with △ F G H;

9, I., bis. the ∠s A C D, A D C; and let the bisecting lines be produced to meet the ⊙ce in B and E;



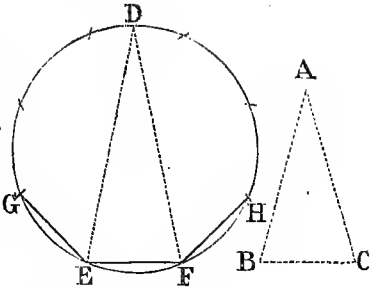
Then the points, A, B, C, D, E, are the angular points of the required pentagon.

IV. Reg. rectil. polygons with an *even* number of sides may also be inscribed in circles *by aid of isosc* △s; but in such isosc. △s, the ∠s at the base are multiples *sesquialter*, as it is termed, of the vert. ∠; *i. e.*, each angle at the base contains 1½ times, or 2½ times, or 3½ times, &c., the magnitude of the vert. ∠. Thus *for the octagon*;

Construct an isosc. △ ABC, of which each ∠ at the base, ∠ B or ∠ C = 2½ times the vert. ∠ A.

By 2, iv., in the given ⊙ DGH insc. a △ DEF eq. ang. with △ ABC;

then the base EF, the chord of the arc EF, will be a side of the octagon to be inscribed in ⊙ DGH.



V. *Formulæ* for determining *the relative magnitudes of the angles of isosc. △s*, to be used *in the construction of regular polygons.*

The vert. ∠ of such isosc △s $= \dfrac{180°}{\text{number of sides}}$

The multiplier of that vert. ∠ for the ∠s at the base $= \dfrac{\text{sides} - 1}{2}$.

Thus in a regular polygon of

5 sides, the vert. ∠ = 36°;	the multiplier is 2;	the ∠s at the base each = 72°			
7	„ = $25°\frac{5}{7}$;	„	3;	„	= $77°\frac{1}{7}$
9	„ = 20°;	„	4;	„	= 80°
11	„ = $16°\frac{4}{11}$;	„	5;	„	= $81°\frac{9}{11}$
13	„ = $13°\frac{11}{13}$;	„	6;	„	= $83°\frac{1}{13}$

And in a regular polygon of

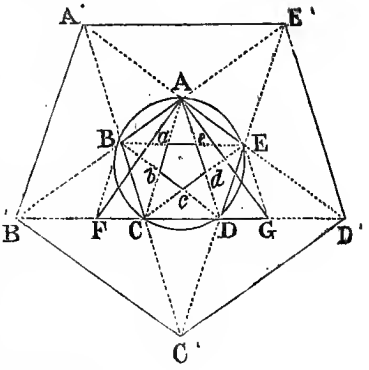
6 sides, the vert. ∠ = 30°;	the multiplier is $2\frac{1}{2}$;	the ∠s at the base each = 75°			
8	„ = $22°\frac{1}{2}$;	„	$3\frac{1}{2}$;	„	= $78°\frac{3}{4}$
10	„ = 18°;	„	$4\frac{1}{2}$;	„	= 81°
12	„ = 15°;	„	$5\frac{1}{2}$;	„	= $82°\frac{1}{2}$
14	„ = $12°\frac{6}{7}$;	„	$6\frac{1}{2}$;	„	= $83°\frac{4}{7}$
16	„ = $11°\frac{1}{4}$;	„	$7\frac{1}{2}$;	„	= $84°\frac{3}{8}$
18	„ = 10°;	„	$8\frac{1}{2}$;	„	= 85°
20	„ = 9°;	„	$9\frac{1}{2}$;	„	= $85°\frac{1}{2}$

APP.—I. *To draw a triangle equal in area to a given pentagon,* ABCDE see on page 118.

C. 1	Pst. 1 & 2	Join AC, AD, and prod. CD;
2	31, I.	through B and E draw BF ∥ AC, EG ∥ AD;
3	Pst. 1.	and join AF, AG;
4	Sol.	then △ AFG = pent. ABCDE.
D. 1	37, I.	∵ △ ACF = △ ACB, and △ ADG = △ ADE;
2	*Add.*	add △ ACD to the equals;
3	Ax. 2.	∴ △ AFG = pent. ABCDE.

II. *The lines* AC, BD, CE, DA, EB, *joining the alternate angles,* A,C; B,D; C,E; D,A; E,B; *of a reg. pentagon,* ABCDE, *will form another reg. pentagon,* a b c d e; *and the points of intersection,* A′, B′, C′, D′, E′, *of the alternate sides of* ABCDE *produced, namely,* AB *and* CD, *or* DE; BC *and* AE, *or* ED; CD *and* AB, *or* AE; DE *and* AB, *or* BC; EA *and* BC, *or* CD; *will also form another reg. pentagon,* A′ B′ C′ D′ E′.

First—The figure $abcde$ is a regular pentagon.



D.1	C. & H.	∵ arc BCDE = arc AEDC, and arc BC = arc AE;
2	27, III.	∴ ∠ BAE = ∠ ABC, and ∠ BAC = ∠ ABE;
3	Ax. 3.	∴ remg. ∠CAE = remg. ∠EBC.
4	H.15,I.&D.3.	and ∵ BC = AE, ∠ B a C = ∠ A a E, and ∠ CAE = ∠ EBC;
5	26, I. 6, I.	∴ A a = B a, and ∠ ABE = ∠ BAC;
6	32, I.	∴ ∠ A ae = 2 ∠ ABE = 2 ∠ AEB = ∠ A ea;
7		and ∴ A a = A c = B a = E e.
8	Remk.	Thus the △s A ac, B ab, C bc, &c., are isosc. △s;
9		they are also equal; ∴ $ae = ab = be = ed = dc$.
10	5, I.	Now, in isosc. △s the ∠s on the other side of the base are equal;
11		∴ ∠ cab = ∠ abc = ∠ bcd, &c.
12		∴ also the fig. $abcde$ is equiangular.
13	D. 9 & 12.	∴ $abcde$ is a regular pentagon. Q. E. F.

Second.,—Also A′ B′ C′ D′ E′ is a reg. pentagon.

D.1	H. 13, I.	∵ ∠ ABC = ∠ BAE, and ∠s A′BA + ABC = ∠s A′AB + BAE;
2	Ax. 3. *Sim.*	∴ ∠ A′AB = ∠ A′BA; and so = ∠ B′CB = ∠ B′BC, &c.:
3	H. 26, I.	Also AB = BC; ∴ △ ABA′ = △ BCB′;
4	*Sim.*	So △s CDC′, DED′ and EAE′ are equal.
5	D. 1-4.	And ∵ BB′, BA′ = E′A, AA′, & ∠ B′BA′ = ∠ E′AA′;
6	4, I. & *Sim.*	∴ B′A′ = A′E′; and so = E′D′ = D′C′ = C′B′;
7	Conc.	∴ the fig. A′ B′ C′ D′ E′ is equilateral.
8	D.	Also the 3 ∠s at A′ = the 3 ∠s at B′ = 3 ∠s at C′, &c.
9	Conc.	∴ the fig. A′ B′ C′ D′ E′ is equiangular;
10	D. 7, & 9.	∴ it is a reg. pentagon. Q. E. F.

Prop. 12.—Prob.

To circumscribe an equilateral and equiangular pentagon about a given circle.

Cor. 11, IV. 17, III. 1, III. Pst. 1

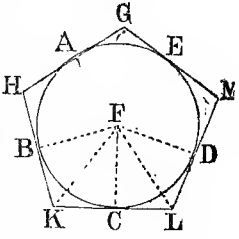
DEM. Ax. 1. Def. 15, I. Ax. 3. 8, I. 4, I. 27, III. Ax. 7. Def. 10, I 26, I. Ax. 6. Def. 4, IV.

28, III. In equal circles, equal st. lines cut off eq. arcs, the gr. = the gr., and the less to the less.

18, III. If a st. line touches a ⊙, the st. line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

47, I. In any rt. angled △ the square on the side opp. to the rt. ∠ is equal to the squares on the sides containing the rt. ∠.

E.1	Dat.	Let A B D be the given ⊙;
2	Quæs.	to desc. a reg. pent. about that ⊙.
C.1	11, IV.	In ⊙ A B D insc. a reg. pent., of which the angular • s are A, B, C, D, E, and the arcs, A B, B C, C D, D E, E A, are equal;
2	17, III.	through A, B, C, &c., draw G H, H K, K L, &c., tangents to the ⊙ A B D;
3	Sol.	then the fig. G H K L M is the pent required.
4	1, III.	Find F the centre of the ⊙ A B D;
5	Pst. 1.	and join F B, F K, F C, F L, and F D.
D.1	C. 1.	∵ A, B, C, D, E, are angular • s of the reg. pent.
2	28, III.	∴ arcs A B, B C, C D, &c., are equal;
3	C. 2, 4.	and ∵ K L is a tang.; and from cen. F, F C is drawn to C;
4	18, III.	∴ F C ⊥ K L, and ∠s F C K, F C L, rt. ∠s.
5	*Sim*	So ∠s F B K, F D L, &c., are rt ∠s;
6	D. 4 & 5.	Now ∵ ∠s F C K and F B K are rt. ∠s;
7	47, I.	∴ $F C^2 + C K^2 = F K^2$, and $F B^2 + B K^2 = F K^2$;
8	Ax. 1.	∴ $F C^2 + C K^2 = F B^2 + B K^2$;
9	Def. 15. Ax. 3.	but $F C^2 = F B^2$; ∴ $C K^2 = B K^2$, and C K = B K.
10	C. and D. 9.	Also ∵ F C = F B, F K com., and C K = B K;
11	8, I.	∴ ∠ C F K = ∠ B F K, and ∠ C K F = ∠ B K F;
12	Remk.	thus F K bisects ∠s B F C and B K C, and ∠ B F C = 2 ∠ B F K, and ∠ B K C = 2 ∠ B K F.



13	*Sim.*	So FL bisects ∠s CFD, CLD.
14	C.1.27,III.Ax.7	Again ∵ arc BC = arc CD, ∴ ∠ BFC = ∠ CFD, and ∠ KFC = ∠ CFL.
15	Def. 10. D. 14.	Also ∵ ∠ KCF = ∠ LCF, FC com., and ∠ KFC = ∠ CFL;
16	26, I.	∵ KC = CL, and ∠ FKC = ∠ FLC.
17	*Sim.*	Now ∵ KC = CL; ∴ KL = 2 KC, and so HK = 2 BK;
18	D.9. D.17. Ax.6.	and ∵ BK = KC, KL = 2 KC, and HK = 2 BK; ∴ HK = KL.
19	*Sim.*	So GH, GM, ML, each = HK, or KL;
20	Conc.	∴ the pent. GHKLM is equilateral.
21	D. 16. 12. 13.	*Lastly*, ∴ ∠ FKC = ∠ FLC, ∠ HKL = 2 ∠ FKC, and ∠ KLM = 2 ∠ FLC;
22	Ax. 6.	∴ ∠ HKL = ∠ KLM;
23	*Sim.*	and so ∠s KHG, HGM, GML, each = ∠ HKL or ∠ KLM;
24	Conc.	∴ the pent. GHKLM is equiangular;
25	D. 20 & 24.	∴ the fig. GHKLM is a reg. pentagon.
26	C. 2.	And each side touches the given ⊙ ABD;
27	Def. 4, IV.	∴ the pent. GHKLM is circumscribed about the ⊙ ABD. Q. E. F.

Sch.—It is a general truth, that, "If the circumference of a circle be divided into any number of parts, the chords joining the points o division shall include a regular polygon, inscribed in the circle; and the tangents drawn through those points shall include a regular polygon of the same number of sides circumscribed about the circle."

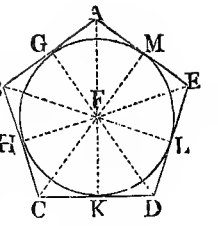
Prop. 13.—Prob.

To inscribe a circle in a given equilateral and equiangular pentagon.

Con.—9, I. 12, I. Pst. 3.

Dem.—4, I. Ax. 1. Ax. 11. All rt. ∠s are equal to one another. 26, I. Cor. 16. III. Def. 5, IV.

E. 1	Datum.	Let ABCDE be the given reg. pentagon;
2	Quæs.	in that pent. to inscribe a circle.



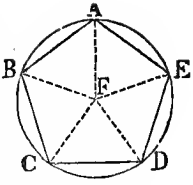
C. 1	9, I.	Bis. ∠s BCD, CDE by CF, DF cutting in F,
2	12, I.	from F draw FG, FH, FK, &c., perps. to AB, BC, &c.;
3	Pst. 1.	join FA, FB, FC, FD, FE;
4	Pst. 3.	with any of the perps. from F, desc. a ⊙ GHL;
5	Sol.	then the ⊙ GHL is the circle required.
D. 1	H. C. 1.	∵ in △s BCF, DCF, BC = CD, CF, com., and ∠ BCF = ∠ DCF;
2	4, I.	∴ BF = FD, and ∠ CBF = ∠ CDF;
3	C. 1.	But ∠ CDF = ½ ∠ CDE,
		and ∠ CDE = ∠ CBA;
4	Ax. 1.	∴ ∠ CDF = ½ ∠ CBA;
5	D. 2 Ax. 1.	and ∠ CBF = ∠ CDF;
		∴ ∠ CBF = ½ ∠ CBA,
		i. e., BF bisects ∠ CBA;
6	*Sim.*	So ∠s BAE, AED are bisected by AF, FE.
7	C. 1, 2.	Now in △s FHC, FKC, ∵ ∠ HCF = ∠ KCF, ∠ FHC = ∠ FKC and FC com.;
8	26, I.	∴ perp. FH = perp. FK;
9	*Sim.*	also FK = FL = FM = FG;
10	D. 8 & 9.	∴ the five st. lines are equal;
11	Remk.	and a ⊙ from F at the distance of one of the five lines, FG, will pass through H, K, L, M;
12	Cor. 16, III.	that ⊙ GHKLM will also touch the st. lines;
13	Def. 5, IV.	∴ ⊙ GHL is inscribed in the given pent. ABCDE. Q. E. F.

PROP. 14.—PROB.

To circumscribe a circle about a given equilateral and equiangular pentagon.

CON. 9, I. Psts. 1 and 3.

DEM. 13, V. AX. 7. 6, I. Def. 6, IV.



E.1	Datum.	Let A B C D E be a reg. pent.;
2	Quæs.	to circumsc. a ⊙ about that pent.
C.1	9, I.	Bis. the ∠s B C D, C D E, by C F, F D;
2	Pst. 1.	from F, the point in which C F, F D, meet, draw F B, F A, F E, to the points B, A, E;
3	Pst. 3.	with any one, as F A, desc. a ⊙ A C E;
4	Sol.	then the ⊙ A C E is circumsc. about the pent. A B C D E.
D.1	13, IV. *Sim.*	As in 13, IV., ∠s C B A, B A E, A E D, are bis. by F B, F A, F E;
2	H. & C. 1.	and ∵ ∠ B C D = ∠ C D E, and ∠ F C D = ½ ∠ B C D, and ∠ C D F = ½ ∠ C D E;
3	Ax. 7 & 6. I.	∴ ∠ F C D = ∠ F D C, and C F = D F.
4	*Sim.*	So F B, F A, F E, each = F C = F D;
5	Ax. 1.	∴ F A = F B = F C = F D = F E.
6	Remk.	Hence, a ⊙ from F, with any one of these sides, as F A for a rad., will pass through the extremities A, B, C, D, E;
7	D. 6.	and ∴ the ⊙ A C E passes through the angr. • s A, B, C, D, E;
8	Def. 6, IV.	∴ the ⊙ A C E is circumscd. about the reg. pent. A B C D E. Q. E. F.

Sch.—I. Generally the method of bisecting two adjoining angles of any reg. polygon may be applied to the circumscribing and inscribing of a circle about, or in that polygon; for, "If any two adjoining angles, as B C D, C D E, of a regular polygon, A B C D E, be bisected, the intersection F of the bisecting lines, C F, D F, shall be the common centre of two circles, the one circumscribing the polygon, the other inscribed in it."

II. Propositions 12, 13, and 14, may be generalised so as to include all regular polygons, and will then embrace the three problems :—

Prob. 1.—*To circumscribe a regular polygon about a given circle.*

To be done by drawing tangents through the angular points of the inscribed polygon.

Prob. 2.—*To inscribe a circle in a regular polygon.*

By bisecting two adjacent angles, and from the point where the angle-bisecting lines meet, drawing a perpendicular to a side of the polygon, and, with that perpendicular for radius, and this point as centre, by describing a circle.

Prob. 3.—*To circumscribe a circle about a given polygon.*

By bisecting two adjacent angles, and from the point where the angle-bisecting lines meet, with one of those lines for radius, describing a circle.—See Hose's Euclid, p. 153.

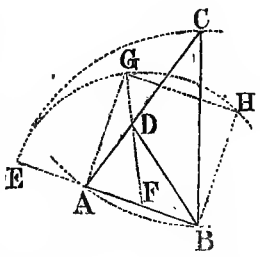
ADDENDA TO 14, IV.

Before dismissing the *regular pentagon* and its intimately allied figure, the *regular decagon*, it will be useful to state some of the *principles* of their construction.

I. To analyze the conditions on which the drawing of those figures depends,

First—Of the *Decagon* :

At the centre there are *ten* equal angles, each = 1-10th ot 360°, or 4-10ths = 2-5ths of 90°. At C draw an ∠ equal 2-5ths of a rt. ∠; and from C, with any rad., make CA = CB, and join A B; then AB the chord of 1-10th of 360° will measure the circle into *ten* equal parts, and consequently is the side of a regular decagon.



Since ∠ C = 1-5th of 180°, and ∠ A = ∠B, ∴ ∠ A and ∠ B of △ A B C each = 2-5ths ot 180°, and each is double of ∠ C.

Bis. ∠ A B C, and the △ B D C is an isosc. △ and the ext. ∠ A D B = 2 ∠ C = ∠ A.

The △ A B D is also isosc., and its ∠s are equal to the ∠s of △ A C B.

Hence △s A B D, A C B, are similar, and (by 4, VI.) the sides about the equal ∠s are proportional; *i.e.* A C : A B :: A B : A D; and A B = B D = C D; ∴ A C : C D :: C D . A D.

Thus when C A, the rad. of a circle, is divided, at D, into extreme and mean ratio (11, II.) the greater segment, C D, will be the side of the decagon inscribed in that circle.

Hence, PROB. 1.—*To find the side of an inscribed decagon, when the rad.*, C A, *of a circle is given.*

By 11, II., divide the rad. C A into extreme and mean ratio in D, and the greater segment C D equals the side of the decagon required.

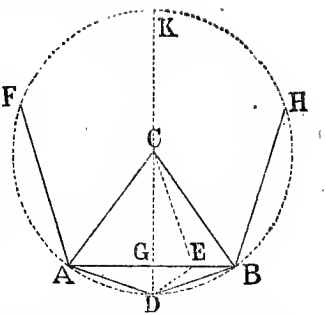
PROB. 2.—*To find the radius of the circumscribing circle when the side* A B, *of the regular decagon is given.*

By 11, II., produce B A, so that the rectangle contained by the whole line produced BE, and the part produced, A E, *i. e.* B E · A E = A B²; the whole line thus produced, B E, is the radius of the circumscribing circle.

SECOND. Of the *Pentagon.*

By joining the alternate angular points of the decagon a corresponding Pentagon will he drawn.

II. To demonstrate, *that the square on the side* A B *of a reg. pentagon inscribed in a circle equals the sum of the squares of the rad.* A C, *and of the side* A D, *of the inscribed decagon; i.e.* A B² = A C² + A D².



Let the sides of the inscribed decagon be AD, DB; join A, B; then AB is the side of the inscribed pent.

Bis. ∠ BCD by CE, and join ED.

Then ∵ DC, CE = BC, CE, and ∠ DCE = ∠ BCE;

∴ (4, I.) DE = EB, and △ CDE = △ CBE.

And in △s BED, ADB, ∵ ∠ EBD is com. ∠ EDB = ∠ BAD, and ∠ BED = ∠ ADB,

∴ the △s BED, ADB, are similar.

Hence (4, VI.) AB : BD :: BD : BE, or $BD^2 = AB \cdot BE$ (1).

Again, ∵ the rad. CA bisects the ∠ BAF, of a regular figure,

∴ ∠ CAE = ½ ∠ BAF of the inscribed pentagon;

And ∵ the *five* angles together = 6 rt. ∠s, ∴ the half of one of the five, namely ∠ CAE = 3-5ths of a rt. ∠.

But, ∵ ∠ ACD = 2-5ths of a rt. ∠, and ∠ DCE = 1-5th of a rt. ∠,

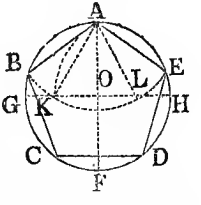
∴ ∠ ACE = 3-5ths of a rt. ∠.

Thus, ∠ CAE = ∠ ACE, and △ AEC is isosceles and similar to △ ACB.

Hence (4, VI.) AB : AC :: AC : AE; or $AC^2 = AB \cdot AE$ (2).

Adding Equations (1) and (2); $BD^2 + AC^2 = AB \cdot BE + AB \cdot AE = (2, II.)\ AB^2$.

Cor.—Hence a *Practical Method* for inscribing a pentagon in a given circle ADBE.



Draw diam. GH at rt. ∠s to diam AF;
bisect the rad. OH in L, and join LA;
make LK = LA, and join KA;
KA is the length of the side of the pentagon required.

N.B.—The rad. OG is cut in extreme and mean ratio in the point K; and OK is the length of the side of an inscribed decagon.

Prop. 15.—Prob.

To inscribe an equilateral and equiangular hexagon in a given circle.

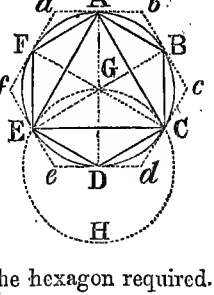
Cor. I, III. Pats. 3. 1. 2.

DEM. Def. 15, I. Ax. 1. 32, I. 26, III. 29, III. Ax. 2. 27, III. Def. 3, IV.

Cor. 5, 1. Every equil. △ is also equiangular.

13, I. The ∠s which one st. line makes with another upon one side of it are either rt. ∠s, or together equal to two rt. ∠s.

15, I. If two st. lines cut one another, the opp. or vert. ∠s shall be equal.



E.1	Datum.	Let ACE be a given circle;
2	Quæs.	to inscribe in it a reg. hexagon.
C.1	1, III.	Find G the cen. of ⊙ ACE, and draw the diam. AD;
2	Pst. 3.	from D, with D G, desc. ⊙ E C H;
3	Psts. 1 & 2.	join E G, C G, and prod. them to B, F;
4	Pst. 1.	join also A B, B C, C D, D E, E F, F A;
5	Sol.	the fig. A B C D E F is the hexagon required.
D.1	C. 1 & 2.	∵ G is cen. of ⊙ A C E, and D of ⊙ E C H;
2	Def. 15, I. Ax. 1.	∴ G E = G D, and D E = G D; ∴ G E = E D;
3	D. 2. Cor. 5, I.	and ∵ △ E G D is equil.; ∴ ∠ E G D = ∠ G D E = ∠ D E G;
4	32, I.	but the 3 ∠s of a △ = 2 rt. ∠s; ∴ ∠ E G D = ⅓ of 2 rt. ∠s.
5	*Sim.*	So ∠ D G C = ⅓ of 2 rt. ∠s;
6	13, I.	and ∵ G C with E B makes ∠s E G C, C G B, = 2 rt. ∠s;
7	Conc.	∴ remg. ∠ C G B = ⅓ of 2 rt. ∠s;
8	D. 4, 5, 7.	∴ ∠ E G D = ∠ D G C = ∠ C G B;
9	15, I.	to these ∠s are equal the vert. ∠s B G A, A G F, F G E;
10	D. 8. 9.	∴ the *six* ∠s are eq.; E G D = D G C = C G B = B G A = A G F = F G E.
11	26, III.	But eq. ∠s stand on eq. arcs,
12	Conc.	∴ the *six* arcs are eq.; A B = B C = C D = D E = E F = F A;
13	29, III.	and eq. arcs have eq. chords;
14	Con	∴ the *six* chords are eq.; and A B C D E F is equil.
15	D 13. *Add.*	Again, ∵ arc A F = arc E D, to each add arc A B C D;

16	Ax. 2.	∴ the whole arc F A B C D = the whole arc E D C B A.
17	C.	Now ∠ F E D stands on arc F A B C D, and ∠ A F E = on arc F D C B A;
18	27, III.	∴ ∠ A F E = ∠ F E D;
19	*Sim.*	So ∠s A B C, B C D, C D E each = ∠ A F E = ∠ F E D;
20	D. 14 and 10.	∴ the hexagon is eq. lat. and eq. angular,
21	Def. 3, IV.	and it is inscribed in ⊙ A C E.

Q. E. F.

Cor. 1.—*The side of a regular hexagon inscribed in a circle is equal to the radius, or semi-diameter, of the circle*; or, in other words, *the chord of* 60° *is equal to the radius.*

D.1	15,IV.D.2.	∵ G E = E D; E D being the side of the hexagon, and G E the rad.;
2	Conc.	∴ E D the side, or chord of 60° = G E the radius.

Cor. 2.—*An equilateral triangle would be inscribed by joining the points* A, E, *and* C, *alternate points in the hexagon.*

Cor. 3.—*Every equil. figure inscribed in a* ⊙ *is also equiangular; for* ∵ *its* ∠*s are contained by the chords of equal arcs, and* (28, III.) *stand on equal arcs,* ∴ *its* ∠*s are all equal, each to each.*

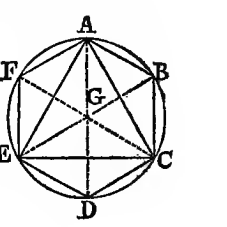
Sch.—1. The opposite sides, A B and D E, or B C and E F, or C D and F A, of a hexagon are parallel; for, ∵ A D meeting A B and D E makes the alt. ∠ B A D = the alt. ∠ A D E, ∴ (27, I.) A B ∥ D E.

2. If, through the angular points of the inscribed hexagon, A, B, C, D, E, F, tangents to the circle be drawn, *a b*, *b c*, *c d*, *d e*, *e f*, *f a*, these will form a regular hexagon circumscribing the circle.

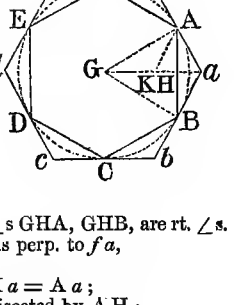
The Proof is the same as in Prop. 12 and 13, IV.

Use and App.—1. *On a given line,* A B, *to describe a regular hexagon.*

C. 1	1, I.	On A B draw an eq. lat. △ A G B;
2	Pst. 3.	from G, with G A, or G B, desc. a ⊙ A C E;
3	Sol.	then A B will exactly divide the ⊙ce into *six* equal parts.
D.1	C. 1.	∵ G A, G B = G C, G B, and ∠ A G B = ∠ B G C;
2	4, I. & *Sim.*	∴ A B = B C; and B C = C D = D E = E F = F A.
3	Conc.	∴ fig. A B C D E F is a regular hexagon.



2. *The inscribed hexagon,* A B C D E F, *in a circle is three fourths of the area of the circumscribed hexagon, a b c d e f.*



E.1	Dat.	Let circumsc. hex., *a b c d e f*, touch the circle A C E, at the ∠s A, B, C, D, E, F, of the inscribed hex. A B C D E F.
C. 1	Pst. 1.	Join G, and any angular · as *a* of the circumscribing hex.;
2	Pst. 1.	join also G A, G B, adjacent points of contact;
3	10, I.	Bis. G *a* in K, and join A K;
4	Conc.	then △ A G H = ¾ of △ A G *a*.
D.1	8, I.	∵ △ G A *a* = △ G B *a*;
		∴ ∠ A G *a* = ∠ B G *a*;
2	4, I. Def. 10, I.	and ∵ △ G A H = △ G B H; ∴ ∠s G H A, G H B, are rt. ∠s.
3	Cor. 1, 15, IV.	But G *a* = G *f* = *f a*; and G A is perp. to *f a*,
4	18, III.	∴ also G A bisects *f a*.
5	C. 2.	and ∵ A K bisects G *a*, A K = K *a* = A *a*;
6	Conc.	∴ △ A K *a* is. eq. lat.; and K *a* bisected by A H;
7	C. 2. 38, I.	but ∵ G K = K *a*, ∴ △ G A K = △ *a* A K;
8	D. 6. 38, I.	and ∵ K H = H *a*, ∴ △ K A H = △ H A *a*;
9	Remk.	thus △ K A *a* = ½ △ G A *a*, and △ H A *a* = ¼ G A *a*;
10	Conc.	∴ △ G A H = ¾ △ G A *a*.
11	*Sim.*	So, by drawing lines from G to the angr. · s of the inscribed and circumscribed hexagons, each △ of the inscrd. hex. = ¾ of each △ of the circumscrd. hex.
12	Conc.	∴ the inscr. hex. A B C D E F = ¾ of the c. scr. hex. *a b c d e f*.

Q. E. D.

3. The area of a reg. hex. A B C D E F is *six times* the area of the eq. lat. △ A G B described on the same st. line, A B.

4. Because the side of a reg. hexagon is equal to the chord of 60°; and the chord of 60° equals the rad.; ∴ ½ *rad.* = *sine of* 30°.

5. The inscribed hexagon and the successive bisections of its ares, have been employed as the groundwork for finding *the approximate ratio of the circumference of a circle to its diameter.* By forming Polygons of any number of sides that are successive bisections of the original 6, as 12, 24, 48, &c., we obtain the *apothem, i. e.*, the perpendicular from the centre on one of the sides, which continually approaches the radius in length; and from a polygon of 1536 sides we deduce the approximate values of the circumference and radius to be 6·283185 and 1.

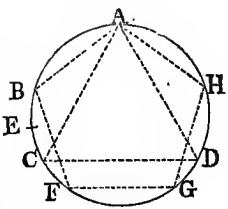
When a *square* is taken as the groundwork of the process, and the *apothems* and sides of successively inscribed polygons of 8, 16, 32, &c., sides are employed; on arriving at the polygon with 32768 sides, the Areas of the inscribed and circumscribed Polygons agree to the seventh place of decimals, 3·1415926, and since the area of the circle is intermediate, this value, 3·1415926, as far as it goes, must also be the area of the corresponding circle. Now, by Use 4, Pr. 41, Bk. I., the *Area of a circle is equal to the rectangle under its circumference and semi-radius, or under its radius and semi-circumference;* therefore the radius being 1, the semi-circumference is 3·1415926; and since the diameter is twice the radius, 3·1415926 to 1 is also the Ratio of the circumference to the diameter.—*See Chambers's Euclid, pp.* 202-221, *and Penny Cyclopædia, Vol. XIX., pp.* 186, 187.

Prop. 16.—Prob.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Con. 1, I. 2, IV. Cor. 6, I. 11, IV.
Cor. 5, I. Every eq. lat. △ is also eq. ang.
28, III. To bisect a given arc of a circle.

Dem. 28, III. 1, IV. 11, IV.

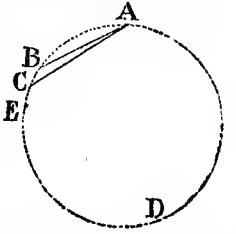


E.1	Dat.	Let A B F D be the given ⊙;
2	Quæs.	in that ⊙ to inscr. a reg. quindecagon.
C.1	1,I.Cor.5,I.	Desc. an eq. ang. and eq. lat. △;
2	2, IV.	and in ⊙ A B F D inscr. △ A C D eq. ang. to it;
3	Cor. 6, I.	then △ A C D will be eq. lat.
4	11, V.	Also insc. in ⊙ A B F D a reg. pent. ABFGH;
5	Sup.	and let one angr. • A be com. to the △ and the pent.
6	30, III.	Bis arc B C in E, and join E C, E B;
7	Sol.	then E C is a side of the quindecagon, to be set round the ⊙ A B F D.
D.1	28, III.	∵ equal chords cut off equal arcs,
2	Conc. 1.	∴ arc A C = arc C D = arc D A;
3	Conc 2.	and ∴ arc A C = $\frac{1}{3}$ ⊙ce.
4	*Sim.*	So also arc A B = $\frac{1}{5}$ ⊙ce.
5	D. 3. 4.	Thus of the 15 eq. arcs in the ⊙ce, arc A C contains 5 and arc A B, 3;
6	Ax. 3.	∴ arc B C, equalling arc A C *minus* arc A B, contains 2 of the 15;
7	C. 6.	∴ arc B E = arc E C = $\frac{1}{15}$ of the ⊙ce.
8	1. IV.	Hence, if st. lines each = chord B E, be placed in succession round the ⊙ A B F D;
9	Conc.	then a reg. quindecagon will be inscribed in the given ⊙.
10	11, IV.	The proof will be similar to that for the pentagon.

COR.—The only reg. st. lined figures which can be placed, side by side, so as to make a continuous plane surface, are the equilateral triangle, the square, and the hexagon.—*See Use* 3. *Pr.* 15, *Bk. I.*

SCH.—1. As in the Pentagon, 12, IV., if through the angular points B, E, C, F, &c., of the inscribed quindecagon, tangents to the circle be drawn, *a regular quindecagon will be circumscribed*: and, as in the Pentagon, 13 and 14, IV., a circle may be inscribed in a given quindecagon, or circumscribed about it.

2. The arc subtending a side of *a regular thirty-sided figure* may be readily found, by placing in a circle, A B C D, from the same point A, the line A B = the side of an inscribed hexagon, and A C = the side of an inscribed pentagon;



then, the arc B C = the arc of a reg. thirty-sided figure;

for 1-5th of a ⊙ *minus* 1-6th of a ⊙ = 1-30th of the same circle.

Also *twice* B C = B E = 1-15th the arc of a regular quindecagon.

USE.—This proposition opens the way for the *construction of other polygons*; for, if we obtain the common measures of 360°, as 2, 3, 4, 5, 6, 8, 9, 10, &c., and divide 360° by any two successive common measures, the difference of their quotients will give an arc of the circle, by which the circle will be exactly measured; consequently, a regular polygon may be inscribed, the sides of which are equal in number to 360 divided by the difference of the quotients arising from the division of 360° by the two successive common measures:

Thus, 1-9th of 360 = 40; 1-10th of 360 = 36; and 40 − 36 = 4; and $\frac{360}{4} = 90$, the number of parts into which the circle would be divided, and the number of the sides of the polygon.

Again, 1-36th of 360 = 10; 1-40th of 360 = 9; and 10 − 9 = 1, the difference of the arcs; and 1 will give a polygon of 360 sides.

OBSERVATIONS ON POLYGONS.

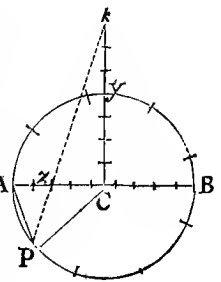
Obs. I.—The only known ways of dividing a circle geometrically are—

1st, into 3, 6, 12, 24, &c., parts—by	Prop.	15, IV.,	&	Prop.	9, I.
2nd, ,, 4, 8, 16, 32, &c. ,,	,,	6, IV.,	,,		9, I.
3rd, ,, 5, 10, 20, 40, &c. ,,	,,	11, IV.,	,,		9, I.
4th, ,, 15, 30, 60, 120, &c. ,,	,,	16, IV.,	,,		9, I.

Obs. II.—Consequently there are many polygons which cannot be inscribed in a circle, except by mechanical contrivances; but the following methods *approximate* so near to the truth, that they may be adopted without sensible error.

1. *In a given circle, as* A B P, *to inscribe any regular rt. lined figure; or to divide the circumference of a circle into any assigned number of equal parts.*

C. 1	U. & A. 34, I.	Divide the diam. A B, into the assigned number of eq. parts, as 9;
2	11, I.	from cen. C. raise a perp. C k;
3	U. & A. 34, I.	div. rad. C y into 4 eq. parts;
4	3, I.	and set off 3 of those parts from y to k;
5	Pst. 1.	join k and z, the second of the divisions from A;
6	Pst. 2, & 1.	prod. k z to ⊙ce, P; and join P C, A P;
7	Sol.	the line A P = the side of required figure;
8	Remk.	and A P set round the ⊙ will divide it into the assigned number of parts.



N.B. It is usual to denote the *circumference*, measuring four rt. angles, by 2π; the *semi-circumference*, by π; and the *number of sides* of the figure, or equal parts of the circle, by n.

Analysis.—The $\angle$ A C P is given $= \frac{2\pi}{n}$;

also $\angle$ C A P or $\angle$ C P A $= 90° - \frac{2\pi}{2n}$;

A C $= \frac{n}{2}$ being given, C $z = \left(\frac{n}{2} - 2\right)$ is given;

$\therefore$ P z can be found; as also $\angle$ A z P, or its equal $\angle$ C z k; hence the compl. of the $\angle$ C z k, namely $\angle$ C k z, can be found. And C z being given, C k can be found; and hence k y can be found;

and $k y = \frac{3}{4}$ C y nearly.—*See Treatise on Mensuration, Irish National Schools*, p. 19. *Demonstrations*, p. 53.

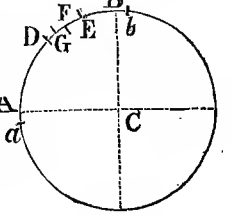
2. Though no exact geometrical rule has been discovered for inscribing many figures, as the heptagon, enneagon, or nonagon, hendecagon, &c., figures of 7, 9, 11, 13, &c., sides,—yet a high degree of accuracy may be obtained by the following *approximative process*, the real errors of which are far less than those which the imperfections of our best instruments entail in all geometrical constructions.

By 9, I. continue the series of bisections of the circle and of its arcs 2, 4, 8, 16, 32, 64, &c., until a number be found greater or less by *one* than a multiple of the number of sides in the required polygon. Of the equal parts thus found, take as many as constitute a multiple part of the required polygon, and let one more of the equal parts obtained by successive bisections be also bisected successively, until its parts are one more than the number of the sides of the polygon;—then *the first multiple part* plus *the second multiple part* will, with sufficient accuracy, be equal to the side of the required polygon.

Thus, *for the Heptagon:*

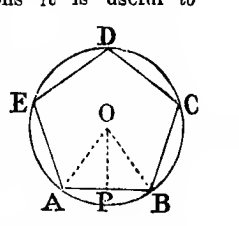
The continued bisection of the ⊙ce, until it is divided into 64 equal parts, gives a number greater by *one* than 9×7, or 63, a multiple of 7.

Now by bisecting the quadrantal arc A B in D, arc D B in E, arc D E in F, and arc D F in G, (A G being equal to 9 out of the 64 parts), the arc A G will be less than a *seventh* part of the circumference by a *seventh* part of one of them, D G.



But the arc D G being small, a *seventh* part of its *chord* may, without any considerable error, be assumed for the *seventh* part of the *arc* itself, being somewhat *less* than the latter;—and if the chord of A a be taken equal to this approximate seventh part, the error of assuming for it the arc A a, which is somewhat *greater* than its chord, will be still less,—so that arc G a will be equal, very nearly, to one-seventh of the circumference,—and the chord of G a, very nearly equal to the side of a regular heptagon inscribed in a circle.—*Geometry, Plane, Solid and Spherical*, p. 121.

Obs. III.—In the construction of polygons it is useful *to ascertain* the magnitude either of *the angle*, A O B at the centre, or of the *angle* A B C, formed by two adjacent sides.



As before, in the formulas, let π denote 2 rt. $\angle$s, or the semi-circumference; 2π, 4 rt. $\angle$s, or the whole circumference; n, the number of sides; and θ the magnitude of one angle, either at the centre, or at the concave boundary.

1*st*. *To find the magnitude of an* $\angle$ *at the centre, as* $\angle$ A O B *of a polygon.*

$n\,\theta$ = sum of all the $\angle$s at the centre, or $2\,\pi$ = 4 rt. $\angle$s, (Cor. 2. 15, I.)

$\therefore \frac{2\,\pi}{n} = \theta$, the magnitude of one $\angle$, as $\angle$ A O B at the centre.

2*nd*. *To find the magnitude of an angle, as* $\angle$ A B C, *formed by two adj. sides of a polygon.*

$n\,\theta$ = the sum of the magnitudes of all the interior $\angle$s;
But, Cor. 1. 32, I., $n\,\theta + 2\,\pi = n\,\pi$;
and by trans. $n\,\theta = n\,\pi - 2\,\pi = (n-2).\ \pi$.
$\therefore \theta = \frac{(n-2)}{n} . \pi$ = the mag. of one $\angle$, as $\angle$ A B C, formed by two adj. sides.

Note.—For n substitute the number of sides, 3, 4, 5, 6, &c.; and for $2\,\pi$, 360°, and the magnitude of the angles is found in numbers.

OBS. IV.—*On a given rt. line,* A B, *to construct a reg. polygon.*

1. By dividing $2\pi = 360°$ by n the number of sides, we obtain the arc A B, or the measure of the $\angle$ at the cen. $\angle$ A O B; and $\triangle$ A O B being isosc., $\angle$s O A B, O B A are equal.

Now $\frac{180° - \angle AOB}{2} = \angle OAB = \angle OBA$;

and $\because \angle ABO = \frac{1}{2} \angle ABC$, $\therefore 180° - \angle AOB = \angle ABC$, one of the angles of the polygon to which all the others are equal.

Hence the Rule, or Formula;

$\pi - \frac{2\pi}{n} = \angle ABC$, one of the eq. $\angle$s of the polygon;

at A and B make $\angle$s O A B, O B A each $= \frac{\angle ABC}{2}$;

then from O, with O A, or O B, desc. a $\odot$;

and in this circle place the st. line A B continually; a polygon of the assigned number of sides will be drawn.

2. By means of Trigonometry the radius of the circumscribing circle is calculated, assuming the side of the polygon to be 1, and the following Table constructed.

No. of Sides.	Name of Polygon.	Radius of Circumscribing Circle.	Angle O A B or O B A.
3	Trigon	·5773503	30°
4	Tetragon	·7071068	45°
5	Pentagon	·8506508	54°
6	Hexagon	1·Side = radius	60
7	Heptagon	1·1523825	$64\frac{2}{7}$°
8	Octagon	1·3065630	$67\frac{1}{2}$°
9	Nonagon	1·4619022	70°
10	Decagon	1·6186340	72°
11	Undecagon	1·7747329	$73_{\frac{1}{11}}$°
12	Dodecagon	1·9318516	75°

Then, the units of length in A B × tabular rad. = units of length in O A or O B, the rad. of the actual circle, in which the st. line A B is to be placed, step by step, so that the polygon may be formed.

Obs. V.—To calculate numerically the *area of any regular polygon*, it is sufficient to find the area of one of the triangles into which the polygon may be divided, and to multiply that area by the number of triangles;

Or, *given a side, as* AB, *and* OP *the perpendicular, from* O *the centre, on* AB;

then, $\frac{n\,AB}{2} \times OP =$ the area of the polygon.

To facilitate the calculation *when only a side, as* AB, *is given*, the following Table has been formed on trigonometrical principles—the side of the polygon being 1.

No. of sides.	Radius of Inscribed Circle.	Area of Polygon, or Multiplier.	Trigonometrical Expression.
3	0·2886751	0·4330127	$= \frac{3}{4}$ tang. 30° $= \frac{1}{4}\sqrt{3}$.
4	0·5	1·	$= \frac{4}{4}$ tang. 45° $= 1 \times 1$.
5	0·6881910	1·7204774	$= \frac{5}{4}$ tang. 54° $= \frac{5}{4}\sqrt{(1+\frac{2}{5}\sqrt{5})}$.
6	0·8660254	2·5980762	$= \frac{6}{4}$ tang. 60° $= \frac{6}{4}\sqrt{3}$.
7	1·0382617	3·6339124	$= \frac{7}{4}$ tang. $64\frac{2}{7}°$.
8	1·2071068	4·8284271	$= \frac{8}{4}$ tang. $67\frac{1}{2}° = 2 \times (1+\sqrt{2})$.
9	1·3737387	6·1818242	$= \frac{9}{4}$ tang. 70°
10	1·5388418	7·6942088	$= \frac{10}{4}$ tang. $72° = \frac{5}{2}\sqrt{(5+2\sqrt{5})}$.
11	1·7028437	9·3656404	$= \frac{11}{4}$ tang. $73\frac{7}{11}°$
12	1·8660254	11·1961524	$= \frac{12}{4}$ tang. $75° = 3 \times (2+\sqrt{3})$.

By using the Table we find,

$AB^2 \times$ tabular area = area of the reg. polygon;

or, $AB \times$ rad. of inscr. $\odot \times \frac{n\,AB}{2} =$ area.

OBS. VI.—DODSON, in his "Calculator," supplies the following Tables for the calculation and construction of any reg. polygon having not more than 12 sides.

1° *When the length of the side* = 1.

Sides.	Radius of Circumscribed Circle.	Radius of Inscribed.	Area.
3	0·5773503	0·2886751	0·4330127
4	0·7071068	0·5	1·
5	0·8506508	0·6881910	1·7204774
6	1·	0·8660254	2·5980762
7	1·1523825	1·0382617	3·6339124
8	1·3065630	1·2071068	4·8284271
9	1·4619022	1·3737387	6·1818242
10	1·6180340	1·5388418	7·6942088
11	1·7747329	1·7028437	9·3656404
12	1·9318516	1·8660254	11·1961524

2° *When radius of circumscribed circle* = 1.

Sides.	Length of Side.	Radius of Inscribed Circle	Area.
3	1·7320508	0·5	1·2990381
4	1·4142136	0·7071068	2·
5	1·1755705	0·8090170	2·3776412
6	1·	0·8660254	2·5980762
7	0·8677674	0·9009689	2·7364102
8	0·7653668	0·9238795	2·8284271
9	0·6840403	0·9396926	2·8925437
10	0·6180340	0·9510565	2·9389263
11	0·5634651	0·9594931	2·9735250
12	0·5176381	0·9659259	3·

3° *When radius of inscribed circle* = 1.

Sides.	Length of Side.	Rad. of Circumscribed Circle.	Area.
3	3·4641016	2·	5·1961524
4	2·	1·4142136	4·
5	1·4530851	1·2360680	3·6327128
6	1·1547005	1·1547005	3·4641016
7	0·9631491	1·1099160	3·3710222
8	0·8284271	1·0823919	3·3137084
9	0·7279405	1·0641776	3·2757315
10	0·6498394	1·0514622	3·2491970
11	0·5872521	1·0422172	3·2298913
12	0·5358984	1·0352760	3·2153904

4° *When the area* = 1.

Sides.	Length of Side.	Rad. of Circumscribed Circle.	Radius of Inscribed Circle.
3	1·5196716	0·8773827	0·4386912
4	1·	0·7071068	0·5
5	0·7623870	0·6485251	0·5246678
6	0·6204033	0·6204033	0·5372849
7	0·5245813	0·6045183	0·5446520
8	0·4550899	0·5946034	0·5493420
9	0·4201996	0·5879764	0·5525172
10	0·3605106	0·5833184	0·5547687
11	0·3267617	5·5799148	0·5564242
12	0·2988585	0·5773503	0·5576775

Use and App.—By the help of these tables, and of the compasses, and a scale of equal parts, the construction of any regular st. lined figure is reduced to a simple calculation, to the drawing of a circle, and to the setting off of equal chords on that circle.

Ex.—To construct a reg. heptagon with an area 144 times greater than the square on any one division of the scale of eq. parts.

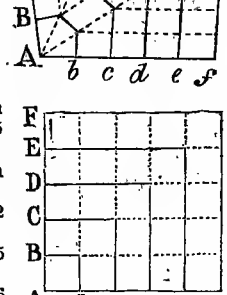
Increase the side and rad., in Table 4. in the proportion of $\sqrt{144}$ to $\sqrt{1}$, or of 12 to 1; then

·5245813 × 12 = 6·2949756 = length of a side of the hept.
·6045183 × 12 = 7·2542196 = „ of rad. of circumscribed ⊙,
·5446520 × 12 = 6·5358240 = „ of rad. of inscribed ⊙,

From the same centre with the two radii, draw two circles; the side found above will measure the larger circle into seven steps; and chords, joining the · s in which the ⊙ce is cut, will touch the inner circle.

Thus, a reg. heptagon will be inscribed, or circumscribed.

OBS. VII.—In Arithmetic *polygonal numbers* are such as are the sums of a series of numbers beginning with *unity* and so increasing as to be representative of the figure of a polygon. These numbers are subdivided into triangular, quadrangular, pentagonal, &c., and may be explained by taking a pentagonal number.



1. Construct a set of pentagons A *c* C, A *d* D &c., double, treble, &c. of A *b* B, in lineal dimension;

Divide the sides of each pent. into parts each = corresponding side A *b* B;

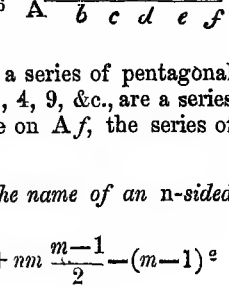
Then, beginning with A, *one* point, and taking in all the points of pent. A *b* B, we have 1 + 4 = 5 points;

Add all the · s of pent. A *c* C that are not in pent. A *b* B, and we have 1 + 4 + 7 = 12 points;

So for pent. A *d* D, 1 + 4 + 7 + 10 = 22 points;

For pent. A *e* E, 1 + 4 + 7 + 10 + 13 = 35 points;

And pent. A *f* F, 1 + 4 + 7 + 10 + 13 + 16 = 51 points &c.



This series 1, 5, 12, 22, 35, 51, &c., is a series of pentagonal numbers, in a way similar to that in which 1, 4, 9, &c., are a series of square numbers. By aid of the Square on A *f*, the series of square numbers will be readily formed.

2. *To find the numbers which bear the name of an* n*-sided figure.*

The *m*th number of the *n*-sided fig. $= 1 + nm\frac{m-1}{2} - (m-1)^2$

Form a series of terms beginning with 1, and, by a com. difference $= n - 2$, increasing in Arithmetical progression;

then the sums of the terms of the arithmetical series form the series of polygonal numbers.

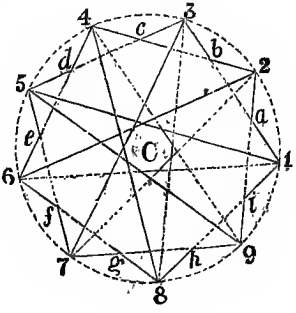
Thus, for Decagonal numbers, in which $n - 2 = 8$;
the series 1, 9, 17, 25, 33, 41, 49 &c. in arithmetical progression;
gives 1, 10, 27, 52, 85, 126, 175 &c. Decagonal numbers.

Some of the polygonal numbers are,

Triangular, 1, 3, 6, 10, 15, 21, &c.
Quadrangular, 1, 4, 9, 16, 25, 36, &c.
Pentagonal, 1, 5, 12, 22, 35, 51, &c.
Hexagonal, 1, 6, 15, 28, 45, 66, &c.
Heptagonal, 1, 7, 18, 34, 55, 81, &c.

For a further account, *see Penny Cyclopædia, Vol.* 16, *p* 364.

Obs. VIII. As it has been observed, Euclid treats only of *regular convex polygons;* though, according to his definition of a reg. st.-lined figure, they are both equilateral and equiangular, he does not name *Star-shaped polygons*, *i.e.* regular polygons with *re-entrant* angles, as *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*. Figures of this kind are described, by first constructing any regular convex polygon, as might be at the points 1, 2, 3, 4, 5, 6, 7, 8, 9; and by then drawing successive diagonals, as 1—3, 2—4, 3—5, &c., so as to cut off that number of sides which is *prime* to the number of sides of the assumed convex polygon; or what leads to the same results, so as to cut off a number of sides which neither measures the number in the given convex polygon, nor is measured by any of its common measures.



To 5 the no. of sides in a pentagon, 2 and 3 are the prime numbers;

7	,,	heptagon, 4 and 3, and 5 and 2	,,
9	,,	nonagon, 2 and 7, and 4 and 5	,,
12	,,	dodecagon, 5 and 7	,,
13	,,	tridecagon 2 and 11, 3 and 10, 4 and 9, 5 and 8, 6 and 7;	

And diagonals cutting off successively in a

Pentagon, 2 and 3 sides,	will form ... 1	star-shaped	polygon;
Heptagon, 4 and 3, or 5 and 2	will form ... 2	,,	polygons;
Nonagon, 2 and 7, or 4 and 5	will form . . 2	,,	polygons;
Dodecagon, 5 and 7,	will form ... 1	,,	polygon;
Tridecagon, 2 and 11, 3 and 10, 4 and 9, 5 and 8, or 6 and 7 sides	will form ... 5	,,	polygons.

A similar process will form other star-shaped polygons.

N.B.—The figure as C at the centre of a star-shaped polygon, formed by the intersection of lines from the angular points, 1, 2, 3, &c., is also a regular convex polygon, with the same number of sides as there are re-entrant angles to the star-shaped polygon.

REMARKS ON BOOK IV.

1. The Problems of which the Fourth Book of Euclid is entirely composed require little classification; they may be brought under the general heads;—

1st, Of rt.-lined regular figures inscribed in circles; Pr. 2, 6, 11, 15 and 16;

2nd, Of rt.-lined regular figures, circumscribed about circles; Pr. 3, 7, 12;

3rd, Of circles inscribed in regular rt.-lined figures; Pr. 4, 8, 13;

4th, Of circles circumscribed about rt.-lined figues; Pr. 5, 9, 14;

5th, Of constructing an isosceles triangle, having each angle at the base double of the vertical angle, Pr. 10.

2. The Use and Application of the Propositions of this Book have been given so much *in extenso*, that it would be superfluous to add to these remarks, except again to challenge attention to the value of theoretical reasoning as the guide in Geometry to most important Practical Results.

GRADATIONS IN EUCLID.

BOOK V.

THE THEORY OF PROPORTION, OR OF THE COMPARATIVE MAGNITUDES OF PLANE FIGURES.

THIS Book is entirely independent of the four books which preceded it. In the main they relate to the Properties of Figures on a plane surface, but the fifth book introduces Properties of a more general kind, and though restricted by Euclid to the comparative magnitudes of right lines, really extends to all kinds of magnitude. It is no longer absolute equality, or inequality, which we have to consider, but the RATIO, or mode of estimating the relative lengths of lines and the magnitudes of figures, and the *proportion*, or setting forth of those relative lengths or magnitudes.

As in the Second Book material assistance for illustrating the properties of Rectangles, was derived from Algebra and Arithmetic, so in this book similar help will be obtained from the same sources. Indeed very many of the terms employed will be already familiar from their use in Arithmetic; and it will be seen that the estimating of Ratios, and the setting forth of proportions, rest so entirely on a numerical basis, that to a very high degree the Fifth Book, or the Theory of Proportion, is an application of Numbers to the purposes of Plane Geometry.

For this reason it will be of advantage to the Learner to be presented with a brief view of the Principles on which are established the Properties of Proportional Numbers and Quantities. If, however, he has already mastered the subject, he may pass over the next few pages, and at once enter on the Theory of Geometrical Proportion.

SOME PROPERTIES OF PROPORTIONAL NUMBERS,

INTRODUCTORY TO EUCLID'S THEORY OF GEOMETRICAL PROPORTION.

We may compare two numbers together, either by their *difference*, or by their *quotient, i.e.* by the number of times which the greater contains the less, or the less measures the greater. When we say $12 - 9 = 3$, we compare 12 and 9 by their difference; but when we say 12 *contains* 9, *one* and *one quarter* times, we compare them by a division of 12 into 9 and a part of 9.

The proportion which one number or quantity bears to another is often called its *Ratio*; the ratio, measured by the *difference* is named an *arithmetical* ratio,—that measured by the quotient,—a *geometrical* ratio.

Proportion is applied, *either* to *an identity of difference* between three or more numbers, as 12, 9, 6, where the common difference or the Arithmetical ratio is 3; *or*, to *identity of relative magnitudes*, as 12 : 9 :: 8 : 6,—where 12 contains 9 just as often as 8 contains 6,—the common quotient or ratio being $1\frac{1}{3}$. When the differences are identical, the numbers are in Arithmetical Proportion; when the contents of each pair of terms are identical, the numbers are in Geometrical Proportion. The term proportion, taken by itself, is usually restricted to numbers in geometrical proportion; and of these we have now to treat.

Identity in the quotients of successive pairs of numbers constitutes Proportion. Take for an example, 15 : 5 :: 36 : 12; the quotient obtained on dividing 15 by 5 is the same as that obtained by dividing 36 by 12; and these four numbers, 15, 5, 36, and 12,—or any other four numbers fulfilling the condition of equality of quotients in each successive pair, form a Proportion, or set of Proportionals.

The *extremes* are the first and last terms in the series; the terms placed between the first and last terms, the *means*; the *antecedent* is the first term of a ratio; the *consequent*, the second term.

The *ratio* may be expressed, either by a whole number or by a fraction; thus in the proportionals, 18 : 6 :: 24 : 8, the constant ratio is 3; in the proportionals 12 : 9 :: 36 : 27 that ratio is $\frac{4}{3}$ or $1\frac{1}{3}$.

If we take two sets of quantities in direct proportion, $a : b :: c : d$ (which may also be written in a fractional form $\frac{a}{b} = \frac{c}{d}$) we can readily exhibit various rules that are employed in modifying a Proportion: they are all dependent on the principle that *resulting equations are equally true whenever the thing which is done on one side of an equation is also done on the other side.* Take as an example $\frac{a}{b} = \frac{c}{d}$.

RULE 1. Multiply each side of the equation by $b \times d$; we obtain $\frac{abd}{b} = \frac{cbd}{d}$, or $ad = cb$; $\therefore$ the product of the extremes = the product of the means; and *conversely*, if the product of the extremes = the product of the means, the quantities are proportional; for, dividing each side by bd, we have $\frac{ad}{bd} = \frac{cb}{bd}$, or $\frac{a}{b} = \frac{c}{d}$, i.e., $a : b$ & $c : d$ are proportionals.

RULE 2. Multiply each side by $\frac{b}{c}$, the sides become $\frac{ab}{bc} = \frac{cb}{dc}$ or $\frac{a}{c} = \frac{b}{d}$, i. e. $a : c = b : d$.

RULE 3. Add *unity* to each side, and the equation is $\frac{a}{b} + 1 = \frac{c}{d} + 1$; or reducing the mixed quantity, $\frac{a+b}{b} = \frac{c+d}{d}$ *i. e.*, $a + b : b = c + d : d$.

RULE 4. Subtract *unity* from each side, we have the equation $\frac{a}{b} - 1 = \frac{c}{d} - 1$; or $\frac{a-b}{b} = \frac{c-d}{d}$ i.e., $a - b : b = c - d : d$.

RULE 5. Any common factor, as m, may be expunged, except from the two extremes, or from the two means; for, if $\frac{ma}{b} = \frac{mc}{d}$, on dividing both sides by m, the resultant is $\frac{ma}{mb} = \frac{mc}{md}$; or, $\frac{a}{b} = \frac{c}{d}$. Or, if $\frac{a}{mb} = \frac{c}{md}$; on multiplying by m, we have $\frac{ma}{bm} = \frac{mc}{md}$; or, again $\frac{a}{b} = \frac{c}{d}$. Or, if $\frac{ma}{mb} = \frac{c}{d}$; $\therefore \frac{ma}{mb} = \frac{c}{d}$; $\therefore \frac{ma}{mb} = \frac{a}{b}$; $\therefore \frac{ma}{mb} = \frac{c}{d}$ is identical with $\frac{a}{b} = \frac{c}{d}$.

RULE 6. In two or more series of Proportionals, the *products* of the corresponding antecedents and consequents also constitute a Proportion, and this is named a *Compound Proportion;* for if $\frac{a}{b} = \frac{c}{d}$ & $\frac{e}{f} = \frac{g}{h}$, on multiplying the terms, on the left hand of the sign =, together, and those on the right hand, $\frac{a}{b} \times \frac{e}{f}$ & $\frac{c}{d} \times \frac{g}{h}$, the resultant will be $\frac{ae}{bf} = \frac{cg}{dh}$.

Three or more series of proportionals are compounded in a similar way.

The *Essential Property*, or *Criterion*, of numbers in true proportion is, that *the product of the extremes is equal to the product of the means;* and reciprocally, if the product of any two numbers equals the products of any other two numbers, the series of numbers constitutes a proportion; thus,

in $15 : 5 :: 36 : 12$; $-15 \times 12 = 180 = 5 \times 36$; $\therefore$ a *true* proportion.
in $14 : 5 :: 35 : 12$; $-14 \times 12 = 168$; $-5 \times 35 = 175$; $\therefore$ a *false* proportion.

From the Essential Property of numbers in true proportion it follows, that *when any three terms are given the fourth may be found;* for on dividing the product of the means by the given extreme, or that of the extremes by the given mean, the other extreme, or the other mean, will be obtained; thus in $6 : 8 :: 24 : x$;

$\therefore 6x = 8 \times 24$; $\therefore x = \frac{8 \times 24}{6} = 32$, and the terms are $6 : 8 = 24 : 32$.

Or, in $7 : x :: 14 : 6$,

$\therefore 7 \times 6 = 14x$, $\therefore = \frac{7 \times 6}{14} = 3$; and the terms are $7 : 3 = 14 : 6$.

If each of the means is the same number, as in $3 : 6 :: 6 : 12$, the product of the extremes equals the square of one of the means; the value of *one of the*

means is therefore *the square root of the product* of the means; thus in $4 : x :: x : 16$,

$\therefore 4 \times 16 = x^2$, $\therefore x = \sqrt{64} = 8$; and the terms are $4 : 8 :: 8 : 16$.

When the value of one of the means equals the square root of the product of the extremes, that number is named a *mean proportional* between two given numbers. Stated algebraically, when $a : x = x : b$, then $x^2 = a\,b$; or $x = \sqrt{a\,b}$; x being the mean proportional between a and b.

There are *other important properties* of numbers in proportion resting on the equation which may be formed from four proportionals, $\frac{A}{B} = \frac{C}{D}$. For instance as shown above,—add 1 to both sides; $\frac{A}{B} + 1 = \frac{C}{D} + 1$, or $\frac{A+B}{B} = \frac{C+D}{D}$; which is the theorem known by the word *componendo*, putting together: *Or*, taking 1 from both sides; $\frac{A}{B} - 1 = \frac{C}{D} - 1$, *i.e.*, $\frac{A-B}{B} = \frac{C-D}{D}$; which is the theorem indicated by *dividendo*, taking apart. Again, on dividing *unity*, first, by $\frac{A}{B}$, and next, by $\frac{C}{D}$, we have $\frac{1}{1} \div \frac{A}{B} = \frac{B}{A}$, and $\frac{1}{1} \div \frac{C}{D} = \frac{D}{C}$; thus giving, *invertendo*, by inversion, $\frac{B}{A} = \frac{D}{C}$; And on multiplying each side by $\frac{B}{C}$, we have $\frac{A}{B} \times \frac{B}{C} = \frac{A}{C}$, and $\frac{C}{D} \times \frac{B}{C} = \frac{B}{D}$; i. e., *alternando*, by taking each other one, $\frac{A}{C} = \frac{B}{D}$.

These Properties, with their variations and combinations we now mention in their order.

1°. *Multiplicando*, or *dividendo*, by multiplying, or dividing by the same number, the two first terms or the two last,—the two antecedents or the two consequents,—the equation and consequently the proportion is undisturbed.

Take the Proportionals $9 : 12 :: 16 : 24$;

	Multiplicando, by 3;	*Dividendo*, by 3.
1 & 2 terms,	$27 : 36 :: 18 : 24$;	$3 : 4 :: 18 : 24$;
3 & 4 „	$9 : 12 :: 54 : 72$;	$9 : 12 :: 6 : 8$;
1 & 3 „	$27 : 12 :: 54 : 24$;	$3 : 12 :: 6 : 24$;
2 & 4 „	$9 : 36 :: 18 : 72$;	$9 : 4 :: 18 : 8$;

In all these changes the product of the extremes remains equal to the product of the means, and the numbers resulting are therefore still proportionals. In fact the ratio of the two first terms $\frac{9}{12}$, or that of the two last $\frac{18}{24}$, is the same fractional number $\frac{3}{4}$,—and fractions undergo no change of value when the terms of the fraction are *both* multiplied or *both* divided by the same number. Also when the product of the extremes and that of the means are each multiplied or divided by the same number, the resulting products or quotients are also equal, and the proportion remains.

2°. *Alternando*, or *invertendo*, by taking the terms alternately, or by inverting the terms, the proportion also remains undisturbed;

Thus in the proportions $\frac{12}{4} = \frac{75}{25}$,

1st, exchange the extremes, $\frac{25}{4} = \frac{75}{12}$; the ratio being $\frac{25}{4}$,

2nd, „ means, $\frac{12}{75} = \frac{4}{25}$; „ „ $\frac{4}{25}$,

3rd, „ means & extremes, $\frac{4}{12} = \frac{25}{75}$; „ „ $\frac{1}{3}$.

But the proportions continue unbroken though the ratio in each case is different; for after the various exchanges the *criterion* is satisfied,—the product of the extremes *does* equal that of the means.

3°. *Componendo vel dividendo*, by putting the two first terms of the proportion together, and the two last; or by separating the two first and the two last,—there arises the proposition, that the *sum* or *the difference of the two first terms* is *to the second*, as *the sum* or *difference of the two other terms* is *to the fourth*: for example,

in the proportion	$24 : 8 :: 45 : 15$,	the *com. ratio* is 3;
componendo by adding	$24 + 8 : 8 :: 45 + 15 : 15$,	„ „ 4;
dividendo by subtracting	$24 - 8 : 8 :: 45 - 15 : 15$,	„ „ 2;

The explanation to be given is, that when we increase or diminish each antecedent by its consequent, we do nothing except increase or diminish by *unity* each of the two ratios; and since these ratios were equal at first, they remain equal after such an increase or diminution: for

changing the means, $24 + 8 : 45 + 15 :: 8 : 15$; (1)

but $24 : 8 :: 45 : 15$; (2)

or, $24 : 45 :: 8 : 15$; (3)

& ∴ two ratios equal to a third are equal to each other,

∴ $24 + 8 : 45 + 15 :: 24 : 45$;

or rather $24 + 8 : 24 :: 45 + 15 : 45$.

Thus *the sum*, or *the difference of the two first terms*, is *to the first term*, as *the sum*, or *difference of the two other terms*, is *to the third term*.

4°. Generally, *addendo vel subtrahendo*, by increasing, or diminishing *the sum*, or *the difference of the antecedents* is *to the sum*, or *the difference of the consequents*, as *any one of the antecedents* is *to its consequent*.

Take the proportion, $24 : 8 :: 45 : 15$;

exchanging the means, $24 : 45 :: 8 : 15$;

by the former property 3°, $24 + 45 : 45 :: 8 + 15 : 15$;

again exchanging the means, $24 + 45 : 8 + 15 :: 45 : 15$;

whence by reason of the common ratio $45 : 15$,

$24 + 45 : 8 + 15 :: 24 - 45 : 8 - 15$;

or, rather exchanging the means,

$24 + 45 : 24 - 45 :: 8 + 15 : 8 - 15$.

i. e., *the sum of the antecedents* is *to their difference*, as *the sum of the consequents* is *to their difference*.

Cor. Let there be a succession of numbers, forming, two by two, equal ratios, *the sum of all the antecedents* is *to the sum of all the consequents*, as *any one antecedent* is *to its consequent*.

Assume the series, $8 : 12 :: 2 : 3 :: 4 : 6$ &c., *or*, $a : b = c : d = e : f$, &c.

the ratios	$8 : 12 :: 2 : 3$,	$a : b :: c : d$;
by 4° give	$8 + 2 : 12 + 3 :: 2 : 3$;	$a + c : b + d = c : d$;
but	$2 : 3 :: 4 : 6$,	$c : d = e : f$;
∴	$8 + 2 : 12 + 3 :: 4 : 6$	$a + c : b + d :: e : f$;
applying 4°.	$8 + 2 + 4 : 12 + 3 + 6 :: 4 : 6$	$a + c + e : b + d + f = e : f$.

and so on, whatever may be the number of equal ratios.

If there be any number of fractions equal in value, as $\frac{8}{12}$, $\frac{4}{6}$, $\frac{2}{3}$, &c., if *the sum* or *the difference of the numerators* and *that of the denominators be taken the resulting fraction is equal in value to each of the given fractions:* thus,

in $\frac{8}{12}$, $\frac{4}{6}$, $\frac{2}{3}$,

$$8 \pm 4 \pm 2 = 14 \text{ or } 2, \text{ and } 12 \pm 6 \pm 3 = 21 \text{ or } 3,$$

the resulting fraction being $\frac{14}{21}$ or $\frac{2}{3}$.

The equation $\frac{8}{12} = \frac{2}{3}$ leads to the proportion $8 : 12 :: 2 : 3$;

whence by the foregoing property, $\dfrac{8 \pm 2}{12 \pm 3} = \frac{10}{15} = \frac{6}{9} = \frac{2}{3}$.

5°. COMPOUND RATIO; *Componendo*, by placing together or combining. In any number of proportions, *if all the corresponding antecedents and consequents be respectively multiplied together, the resulting products will be in proportion.* For example,

$$\begin{array}{ll} 3 : 8 :: 12 : 32, & \text{or, } a : b :: c : d; \\ 7 : 15 :: 28 : 60, & e : f :: g : h; \\ 40 : 12 :: 50 : 15, & i : k :: l : m, \text{ \&c.} \end{array}$$

being proportionals they may be represented by

$$\frac{3}{8} = \frac{12}{32}, \qquad \text{or, } \frac{a}{b} = \frac{c}{d};$$

$$\frac{7}{15} = \frac{28}{60}, \qquad \frac{e}{f} = \frac{g}{h};$$

$$\frac{40}{12} = \frac{50}{15}, \qquad \frac{i}{k} = \frac{l}{m}, \text{ \&c.};$$

on multiplying the corresponding sides of these equations, there will result the equal products,

$$\frac{3 \times 7 \times 40}{8 \times 15 \times 12} = \frac{12 \times 28 \times 50}{32 \times 60 \times 15} \qquad \text{or, } \frac{iac}{bfk} = \frac{cgl}{dhm};$$

$$i.e., \quad \frac{840}{1440} = \frac{16800}{28800}; \qquad \text{or, } 840 : 1440 :: 16800 : 28800,$$

satisfying the criterion of proportion that $840 \times 28800 = 1440 \times 16800$.

If we divide $\frac{16800}{28800}$ by 20, we have $\frac{840}{1440} = \frac{840}{1440}$, an *identical proportion*, the terms of the two ratios being the same.

N.B.—The *constant ratio* of the preceding proportion, namely $\frac{840}{1440}$ is equal to the product of the three constant ratios of the given proportions.

Thus the three constant ratios being $\frac{3}{8}$, $\frac{7}{15}$, $\frac{40}{12}$, or $\frac{10}{3}$, we have for their product $\frac{210}{360}$; or suppressing the common factor 30, $\frac{7}{12}$, to which the fraction $\frac{840}{1440}$ may be reduced by suppressing the common factor 120.

The ratio $\frac{7}{12}$, thus arising from the multiplication of several other ratios, is named the *compound ratio.*

6°. From the theory of compound proportion it follows, *when four numbers are in proportion,* 1st, *their squares, cubes, and other like powers are also in proportion,* and 2nd, *their square roots, cube roots, and other like roots are in proportion.*

1st. Take $a : b :: c : d$, or, 4 ; 6 :: 8 : 12;
squaring; $a^2 : b^2 :: c^2 : d^2$, 16 : 36 : 64 : 144;
cubing, $a_3 : b_3 :: c^3 : d_3$, 64 : 216 : 512 : 1728.

Thus there results a series of proportions which, multiplied in the order of the antecedents and consequents, will give products also in proportion.

2nd. Take $a^2 : b^2 :: c^2 : d^2$, or, 4 : 9 :: 16 : 36;
extracting the $\surd$, $a : b :: c : d$, or, 2 : 3 :: 4 : 6;

$\because$ the ratio $\frac{a^2}{b^2} = \frac{c^2}{d^2}$ or $\frac{4}{9} = \frac{16}{36}$, $\therefore$ the ratios of the roots are equal.

But to extract the square root of a fraction, as $\frac{4}{9}$ or $\frac{16}{36}$, we extract the square root of the numerator and that of the denominator, which gives $\frac{\surd 4}{\surd 9} = \frac{2}{3}$ and $\frac{\surd 16}{\surd 36} = \frac{4}{6}$

7°. *Irrational numbers*, or numbers without a perfect root, and *Incommensurables*. When quantities as a, b, c, d, or numbers, as 2, 3, 8, 12, are *not* perfect squares, the quantities $\surd a$, $\surd b$, $\surd c$, $\surd d$, or $\surd 2$, $\surd 3$, $\surd 8$, $\surd 12$, are named *irrational* quantities or numbers; and when quantities or numbers are so related, that, although one of them may be represented in the terms of a certain unit, the other cannot be so represented, such quantities or numbers are named *incommensurable*. The diameter of the circle in relation to the circumferenee, and the diagonal of a square of which the side is unity in relation to that side, are familiar examples. Also in numbers, $\surd 2$, $\surd 3$, $\surd 15$, &c., are incommensurable with unity, for there is no mixed number, nor fraction, exactly equivalent to them. Proportion however exists between incommensurable numbers, for $\surd 2 : \surd 3 = \surd 8 : \surd 12$; and we are led to consider *ratios*, which must in general be regarded as *irrational*, because between numbers without a perfect root: but by *ratio* we must bear in mind that we mean the *proportion* between quantities or numbers;—and by *irrational*, the non-existence of an exact root. Now the question arises, can we apply to proportions of this kind, *i.e.*, between irrational and incommensurable quantities or numbers, all the properties which have just been established?

The answer is affirmative, if we remember that an irrational number may always be replaced *in the mind* by an exact fractional number which only differs from the proposed number by a quantity so small that, when we neglect it, we need not have any regard to the mistake committed; and it is between the commensurable numbers substituted for the irrational magnitudes that the ratios are judged to be established.

As to the ratios between exact fractional numbers, it is easy to understand, that according to the rule for the division of fractions. they may always be replaced by the ratios between whole numbers; thus, the ratio of $\frac{3}{7}$ to $\frac{5}{11}$, being the quotient arising from the division of $\frac{3}{7}$ by $\frac{5}{11}$ is equal to $\frac{3}{7} \times \frac{11}{5}$, or to $\frac{33}{35}$, that is to say, to the ratio of 33 to 35.

Again, the ratio of $\frac{7}{8}$ to $\frac{15}{23}$ is equal to $\frac{7}{8} \times \frac{23}{15}$, or to the ratio of 161 to 120.

Thus, all the properties of proportion are true with respect to numbers, whatever may be the numbers about which we reason. Similar truth and certainty will appear even in a higher degree when we consider;

EUCLID'S

THEORY OF GEOMETRICAL PROPORTION.

By some writers this fifth book of Euclid has been named *the Elements of Mathematical Logic.* The other books have shown us the absolute properties, the equality or inequality of Plane Geometrical figures;—*this* book enables us to institute a comparison between them. The Definitions and Propositions indeed are so expressed, as if they applied only to such plane figures; but they extend equally to lines, surfaces, amd solids, and to every species of quantity. Whatever be the science or branch of knowledge that depends upon Proportion, it is founded on the Principles contained in the fifth book of Euclid. We take, for instance, Practical Geometry;—nearly all its measurements are calculated by the Doctrines of Proportion,—the survey of an estate, or of a whole kingdom, is carried out by the application of those doctrines: and in Arithmetic, Astronomy, Statics, &c., the use of this book is indispensable. "In fine," says an old writer, "one may affirm, that if one should take away the knowledge of the Propositions that this Book giveth us, the remainder would be of little use."

The Definitions and Propositions of the fifth book may be extended to every species of quantity and magnitude; and may be easily applied to *number;* but though our clearest notions of ratio and proportion are derived from numbers, Number, as we have said before, is only to a small extent the actual subject of Geometry.

N.B.—The text of the Definitions and Propositions is from Simson's Edition;—variations are given in the notes, and if between brackets, or " " they are from the Greek text of Euclid.

DEFINITIONS.

I.—A less magnitude is said to be a *part* of a greater magnitude, when the less measures the greater; that is when the less is contained a certain number of times exactly in the greater.

"A part is a magnitude of a magnitude,—the less of the greater, when it exactly measures the greater."—EUCLID.

"For the clearer vnderstandyng of a parte, it is to be noted, that a part is taken in the Mathematicall Sciences two maner of wayes. One way a part is a lesse quantitie in respect of a greater, whether it measure the greater or no. The second way, a part is onely that lesse quantitie in respect of the greater which measureth the greater."—BILLINGSLEY, fol. 126.

By *part* is meant an *aliquot* part, or *submultiple*,—not any portion of a whole or greater magnitude, but *the* portion, which if repeated will exactly make up that whole, or greater magnitude; thus, if A represents a line = 3 inches, and B a line = 9 inches, A = 3 is an aliquot part or submultiple of B = 9;—A repeated exactly measures B, and B is the multiple of A. In numbers 4 and 6 are submultiples of 24; and the quantities, *a*, *b*, *c*, *d*, of *a b c d*.

Aliquant part has been used for the less magnitude, which though repeated does not exactly make up the greater; thus a measure of 3 feet is an *aliquant* part of 7 feet.

The magnitudes compared must be *of the same kind*,—lines, or surfaces, or solids,—weight, or time, &c. Lines bound surfaces, but are no part of them; an hour cannot be measured by an ounce,—and a penny is no submultiple of a mile.

One magnitude *measures* another when it is contained in that other magnitude an exact number of times.

And a magnitude which is a measure of two or more magnitudes is named the *common measure* of those magnitudes.

II.—A greater magnitude is said to be a *multiple* of a less, when the greater is measured by the less, that is, "when the greater contains the less a certain number of times exactly."

"The greater magnitude is a multiple of the less, when it is exactly measured by the less."—EUCLID.

"The *Multiplex* is a great quantity compared with a less which it contains precisely some number of times."—DE CHALES.

"By this worde *multiplex*," says BILLINGSLEY, fol. 127, "which is a terme proper to arithmitike and number, it is easy to consider that there can be no exact knowledge of proportion and proportionalitie, and so of this fifth booke, wyth all the other bookes followyng, without the ayde and knowledge of numbers." If excuse be needed, this is the reason why we have prefixed, "*Some Properties of Proportional Numbers* to EUCLID's otherwise matchless "*Theory of Geometrical Proportion*."

In things of the same kind every greater magnitude contains the less; thus a vessel of 27 cubic inches contains another vessel of 13 cubic inches;—but the greater is a multiple of the less only when it is measured exactly by the less,—*i. e.*, by its submultiple; thus a piece of cloth 27 yards in length is the multiple of another piece 9 yards in length,—3 repetitions of the less exactly making up the greater.

Equimultiples are magnitudes containing their respective aliquot parts the same number of times; thus *one yard* and *one barrel* are equimultiples of *an inch* and *a gallon*;—for the yard contains the inch 36 times, and the barrel also contains the gallon the same number of times.

All magnitudes for which an exact common measure, or submultiple, can be found are *Commensurable* magnitudes, as the radius and diameter of a circle, and the angle of an equilateral triangle, of a square, or of a reg. hexagon, and the space round any point: and those magnitudes are *Incommensurable* which have no common measure; thus the diagonal of a square represented in numbers, and the side of that square have no common measure;—for if the side contains 100 units, the diagonal will contain less than 142 and more than 141;—what is the submultiple of the one is not the submultiple of the other. The same is also true,—1°. of the diameter and circumference of a circle; 2°. of the diagonal and side of a cube; 3°. of the segments of a line cut in extreme and mean ratio. None of these have a common measure, neither have they a common multiple.

III.—*Ratio* is a mutual relation of two magnitudes of the same kind to one another in respect of quantity.

A mistake in translating Euclid's κατα πηλικότητα "in respect of quantity," has tended to confuse this definition. The *how great* one thing is when compared with another is the hinge on which the definition turns. Euclid is speaking of two magnitudes with respect to the spaces which they occupy, whether length, or area, or bulk, and his meaning therefore is better expressed by saying that "Ratio is the relative size which two magnitudes of the same kind have to one another with respect to the space which they occupy." A square of six acres in area, though greater than a square of four acres in area, is, when compared with four acres, a less magnitude than a square of three acres is, when compared with a square of one acre; the space which the six acres occupy is only one and a half times larger than the space which the four acres occupy,—but the space comprised in three acres is three times larger than the space comprised in one acre. Thus the *how great* one thing is, when compared with another, is the essential idea which belongs to EUCLID'S definition of ratio.

Certainly *the how great* is best expressed by numbers; an algebraical notation, m times, or n times, or $\frac{m}{n}$ times, may denote generally that one magnitude B is m times A, or $\frac{m}{n}$ times A; but "the particular ratio of two given magnitudes, whether commensurable or otherwise, can be" expressed or "conceived only by means of the numbers which denote how often the same magnitude is contained, or nearly contained, in each." Without these numbers we form no idea of the relative magnitude of the two given magnitudes,—for numbers constitute, either its exact, or its proximate measure. *See* GEOM. PL. SOL. & SPHER. p. 32.

A ratio is expressed by *two terms*, as A : B, or 6 : 12; the foundation from which the comparison proceeds, A, is named the *antecedent*, and the term to which the comparison extends, B, the *consequent*.

In two commensurable magnitudes the *numerical ratio* of one to the other, "is a certain number, whole or fractional, which expresses *how many*, and *what* parts of the second are contained in the first: for example, if

the common measure of A and B be contained in A *five* times, and in B *six* times, or, which is the same thing, if A contains $\frac{5}{6}$ths of B, then A is said to have to B the numerical ratio '5 to 6,' which is written 5 : 6, or, in the fractional form, $\frac{5}{6}$."

The *measure of a ratio*, or how great one magnitude is when compared with another of the same kind, is determined, by ascertaining *how often* the first magnitude is contained in the second, or *what part* the first magnitude is of the second; if one line, A B, contains 12 units of length, and another line, C D, 4 of the same units, the measure of their relative magnitudes $= 12 \div 4$, or 3; and if one line, E F, contains 3 units, and another G H, 12 units, the measure $= 12 \div 3$, or $\frac{1}{4}$.

When A = B, the ratio A : B is one of *equality*; when A is > B, the ratio is of *greater inequality*; and when A is < B, it is of less *inequality*.

The *inverse* or *reciprocal ratio* arises from changing the order of its two terms; as, instead of A : B, or 5 : 6, making the ratio B : A; 6 : 5 or $\frac{6}{5}$.

IV.—Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

"Magnitudes are said to have a ratio to one another, which are able on being multiplied to exceed one another."—EUCLID.

"In Geometry, *multiplication* is only a repeated addition of the same magnitude, and *division* is only a repeated subtraction."—POTTS. The successive foldings up of a string, or of a piece of cloth, in folds of the same length, is geometrical multiplication; the successive unfoldings, or cuttings off, of pieces of the same length, is geometrical division.

By this definition is excluded the comparison between any two magnitudes of which one is *finite* and the other *infinite*;—for no addition of finite things can ever equal, much less exceed, the infinite; and for the same reason we cannot institute a comparison between two magnitudes of which one is infinitely small and the other infinitely large.

V.—*Definition of Proportion.*—The first of four magnitudes is said to have the *same ratio* to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

"Magnitudes are said to be in the *same ratio*, the first to the second, and the third to the fourth,—when the equimultiples of the first and third, being at the same time compared with the equimultiples of the second and fourth, each with each, are, whatever the multiplication may be, together greater, or together equal, or together less."—EUCLID.

The whole reasoning of the fifth book rests on this celebrated definition as its foundation;—it is indeed, *the* definition which supplies a *criterion* for determining the equality of two ratios.

We shall exemplify its meaning by taking four magnitudes A, B, C, D; and of A and C equimultiples E, G, five times A, C; and of B and D equimultiples F, H, twice B, D: also K, M, four times A, C; and L, N, double of B, D; finally, O, Q, thrice A, C; and P, R, four times B, D; now,

A.	B.	C.	D.
2.	4.	3.	6.
E.	F.	G.	H.
10.	8.	15.	12.
K.	L.	M.	N.
8.	8.	12.	12.
O.	P.	Q.	R.
6.	16.	9.	24.

$\because$ E $>$ F, and G $>$ H; K $=$ L, and M $=$ N; O $<$ P, and Q $<$ R;
$\therefore$ A : B $=$ C : D.

Or, Let $a : b = e : d$; and let a, c, each be taken m times, b and d each n times;

then, 1. If $m\,a > n\,b$; also $m\,c > n\,d$;
2. If $m\,a = n\,b$; „ $m\,c = n\,d$;
3. If $m\,a < n\,b$; „ $m\,a < n\,d$.

Thus the *test* of the equality of the ratios $a : b$ and $c : d$, is, that any equimultiples of a and c are always greater than, equal to, or less than the equimultiples of b and d.

The excellence of Euclid's definition consists in its *applicability* to *all* magnitudes, to *incommensurables* as well as to *commensurables*. See *Study of Mathematics*, pp. 85, 86. Applied to incommensurables indeed the second part of the test does not occur,—for in that case no multiple of one quantity can be found exactly equal to a multiple of the other; but when quantities are commensurable, the second part is as applicable as the first and third.

LARDNER'S comment, in elucidating the meaning of the text, establishes that *Ratios are the same*, 1°. if their consequents are equimultiples of their antecedents; 2°. if their antecedents be equimultiples of their consequents; 3°. if any equimultiples whatever of the antecedents are also equimultiples of their consequents; or 4° when the ratios of every pair of equimultiples of their antecedents to every pair of equimultiples of their consequents are ratios of the same species of inequality,—*i. e.*, if A and a be multiplied by the same number, the results must be either both greater, equal to, or less than, the results obtained by multiplying B and b by any number.

"The best commentary," says the *Manual of Euclid, pt.* II., *p.* 29, "is in the first and last Propositions of the Sixth Book;" where in applying the test of equality of ratios, it is shewn from the nature of the case considered, that what is proved of one set of multiples must necessarily be true of all.

VI.—Magnitudes which have the same ratio are called *proportionals*.

The same ratio is expressed by the sign :: or $=$; thus A : B :: C : D, or A : B $=$ C : D; or in the manner of a fraction $\frac{A}{B} = \frac{C}{D}$, indicating the division of the antecedent by the consequent.

VII.—When of the equimultiples of four magnitudes, (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a *greater ratio* than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a *less ratio* than the first has to the second.

A *greater ratio* exists, if the first magnitude contains any aliquot part of the second a greater number of times than the third contains the like aliquot part of the fourth; for example, in the four quantities, a, b, c, d, let $ma > nb$, but $mc < nd$, then $a : b > c : d$; or, let $ma < nb$ but $mc > nd$ then $a : b < c : d$. Or, take four numbers 101, 10, 200, 20; then, ∵ 101 contains one hundred and one times the tenth part of 10,—but 200 contains only one hundred times the tenth part of 20, ∴ the ratio 101 : 10 > the ratio 200 : 20; and ∴ 101 is a greater number with respect to 10 than 200 is with respect to 20.

VIII.—Analogy or Proportion is the similitude of the ratios.

"Analogy is the sameness (identity) of the ratios."—EUCLID.

Analogy is a *reasoning out* of two sets of comparisons and a declaring of their identity; thus, the admiral and the fleet are one set of relations,—the general and the army another,—and our reasoning is an analogy, when we declare, *as* an admiral to his fleet, *so* is a general to his army.

Ratio is the comparison instituted between two magnitudes,—Analogy, the comparison between two ratios,—the analogy being complete or perfect when the ratios are identical. Take four magnitudes A, B, C, D, and from them form the ratios A : B and C : D; if the one ratio equals the other, there exists a proportion between the magnitudes,—A : B :: C : D. This proportion does not establish the absolute, but the relative magnitudes.

IX.—Proportion consists in three terms at least.

The middle term being repeated, there are in reality four terms, without which a proportion cannot be established; thus, $a : b :: b : c$, or 2 : 4 :: 4 : 8.

X.—When three magnitudes are proportionals, the first is said to have to the third the *duplicate ratio* of that which it has to the second.

A duplicate, or two-fold ratio is a ratio compounded of two equal ratios; thus if $a : b : c$, then $a : c = \frac{a}{b} \times \frac{b}{c}$; or 2 : 4 : 8; then $\frac{2}{8} = \frac{2}{4} \times \frac{4}{8} = \frac{8}{32} = \frac{1}{4}$.

The duplicate ratio is expressed *algebraically* and *arithmetically* by the ratio of their squares; thus, $a : b : a ; - a : c = a^2 : b^2$,
or 2 : 4 : 8; 2 : 8 = 4 : 16.

Such magnitudes are in *continued proportion*;—for magnitudes are continued proportionals, when every two terms have always the same ratio; or when the first has the same ratio to the second as the second has to the third,—and the second to the third the same as the third to the fourth; thus, $a : b : c : d : e$, &c., or 2 : 4 : 8 : 16 : 32, &c., the common ratio being 2.

In continued proportionals the first and last terms are the *extremes*, the intermediate are the *means*.

A *mean proportional* is a magnitude between two other magnitudes, forming with them a continued proportion; and the *third proportional* is the magnitude in continued proportion with the two other magnitudes.

A *double Ratio* and a *duplicate Ratio* must not be confounded. A double, triple, &c., Ratio is so called when the antecedent is double, triple, &c., of the consequent; a duplicate Ratio is a ratio compounded of two equal ratios, as in the proportionals, 2 : 4 : 8, or 3 : 9 : 27;—2 : 8, or 3 : 27 being compounded of 2 : 4, and 4 : 8;—or of 3 : 9, and 9 : 27;—in the one instance it is the half of a half,—in the other the third of a third. The half of a half is the square of $\frac{1}{2}$,—the third of a third is the square of $\frac{1}{3}$; 2 is $\frac{1}{2}$ of $\frac{1}{2}$, or $\frac{1}{4}$ of 8,—and 3 is the $\frac{1}{3}$ of the $\frac{1}{3}$, or $\frac{1}{9}$ of 27.

In like manner 8 : 2 is a duplicate ratio of 8 : 4 because 8 is the double of 4.

Duplicate ratio is a species of compound ratio, of which instances occur in propositions 19 and 20, bk. vi.

XI.—When four magnitudes are continual proportionals, the first is said to have to the fourth, the *triplicate ratio* of that which it has to the second, and so on, *quadruplicate*, &c., increasing the denomination still by unity in any number of proportionals.

"When four magnitudes are proportional, the first is said to have to the fourth a ratio triplicate of that which it has to the second; and so on successively in order, as far as the analogy (or proportion) may extend."
—EUCLID.

Triplicate ratio, compounded of three equal ratios, is the ratio of the cubes; thus, if A : B : C : D, or 2 : 4 : 8 : 16;—then $A : D = A^3 : B^3$, or 2 : 16 = 8 : 64. The ratio 2 : 16 is triplicate of the ratio of 2 : 4; for 2 is $\frac{1}{2}$ of $\frac{1}{2}$, or $\frac{1}{8}$ of 16.

Triplicate ratio is a species of compound ratio, instances of which are found in the 11th and 12th books of Euclid on the Geometry of Solids.

A numerous list of the kinds of proportion might be given; as, Duple proportion, triple, quadruple, quintuple,—Sesquialter, sesquitertia, sesquiquarta, — Super-partiens, super-bipartiens, super-tripartiens, &c., — Multiplex, super-particular, &c., &c.;—but most of them are rather ingenious puzzles than of practical utility.

DEF. A—*Of Compound Ratio.*—When there is any number of magnitudes of the same kind, the first is said to have to the last of these the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D, be four magnitudes of the same kind, the first, A, is said to have to the last, D, the ratio compounded of the ratio of A to B, and of the ratio B to C, and of C to D; or the ratio of A : D is said to be compounded of the ratios of A : B, B : C, and C : D.

So 4, 5, 3, 11 are four numbers, and $4 : 11 = \frac{4}{5} \times \frac{5}{3} \times \frac{3}{11} = 60 : 165$.

And if $A : B = E : F$, $B : C = G : H$, $C : D = K : L$; then $A : D$ is a ratio compounded of ratios, which are the same with the ratios of $E : F$, $G : H$, and $K : L$. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L; thus,

If $A : B = E : F$; $B : C = G : H$; & $C : D = K : L$; then, $A : D = E : F$, of $G : H$, of $K : L$.

Or, if $3 : 4 = 6 : 8$; $4 : 5 = 8 : 10$; and $5 : 7 = 10 : 14$;
then, $\frac{3}{7} = \frac{6}{8}$ of $\frac{8}{10}$ of $\frac{10}{14} = 6 : 14$.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L. Thus

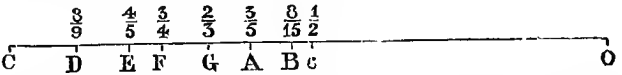
If $A : D = M : N$; then $M : N = E : F$, of $G : H$, of $K : L$.

Or, if $3 : 7 = 21 : 49$; then $21 : 49 = \frac{5}{8} \times \frac{8}{10} \times \frac{10}{14} = \frac{480}{1120} = 3 : 7$.

It may be observed, 1°., that the product of the fractions which represent numerical ratios corresponds to the compound ratios of magnitudes; 2°. in ratios compounded by this definition, the second term of each ratio is the same as the first term of the following ratio,—the consequent of the former becoming the antecedent of the latter.

DEF. B—There is a species of *Progression in the lengths of chords* of the same thickness and degree of tension, which produce the musical sounds of a certain note, of its fifth, and of its octave. Thus, if a musical string, C O, be divided so that its parts are in the proportion to one another as the numbers, $1, \frac{8}{9}, \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{8}{15}, \frac{1}{2}$, the

vibrations of the respective parts, C O, D O, E O, F O, G O, A O, B O, *c* O, will yield the eight sounds



to which musicians give the names, C, D, E, F, G, A, B, *c*. Of these parts of a musical string, thus divided, it is the property, that *the first* is to *the third*, as *the difference between the first and second* is *to the difference between the second and third.* Thus $1 : \frac{4}{5} = \frac{1}{9} : \frac{4}{45}$.

Three straight lines, therefore, are said to be in *Harmonical Progression*, when the first is to the third as the difference of the first and second is to the difference of the second and third.

And when three lines, C O, D O, and E O are in harmonical Progression, D O is named a *harmonical mean* between C O and E O; and E O is named the *third harmonical progressional* to C O and D O.

In the same way magnitudes of any other kind are in harmonical progression, when the first : third : : the dif. between the first and second : the difference between the second and third; thus,

$$A : B = A \sim B : B \sim C.$$

And *any number* of lines or magnitudes are in harmonical progression, when every successive three are in harmonical progression.

XII.—In proportionals the antecedent terms are called *homologous* to one another,—as also the consequents to one another.

"Magnitudes are said to be homologous,—the antecedents to the antecedents, and the consequents to the consequents."—EUCLID.

Homologous magnitudes are those which correspond in the proportion; thus in A : B = C : D, A and C being antecedents, are homologous; and B and D, being consequents; but in A : B : C : D, A and C, B and C, B and D are homologous,—for the full statement of the progressionals is, A : B=B : C = C : D,—where B and C are at one time consequents, and at another time antecedents.

TECHNICAL WORDS TO DENOTE CHANGES IN THE ORDER OF PROPORTIONALS.

1. FOR FOUR PROPORTIONALS.

XIII.—*Permutando*, or *alternando*, by permutation, or alternately. This word is used when there are four proportionals, and it

is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth; as is shown in Prop. 16 of this fifth book. Thus,

If A : B = C : D, or 1 : 2 = 3 : 6;
then *alternando* A : C = B : D; or 1 : 3 = 2 : 6.

"Alternate ratio is the comparison of the antecedent to the anteeedent, and of the consequent to the consequent."—EUCLID.

XIV.—*Invertendo*, by inversion; when there are four proportionals, and it is inferred that the second is to the first, as the fourth to the third. Prop. B. Book V. Thus,

If $\frac{A}{B} = \frac{C}{D}$, or $\frac{1}{2} = \frac{3}{6}$; then *invertendo*, $\frac{B}{A} = \frac{D}{C}$, or $\frac{2}{1} = \frac{6}{3}$

"Inverse ratio is the taking or comparing of the consequent as antecedent with the antecedent as consequent."—EUCLID.

In all changes of the order of Proportionals it is *essential*, that if one of the means be exchanged for an extreme, the other mean must also change places with the other extreme; and that if one of the extremes be placed as a mean, the other extreme must be placed as the other mean; thus, if A : B = C : D, or 1 : 2 = 3 : 6; then A : C ≠ D : B, or $\frac{1}{3} \neq \frac{6}{2}$.

XV.—*Componendo*, by composition; when there are four proportionals, and it is inferred that the first together with the second, is to the second as the third together with the fourth, is to the fourth. Prop. 18. Book V. Thus,

If A : B = C : D; then A + B : B = C + D : D.
Or, 1 : 2 = 3 : 6; then 1 + 2 : 2 = 3 + 6 : 6.

"The Synthesis, or Composition of ratio, is the taking of the antecedent along with the consequent as one term, in comparison with the consequent."—EUCLID.

Generally, the sum of the first *and* second is to the first *or* second, as the sum of the third *and* fourth to the third *or* fourth; thus, if $\frac{A}{B} = \frac{C}{D}$ then $\frac{A + B}{A \text{ or } B} = \frac{C + D}{C \text{ or } D}$; or, if $\frac{2}{3} = \frac{4}{6}$, then $\frac{5}{2} = \frac{10}{4}$, and $\frac{5}{3} = \frac{10}{6}$.

XVI.—*Dividendo*, by division; when there are four proportionals, and it is inferred that the *excess* of the first above the

second is to the second, as the *excess* of the third above the fourth is to the fourth. Prop. 17, Bk. V. Thus,

If A : B = C : D, A being greater than B, and C than D; then A − B : B = C − D : D;

or, if 3 : 1 = 6 : 2; then, 3 − 1 : 1 = 6 − 2 : 2.

"The *Diæresis* or Division of ratio is the taking of the excess whereby the antecedent exceeds the consequent in comparison with the consequent." EUCLID.

Geometrical Division, as we have seen, is the successive subtraction of a less magnitude from a greater.

Generally also,—the difference of the first *and* second is to the first *or* second, as the difference of the third *and* fourth to the third *or* fourth; thus, if $\frac{A}{B} = \frac{C}{D}$, or $\frac{2}{6} = \frac{8}{24}$; then, $\frac{A \sim B}{A \text{ or } B} = \frac{C \sim D}{C \text{ or } D}$;

Or, $\frac{2 \sim 6}{2 \text{ or } 6} = \frac{8 \sim 24}{8 \text{ or } 24}$.

XVII.—*Convertendo* by conversion; when there are four proportionals, and it is inferred that the first is to the *excess* above the second, as the third to its *excess* above the fourth. Prop. E., Bk. V. Thus, if A : B = C : D, or 2 : 1 = 6 : 3; then A : A − B = C : C − D, or 2 : 2 − 1 = 6 : 6 − 3.

"An *anastrophe* or *reversion* of the ratio is the taking of the antecedent in comparison with the excess whereby the antecedent exceeds the consequent."—EUCLID.

Combining several definitions into one,—*conjungendo*, the sum or difference of the first and second is to the first or second, as the sum or difference of the third and fourth is to the third or fourth;—*or* the sum of the first and second is to their difference, as the sum of the third and fourth to their difference.

2°.—FOR ANY NUMBER OF PROPORTIONALS ABOVE TWO.

XVIII.—*Ex æquali* (*sc. distantia*) or *ex æquo*, from equality of distance; when there is any number of magnitudes more than two, and as many others such that they are proportionals when taken two and two of each rank; and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others. Of this there are the two kinds, in definitions 19 and 20, which arise from the different order in which the magnitudes are taken two and two.

"The *ratio of equal distance*, (or of intervals) is when there are several magnitudes and others equal to them in number, taken two and two in the same ratio, and it arises that, as in the first set of magnitudes the first is to the last, so in the second set of magnitudes the first is to the last." "Otherwise, a taking of the extremes for comparison by a removal of the means."—EUCLID.

Thus, assuming $a : b = d : c$; $b : c = e : f$; $r : s = k : y$; $s : t = y : z$;
1st Series. $a, b, c \ldots r, s, t,$ or 2, 4, 8, ... 6, 12, 24;
2nd „ $d, e, f, \ldots k, y, z,$ 3, 6, 12, ... 9, 18, 36;
then $a : t = d : z,$ or 2 : 24 = 3 : 36.

XIX.—*Ex æquali*, or *ex æquo ordinate*. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank: so is the second to the third of the other; and so on in order; and the inference is as mentioned in the preceding definition; hence this is called *ordinate proportion*. It is demonstrated in Prop. 22, Bk. V.

"*Arranged Analogy*, or *Proportion* is when antecedent is to consequent, as antecedent to consequent; and also as consequent to some one thing, so is consequent to some other thing."—EUCLID.

Let A : B : C : D, 12 : 6 : 18 : 36; and E : F : G : H, 6 : 3 : 9 : 18; then A : D :: E : H; 12 : 36 = 6 : 18.

XX.—*Ex æquali in proportione perturbatâ seu inordinatâ*, or *ex æquo perturbate*, from equality in perturbate or disorderly proportion. Archimedes *de sphærâ et cylindro*. Prop. IV, Lib. 2. This term is used when the first magnitude is to second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank so is the third from the last to the last but two of the second rank; and so on in a cross order; and the inference is as in the 18th definition. It is demonstrated in Prop. 23, Bk. V.

"*Disturbed Analogy*, or Proportion is when, on there being three magnitudes and others equal to them in number, it comes to pass, that, as in the first magnitudes, antecedent is to consequent, as in the second magnitudes antecedent to consequent; so, in the first magnitudes, as the consequent is to some other thing, so, in the second magnitudes some other thing is to the antecedent."—EUCLID.

Let one series of magnitudes be A, B, C, 4, 8, 2; and another D, E, F, 12, 3, 6;

so that	A : B = E : F,	*or*	4 : 8 = 3 : 6;
and	B : C = D : E,	and	8 : 2 = 12 : 3;
then	A : C = D : F;	then	4 : 2 = 12 : 6;

i.e., the magnitudes being taken in a cross order are therefore said to be in disturbed or disordered Proportion, though in reality the Proportion is as exact as in any other case of proportion.

"Both this and the former inference come under one general principle, *scil.*, that ratios which are compounded of equal ratios are equal."—LARDNER.

The definitions, *ex æquo ordinate* and *ex æquo perturbate*, may readily be extended to any number of magnitudes, compared with an equal number of other magnitudes.

POSTULATES.

Let it be granted,

1.—That a given magnitude may be so increased that any required multiple of it may be taken.

2.—That any given multiple of a magnitude may be divided into parts, each of which is equal to that magnitude.

AXIOMS.

1.—Equimultiples of the same, or of equal magnitudes, are equal to one another.

2.—Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

3.—A multiple of a greater magnitude is greater than the same multiple of a less.

4.—That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

ADDITIONAL ALGEBRAIC EXPRESSIONS, &c.

N.B.—The capital letters A B, C D, E F, &c; or A, B, C, D &c., denote either lines, or other magnitudes of a like kind.

An *M* denotes magnitude,—*Ms*, magnitudes.

m	multiple.	n	another multiple.
m A &c.	multiple of A, &c.	$m + n$	the sum of the quantities m & n.
m A, m B, &c.,	equimultiples of A, B, &c.	$m\,n$	the product of $m \times n$.
m (A + B)	multiple of (A + B).	$m\,n$ A	a multiple of A by $m\,n$.
m (A − B)	multiple of (A−B).	$(m + n)$ A	,, ,, A by $(m+n)$.
m (A+B−C)	multiple of the excess of (A+B) above C.	*pt.*	part.
		sub-m.	submultiple.

The signs $>$, $\not>$, $<$, $\not<$, between ratios, as A : B $>$ C : D, or A : B $\not>$ C : D, or A : B $\not<$ C : D, or A : B $<$ C : D, denote that he one ratio is less than, or not less than, greater or not greater than the other, according to the sign.

Prop. I.—Theor.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same shall all the first magnitudes be of all the other.

Con. Pst. 2, V.—Auy given multiple of a magnitude may be divided into parts each of which is equal to that magnitude.

Dem. Ax. 2, V.—Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

Def. 2, V.—A greater magnitude is said to be a multiple of a less, when the greater is measured by the less.

Case I.—*Let the number of magnitudes in each set be two.*

E.1	Hyp.	Let A B, C D, &c., be equims. of, E, F, &c., each of each,
2	Conc.	then mult. A B of E = mult. (A B + C D) of (E + F).
D.1	Hyp.	∵ A B, C D equims. of E, F;
2	Ax. 2.	∴ mags in A B, each = E, *equal* Ms, in C D, each = F.
C.	Pst. 2, V.	Divide A B into A G, G B, each = E, and C D into C H, H D, each = F.
D.3	H.	∴ no. of Ms C H, H D = no. of other Ms A G, G B.
4	C.&Ax. 2, V.	∵ A G = E, and C H = F, ∴ A G + C H = E + F;
5	C.&Ax. 2, V.	and ∵ G B = E, and H D = F, ∴ G B + H D = E + F;
6	Conc.	∴ no. of Ms in A B, each = E *equal* no. of Ms in A B + C D, each = E + F.
7	Def. 2, V.	∴ the mult. A B is of E, that same mult A B + C D is of E + E.

Case II.—*Let the number of magnitudes in each set be more than two;* the same demonstration, which has been applied to two, holds for any number of magnitudes.

Conc. | ∴ *If any number of magnitudes.* &c. Q. E. D.

Alg. & Arith. Hyp.—Let A = 24, B = 21, C = 18, &c., be equimults. say m times, or 3, of $a = 8$, $b = 7$, & $c = 6$.

Alg.—Then ∵ A $= ma = a + a + a$; B $= mb = b + b + b$;
& C $= mc = c + c + c$;
+ the Equals, A + B + C $= ma + mb + mc = m(a + b + c)$;
∴ A + B + C same mult, m, of $a + b + c$, as A, B, C,
are respectively of a, b, c.

Arith.—∵ $24 = 3 \times 8 = 8 + 8 + 8$; $21 = 3 \times 7 = 7 + 7 + 7$;
& $18 = 3 \times 6 = 6 + 6 + 6$;
+ the Equals; $24 + 21 + 18 = (3 \times 8) + (3 \times 7) + (3 \times 6)$
$= 3(8 + 7 + 6)$;
∴ 24 + 21 + 18, or 63, the same mult. of 8 + 7 + 6, or 21, that 24 is of 8, 21 of 7, and 18 of 6.

Cor.—Hence, if m be any number, mA + mB + mC $= m$ (A + B + C), i.e., *the sum of the equimultiples = the equimultiple of the sum.*

Sch.—*If to a multiple of a magnitude by any number a multiple of the same magnitude by any number be added, the sum will be the same multiple of that magnitude that the sum of the two numbers is of unity.*

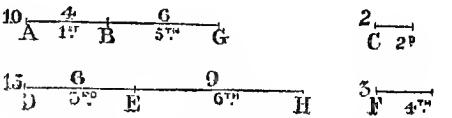
E. 1	Hyp.	Let A $= m$ C and B $= n$ C;
2	Conc.	then A + B $= (m + n)$ C.
D. 1	H. 1, V.	∵ A $= m$ C, ∴ A = C + C + C &c. repeated m times;
2	H. 1, V.	and ∵ B $= n$ C, ∴ B = C + C + C &c, „ n „ ;
3	*Add.* Ax. 2, I.	Adding equals, A + B = C taken $m + n$ times; i.e. A + B $= (m + n)$ C;
4	Conc.	∴ A + B contains C as often as there are units in $m + n$.

Cor. 1—Thus, if there be any number of multiples whatever, as A $= m$ E, B $= n$ E, C $= p$ E &c., it is shown that A + B + C $= (m + n + p)$ E.

Cor. 2.—Hence also, ∵ A + B + C $= (m + n + p)$ E;
and ∵ A $= m$ E, B $= n$ E, and C $= p$ E;
∴ m E + n E + p E $= (m + n + p)$ E.

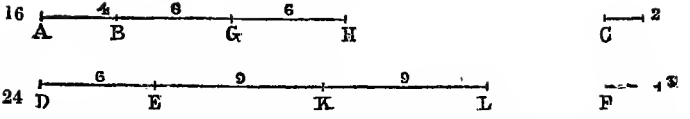
Prop. II.—Theor.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first, together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.



E.1	Hyp.	1	Let A B the 1st $= m$ C the 2nd, & D E the 3rd $= m$ F the 4th;
2	„	2	& B G the 5th $= n$ C the 2nd, and E H the 6th $= n$ F the 4th;
3	Conc.		then A G $=$ A B + B C $= (m + n)$ C; & D H $=$ D E + E H $= (m + n)$ F.
D.1	Hyp.	1	$\because$ A B $= m$ C, and D E $= m$ F;
2	Def. 2, V. obs.		$\therefore$ magns. in A B (each $=$ C) $=$ magns. in D E (each $=$ F);
3	*Sim*		So magns. in B G (each $=$ C) $=$ magns. in E H (each $=$ F);
4	Ax.	2	$\therefore$ magns. in A G (each $=$ C) $=$ magns. in D H (each $=$ F);
5	Def. 2, V.		$\therefore$ A G same mult. of C that D H is of F;
6	Remk.		i. e. A G (the 1st + 5th) same mult. of C the 2nd, that D H (the 3rd + 6th) is of F the 4th.
7	Rec.		*If therefore the first be the same, &c.*, Q. E. D.

Cor.—Hence, if any no. of magns. A B, B G, G H, be mults. of C, and as many, D E, E K, K L, the same *m*s of F, each of each;
then, A H i.e. (A B + B G + G H) the same *m* of C that D L, i. e. (D E + E K + K L) is of F.



Alg. & Arith. Hyp.—Take 6 quantities A = 24, B = 8, C = 21, D = 7, E = 16, F = 14, $m = 3$, and $n = 2$.

Alg. ∵ A the 1st = m B, and C the 3rd = m D.
E the 5th = n B, and F the 6th = n D;
+ *equals*, then A + E = $(m + n)$ B and C + F = $(m + n)$ D.
Now A + E contains B, $(m + n)$ times, & C + F contains D, $(m + n)$ times;
∴ Def. 2, V. A + E is the same mult. of B that C + F is of D.

Arith. ∵ $24 = 3 \times 8$, and $21 = 3 \times 7$;
$16 = 2 \times 8$, „ $14 = 2 \times 7$:
+ *equals*, $24 + 16 = (3 + 2)\,8$, and $21 + 14 = (3 + 2)\,7$.
But 40 contains 8, *five* times, and 35 contains 7, *five* times.
∴ Def. 2, V. 40 the same mult. of 8 that 35 is of 7.

SCHOL.—Allied to this Proposition is the Theorem": *If the first of three magnitudes contain the second as often as there are units in a certain number;—and if the second contain the third also as often as there are units in a certain number, the first will contain the third as often as there are units in the product of these two numbers.*"

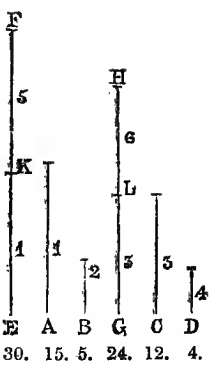
E. 1	Hyp.	Let A = m B, and B = n C;
2	Conc.	then A = mn C.
D. 1	H.	∵ B = n C, ∴ m B = m (n C + n C &c.);
2	2, V.	but m (n C + n C &c.) = $m \left\{ (n + n + \text{\&c.})\ \text{C} \right\}$,
3	Remk.	and $n \times m = mn$;
4	Ax. 1, V.	∴ m B = mn C.
5	H. Ax. 1, I.	But A = m B, ∴ A = mn C.

PROP. III.—THEOR.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth.

Con. Pst. 2, V.—Dem. Def. 2, V.—Cor. 2, V.

E. 1	Hyp. 1.	Let A the 1st = *m* B the 2nd; and C the 3rd = *m* D the 4th;
2	,, 2.	Also, let E F = *n* A, and G H = *n* C;
3	Conc.	then E F the same *m* of B as G H is of D.
D. 1	Hyp. 2.	∵ E F same *m* of A as G H of C;
2	Def. 2, V.	∴ as many magns. in E F, each = A, as in G H, each = C.
C. 1	Pst. 2, V.	Divide E F into E K, K F, each = A,
2	,, V.	and G H into G L, L H, each, = C;
3	Conc.	the no. of *Ms* in E K, K F = no. of *Ms* in G L, L H.
D. 3	Hyp. 1.	∵ A same *m* of B, that C is of D;
4	C. 1.	and ∵ E K = A, and G L = C;
5	Def. 2, V.	∴ E K same *m* of B, that G L is of D;
6	*Sim.* 1.	and ∴ K F same *m* of B, that L H is of D.
7	,, 2.	and so, if more *Ms* in E F & G H each = A, C.
8	Cor, 2, V.	Hence ∵ the 1st E K same *m* of the 2nd B, as 3rd G L, of 4th D;
9		and ∵ the 5th K F same *m* of the 2nd B, as 6th L H of 4th D;
10	2, V.	∴ E F (1st + 5th) same *m* of the 2nd B, as G H (3rd + 6th) of 4th D.
11	Conc.	If, *therefore the first be the same, &c.*



Q. E. D.

Cor. If A, A′ be equimults. of B, B′ and also of C, C′; and if B be a *m* of C, the other B′ shall be the same *m* of C′.

Alg. & Arith. Hyp.—Take $a = 12$, $b = 4$, $c = 15$, $d = 5$; $m = 3$ & $n = 2$.

Alg.—Let $a = mb$, and $c = md$; then $na = mnb$, and $nc = mnd$; *i. e.*, the equims. $n\,a$ & $n\,c$ of the 1st and 3rd. are mults. of the 2nd and 4th.

Arith. ∵ $12 = 3 \times 4$, and $15 = 3 \times 5$, ∴ $2 \times 12 = 6 \times 4$, & $2 \times 15 = 6 \times 5$;
i. e. the equims. 24 & 30, of the 1st 12, and 3rd 15, are equims. of the 2nd 4, and of the 4th 5.

SCH.—"If any equimultiples m A, m C, be taken of the antecedents of an analogy, A : B :: C : D, and any equimultiples, n B, n D, of the consequents, these multiples, taken in the order of the terms, are proportional," i. e. m A : n B :: m C : n D.

C. 1	Pst. 1, V.	Of m A, m C take equims. p times, and of n B, n D, equims. q times;
D. 1	3, V.	then ∵ m A, m C, contain A and C, p m units of times;
2	Conc.	∴ equims. m A, m C by p are equims. of A and C, and equal $p\,m$ A, $p\,m$ C.
3	*Sim*	So n B × q and n D × q = $q\,n$ B and $q\,n$ D.
4	H. and D. 2.	Since A : B :: C : D, and equims. of A and C are $p\,m$ A, $p\,m$ C,
5	D. 3.	and ∵ equims. of B, D are $q\,n$ B, $q\,n$ D,
6	Def. 5, V.	∴ if $p\,m$ A > $q\,n$ B, $p\,m$ C > $q\,n$ D; if =, =, and if <, <.
7	D. 2.	But $p\,m$ A, $p\,m$ C are also equims. of m A and m C,
8	D. 5.	and $q\,n$ B, $q\,n$ D also equims. of n B and n D;
9	Def. 5, V.	∴ m A : n B :: m C : n D;

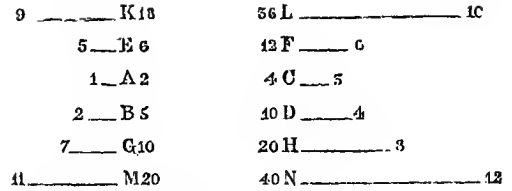
COR. When $n = 1$, then m A : B = m C : D.

PROP IV.—THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equal multiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth; viz., "the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.

CON.—Pst. 1, V. DEM. 3, V. Def. 5, V.

E. 1	Hyp. 1.	Let A : B = C : D;
2	,, 2.	and E, F be any equims. of A and C,
3	,, 3.	and G, H any equims. of B and D;
4	Conc.	then E : G = F : H.



C. 1	Pst. 1, V.	Of E and F take any equims. K, L;
2	,,	and of G and H ,, ,, M, N.
D. 1	Hyp. 2	∵ E is the same *m* of A as F of C,
2	C. 1.	and K, L are equims. of E and F;
3	3, V.	∴ K same *m* of A, that L is of C;
5	*Sim.*	So M same *m* of B, that N is of D.
5	Hyp. 1.	And ∵ A : B = C : D;
6	D. 3.	and K, L are equims. of A and C;
7	D. 4.	and M, N equims. of B and D;
8	Def. 5, V.	∴ K, >, = or < M, so L >, = or < N.
9	C. 1.	But K, L are equims of E and F;
10	C. 2.	and M, N equims. of G and H;
11	Def. 5, V.	∴ E : G = F : H;
12	Conc.	Therefore *if the first of four magnitudes &c.* Q. E. D.

Cor. 1.—Likewise, *if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples whatever of the first and third shall have the same ratio to the second and fourth;* and in like manner, *the first and third shall have the same ratio to any equimultiples whatever of the second and fourth.*"

Or, "If 4 Ms be proportional, then I°, any equims. being taken of the 1st and 3rd, the *m* of the 1st : 2nd = *m* of 3rd : 4th; and II°, any equims. being taken of 2nd and 4th, the 1st : *m* of 2nd = 3rd : *m* of the 4th."

E. 1	Hyp. 1.	Let A : B = C : D
2	Hyp. 2.	and let E, F be any equims. of A & C;
3	Conc.	then E : B = F : D.
C.	Pst. 1, V.	take of E, F any equims. K, L, and of B, D equims. G, H.

D. 1	4, V.	As before K same *m* of A, that L is of C.
2	Hyp. 1.	And ∴ A : B = C : D,
3	C.	and K, L are equims, of A & C,
4	C.	and G, H are equims. of B & D;
5	Def. 5, V.	∴ K > = or < G, so L > = or < H.
6	C.	But K, L are equims of E, F, & G, H any of B, D;
7	Def. 5, V.	∴ E : B = F : D.
8	*Sim.*	In the same way, A : G = C : H

COR. 2. ∵ in Dem. 8, Pr. 4, V., if K > = or < M, L > = or < N;
∴ if M > = or < K, N > = or < L. Hence G : E = H : F. Therefore, *if four magnitudes are proportionals, they will be proportional* by inversion.

N.B.—This Cor. is not in its proper place; it correctly forms Prop. B, V.

COR. 3. If A : B = C : D,—and *if any like parts* of A and C *be taken*, as $\frac{A}{2}$, $\frac{C}{2}$, *and also any like parts* of B and D, as $\frac{B}{3}$, $\frac{D}{3}$, *these like parts will also be proportional*; i. e., $\frac{A}{2} : \frac{C}{2} = \frac{B}{3} : \frac{D}{3}$

Alg. & Arith. Hyp.—Let $a = 2$, $b = 5$, $c = 4$, $d = 10$, $m = 2$ & $n = 3$.

Alg.—∵ $a : b = c : d$, then $ma : nb = mc : nd$. For, 3, V. equims. of ma and mc are equims. of a & c; and equims of nb, nd also equims. of b & d.

But, Def. 5, V., a & $c < = \text{or} > b$ & d, ∴ equims. of a & $c < = \text{or} > b$ & d, and also $< = \text{or} >$ equims. of b & d.
Hence, Def. 5, V. $ma : nb = mc : nd$.

Arith. ∵ $2 : 5 = 4 : 10$, then $2 \times 2 : 3 \times 5 = 2 \times 4 : 3 \times 10$. Now equims. of 4 & 8 are equims. of 2 & 4; and equims of 15 & 30 also equims. of 5 & 10.

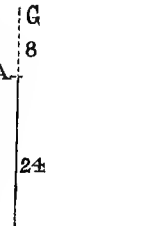
But, Def. 5, V. ∵ 2 & $4 < = \text{or} > 5$ & 10 ∴ equims of 2 & $4 < = \text{or} >$ 5 & 10, and also $< = >$ equims. of 5 & 10. Hence, Def. 5, V. $4 : 15 = 8 : 30$.

APPL.—From Cor. 3, arises the rule in simple proportion in arithmetic, of dividing the 1st and 2nd terms by any common measure, and using the *resulting* instead of the original numbers.

Prop. V.—Theor.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder is the same multiple of the remainder, that the whole is of the whole.

Con.—Pst. 1, V. Dem. 1, V. Ax. 1, V. Equimultiples of the same or of equal magnitudes are equal to one another.
Ax. 3. 1.—If equals be taken from equals the remainders are equal.



E. 1	Hyp. 1	Let A B be the same *m* of C D that A E, taken from A B, is of C F, taken from C D;
2	Conc.	then the rem. E B is the same *m* of rem. F D, as the whole A B of the whole C D.
C	Pst. 1, V.	Take A G same *m* of F D, that A E is of C F.
D. 1	C.	∵ A G, A E are equims. of F D and C F,
2	1, V.	∴ A G, A E, *i. e.*, E G, same *m* of C F, F D, *i.e.*, C D, that A E is of C F.
3	H.	but A E same *m* of C F, that A B is of C D;
4	1, V.	∴ E G same *m* of C D that A B is of C D,
5	Ax. 1, V.	∴ E G = A B;
6	*Sub* & Ax. 3, I.	from each take A E; ∴ rem. A G = rem. E B.
7	C.	∵ A E same *m* of C E that A G is of F D,
8	D. 6	and A G = E B;
9	1, V.	∴ A E same *m* of C F that E B is of F D;
10	H.	But A E same *m* of C F that A B is of C D;
11	1, V.	∴ E B same *m* of F D that A B is of C D.
12	Conc.	*Therefore if one magnitude be the same, &c.*

Q. E D.

Alg. & Arith. Hyp.—Let A = 32, B = 8, C (a part of A) = 24, and D (a part of B) = 6, $m = 4$.

Alg.	Let A = m B.	*Arith.*	Let 32 = 4 × 8.
	C = m D;		24 = 4 × 6.
Subt.	A − C = m (B − D),	Subt.	8 = 4 (8 − 6);
Thus	A − C is m times (B − D,	Thus	8 is 4 times 2;
as	A is m „ B.	as	32 was 4 „ 8.

So the rem. is the same m of the rem. as the whole is of the whole.

Or, Let A − B = D; to both sides add B,—then A = B + D;
$\therefore$ (1, V.) m A = m B + m D;
Subtract m B, and m A − m B = m D;
but D = A − B; $\therefore$ m A − m B = m (A − B);
Thus the rem. is the same m of the rem. that m A is of A.

SCH. "*If from a multiple of a magnitude by any number, a multiple of the same magnitude by a less number be taken away, the remainder will be the same multiple of that magnitude that the difference of the numbers is of unity.*

E. 1	Hyp.	Let m A, n A be mults. of A, m being $> n$;
2	Conc.	then m A − n A = $(m - n)$ A.
C.	*Sum.*	Let $m - n = q$; then $m = n + q$.
D. 1	C & 2, V.	Here m A = n A + q A;
2	*Sub.*	from both take n A; then m A − n A = q A;
3	Conc.	$\therefore$ m A − n A = $(m - n)$ A.

COR. When the difference of the two numbers is equal to unity, or $m - n = 1$, then m A = n A = A; or 2 A − A = A.

PROP. VI.—THEOR.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them.

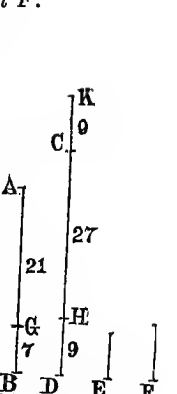
CON. Pst. 2, I.—3, I. From the gr. of two lines to cut off a part equal to the less. Pst. 1, V.

DEM. Ax. 1, V.—Ax. 3, I. Ax. 1, I.—2, V.

E. 1	Hyp. 1.	Let A B, C D be equims. of E F;
2	„ 2.	and let A G taken from A B be an equim. of E,
3	„ 3.	and C H from C D an equim. of F;
4	Conc.	then rems. G B, H D either = E F, or are equims. of them.

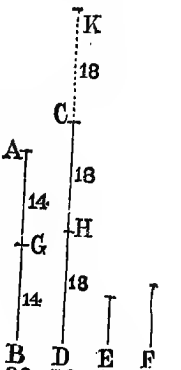
CASE. I.—Let $GB = E$, *then* HD *shall equal* F.

C.	Pst. 2 I & 3 I	Make C K = F.
D. 1	H. 2	∵ A G same *m* of E, that C H is of F;
2	H. & C.	and ∵ G B = E, and C K = F;
3	Conc.	∴ A B same *m* of E, that K H is of F.
4	H.	But A B same *m* of E, that C D is of F.
5	Conc.	∴ K H same *m* of F that C D is of F.
6	Ax. 1 V.	∴ K H = C D,
7	*Sub* Ax. 3 I.	take away C H, ∴ rem. K C = rem. H D;
8	C. Ax. 1 I.	but K C = F, ∴ H D = F.



CASE II.—Let G B be a *m* of E, then H D same *m* of F.

C.	Pst. 1, V.	Of F take C K the same *m* that G B is of E.
D. 1	H.	∵ A G same *m* of E that C H is of F.
2	C.	and G B same *m* of E that K H is of F;
3	2, V.	∴ A B same *m* of E that K H is of F.
4	H.	But A B same *m* of E that C D is of F,
5	Conc.	∴ K H same *m* of F that C D is of F,
6	Ax. 1, V.	∴ K H = C D.
7	*Sub.* Ax. 3, I.	Take away C H, ∴ rem. K C, = rem. H D.
8	C.	And ∴ G B same *m* of E that K C is of F,
9	D. 7	and K C = H D;
10	Conc.	∴ H D same *m* of F that G B is of E.
17	Rec.	*If*, therefore, *two magnitudes, &c.* Q. E. D.



Alg. & Arith. Hyp.—Let $m > n$ express any integers, as 4 and 3; $A = 28$, $B = 36$, $C = 7$, $D = 9$.

Alg.—Let A, B be equims. of C & D	*Arith.* 4 Let 28 & 36 be equims. of 7 & 9,
$A = mC$, $B = mD$,	$28 = \quad \times 7$; $36 = 4 \times 9$
$A > nC$, $B > nD$;	$28 > {}^{3} \times 7$; $36 > 3 \times 9$;
Subt. $A - nC = mC - nC = (m-n)C$;	Subt. $28 - 21 = 4 \times 7 - 3 \times 7 = (4-3)7$.
$B - nD = mD - nD = (m-n)D$.	$36 - 27 = 4 \times 9 - 3 \times 9 = (4-3)9$.
Suppose $m - n = 1$.	Here $4 - 3 = 1$.
1°. $A - nC = C$,	1°. $28 - 21 = 7$.
$B - nD = D$,	$36 - 27 = 9$,
2°. *Or*, equims of C & D	2°. *Or*, equims. of 7 & 9.
$(m-n)$ C and $(m-n)$ D.	as $(5-2)$ 7, and $(5-2)$ 9

SCH.—The six preceding Propositions are chiefly useful for establishing, by the method of Equimultiples, the Propositions which follow. When this method is not employed some have adopted the Postulate,—*Three Magnitudes, A, B, C, being given, let it be granted that there is a 4th magnitude, we may call it x, to which C has the same ratio, as A to B; i. e.* $A : B = C : x$.

PROP. A,—THEOR.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then if the first be greater than the second the third is greater than the fourth; and if equal, equal; if less, less.

CON.—Pst. 1, V. DEM.—Def. 5, V.

E.1	Hyp.	Let $A : B = C : D$,
2	Conc.	then if $A > = \text{or} < B$, $C > = \text{or} < D$.
C	Pst. 1, V.	Take any equims., as 2 A, 2 B, 2 C, 2 D.
D.1	H.	$\because A : B = C : D$,
2	C.	and of the 1st and 3rd equims. 2 A, 2 C are taken, and of the 2nd and 4th equims. 2 B, 2 D;
3	Def. 5, V.	$\therefore$ 2 C is $> = \text{or} < 2$ D, as 2 A is $> = \text{or} <$ 2 B.
4	,,	But 2 C is $> = \text{or} < 2$ D as C is $> = \text{or} <$ D;
5	*Sim.*	and so, 2 A is $> = \text{or} <$ 2 B, as A is $> = \text{or} <$ B;
6	Conc.	$\therefore$ C, is $> = \text{or} <$ D, as A is $> = \text{or} <$ B. Q. E. D.

Or, more briefly;

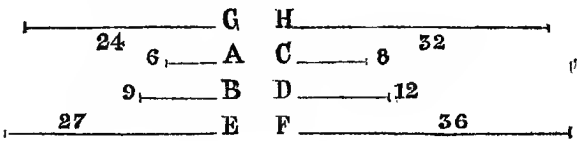
D.1	Def. 5, V.	If 2 A > 2 B, 2 C > 2 D;
2	,,	but if A > B, 2 A > 2 B;
3	Conc.	∴ 2 C > 2 D, and ∴ C > D.
4	*Sim.*	So if A = or < B, C = or < D. Q. E. D.

USE.—SIMSON added this and the next three Propositions. Prop. A is required for the demonstration of 25, V; 21, VI; 34, XI, and 15, XII, and is often employed by Geometers.

PROP. B.—THEOR.

Invertendo, by inverting.—*If four magnitudes are proportionals, they are proportionals also when taken* inversely.

CON. Pst. 1, V. DEM. Def. 5, V.



E. 1	Hyp.	Let A : B = C : D;
2	Conc.	then B : A = D : C.
C. 1	Pst. 1, V.	Of B,D, take any equims. E,F;
2	,,	and of A,C, any equims. G,H.

First.—Let E > G. *i. e.*, G < E.

D. 1	H.	∵ A : B = C : D,
2	C. 2.	and of the 1st A, and 3rd C, are equims. G & H,
3	C. 1.	and of the 2nd B, and 4th D, ,, E & F;
4	Def. 5. V.	∴ H > = or < F, as G > = or < E;
5	Conc.	If ∴ E > G, F is > E.

Second.—So, if E = or < G, F = < H.

6	C.	But E,F any equims. of B,D; & G,H of A,C;
7	Def. 5, V.	∴ B : A = D : C.
8	Rec.	Therefore, *if four magnitudes, &c.* Q. E. D.

Alg. & Arith. Hyp. Let A = 6, B = 9, C = 8; D = 12; $m = 4$, $n = 3$.

Alg. Let m A & mC be equims. of A & C.
nB & nD „ B & D;
∵ A : B = C : D
Def. 5, V. ∴ mA, mC > = or < nB, nD.
If mA, mC > nB, nD;
nB, nD < mA, mC;
If mA, mC < Bn, nD D;
nB, nD > mA, mC;
∴ Any equims. of B, D > = or < A,D.
Def. 5, V. ∴ B : A = D : C.

Arith. Let 4×6, & 4×8 be equims. of 6×8,
3×9, & 3,×12 „ of 9 & 12.
∵ 6 : 9 = 8 : 12.
∴ 24, 32 < 27 & 36.
i. e. 27 & 36 > 24 & 32.
∴ equims. of 9 & 12 > those of 6 & 8.
∴ 9 : 6 = 12 : 8.

SCH. 1. The Proposition may be stated,—"The reciprocals of equal ratios are equal to one another."

2. By an inaccuracy Prop. B has been placed by some as a Corollary to Fr 4. V.

PROP. C.—THEOR.

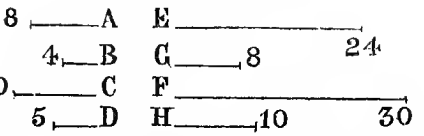
If the first be the same multiple of the second, or the same part (i. e. measure, or measure submultiple) *of it that the third is of the fourth, the first is to the second, as the third is to the fourth.*

CON. Pst. 1 V. DEM. 3 V.—Def. 5 V.—Def. 1 V.—Def. 2 V., P. B. V.

There are *two* Cases of this Prop. according as A and C are multiples or parts of B and D.

CASE I.—*Let A & C be multiples of B & D.*

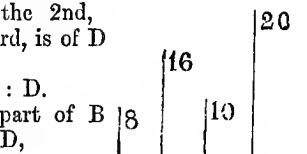
E.	1	Hyp.	Let A 1st be same m of B 2nd, that C 3rd is of D 4th,
	2	Conc.	then A : B = C : D.
C.	1	Pst. 1. V.	Take of A, C any equims. E, F;
	2	„	and of B, D any equims. G, H.



D. 1	H. & C.	∵ A, C equims. of B, D and E, F of A, C;
2	3, V.	∴ E, F equims. of B, D.
3	C.	But G, H equims. of B, D;
4	Conc.	∴ if E > *m* of B than G is of B, F is a > *m* of D than H of D;
5	Remk.	*i. e.* if E > G, F > H.
6	*Sim.*	So, if E = or < G, F = or < H.
7	C.	But E, F are equims. of A, C, and G, H of B, D;
8	Def. 5, V.	∴ A : B = C : D. Q. E. D.

CASE II. *Let A & C be parts, or sub-multiples, of B & D.*

E. 1	H.	Let A the 1st, be the same part of B the 2nd, that C the 3rd, is of D the 4th;
2	Conc.	then A : B = C : D.
D. 1	Def. 1, V.	∵ A the same part of B that C is of D,
2	Def. 2, V.	∴ the same *m* of A that D is of C;
3	Case 1.	∴ B : A = D : C.
4	B. V. *Invert.*	∴ A : B = C : D. Q. E. D
5	Rec.	Therefore, *if the first be the same multiple, &c.*



Alg. & Arith. Hyp. Take A = 2 B = 8, and C = 2 D = 10; *n* = 3 & *p* = 4.

Alg. Take *n* A, *n* C any equims. of A, C;
p B, *p* D „ of B, D,
Then ∵ A same *m* of B, that C is of D;
and *n* A „ „ of A, „ *n* C is of C;
∴ *n* A „ „ of B, „ *n* C is of D,
i. e. *n* A, *n* C are equims of B, D.
But *p* B, *p* D „ „ of B, D;
∴ If *n* A > *m* of B, than *p* B is of B,
n C > *m* of D, „ *p* D of D;
i. e., if *n*A > *p*B, *n*C > *p*D.
So, if *n* A = or < *p*B, *n*C = or < *p*D.
But *n*A, *n*C equims. of A, C,
*p*B, pD, „ of B, D,
∴ by Def. 5. V. A : B = C : D.

Arith. 3×8, & 3×10 are equims. of 8 & 10;
4×4 & 4×5 are equims. of 4 & 5;
Then ∵ 8 is the same m of 4 that 10 is of 5;
24 is the same m of 8 that 30 is of 10;
∴ 24 is the same m of 4 that 30 is of 5;
i. e., 24 & 30 are equims. of 4 & 5.
But 16 & 20 „ „ 4 & 5;
∴ If $24 > m$ of 4, than 16 is of 4,
$30 > m$ of 5, „ 20 of 5,
i. *e.*, if $24 > 16$, $30 > 20$.
So, if $27 =$ or < 16, $30 =$ or < 20.
But 24 & 30 are equims. of 8 & 10,
16 & 20 „ 4 & 5.
∴ by Def. 5, V. $8 : 4 = 10 : 5$.

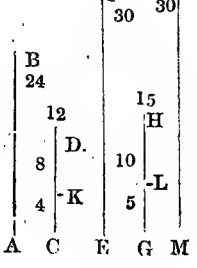
SCH.—The 7th, 8th, 9th, and 10th books of EUCLID's Entire Work treat of Arithmetic and the doctrine of Incommensurables; and the 20th Def. Book VII., gives a definition of quantities which are proportional; but "most of the commentators" says SIMSON, and POTTS repeats the words, "judge it difficult to prove that four magnitudes which are proportionals according to the 20th def. of the 7th book; are also proportionals according to the 5th def. of the 5th book. The Demonstration, however, is as follows; from SIMSON'S Notes, page 317.

I°. *As to four magnitudes in proportion according to Def. 5, V.*

CASE 1.—Let A, B, C, D be four *Ms*, such that $A = m$ B, or $\frac{B}{p}$; and $C = m$ D, or $\frac{D}{p}$; then, by Pr. C, bk. V; $A : B = C : D$.

CASE 2.—In A B let there be the same parts of C D as there are of G H in E F, then also $AB : CD = EF : GH$; For

C. 1	Pst.2,V.	Let C K be of C D the same *pt.* that G L is of G H,
2	„	and A B same *m* of C K that E F is of G L;
D.1	Pr. C,V.	∴ $AB : CK = EF : GL$;
2	C.	And ∵ C D, G H are equims. of C K, G L, the 2nd and 4th,
3	Cor.4,V.	∴ $AB : CD = EF : GH$



II°.—And, *if four magnitudes are proportionals according to the 5th def. of Book V, they are also proportionals according to the 20th def. of Book VII.*

CASE 1.—Let A : B = C : D.
then, by Pr. D, book V, if A is any *m*, or *pt* of B, C is the same *m* or *pt* of D.

CASE 2.—Let A B : C D = E F : G H, as in the foregoing figure:
then if A B = any *pt* of C D, E F = the same *pt* of G A.

C 1	Pst. 2, V.	Take C K a *pt* of C D, and G L the same *pt* of G H;
2	Pst. 1, V.	and let A B be a *m* of C K, and E F the same *m* of G L;
3	Pst. 2, V.	take M the same *m* of G L that A B is of C K;
D 1	Pr. C, V.	∵ A B : C K = M : G L,
2	C	and C D, G H = *m* C K, *m* G L:
3	Cor. 4, V.	∴ A B : C D = M : G H;
4	H.	And A B : C D = E F : G H,
5	9, V.	∴ M = E F
6	Conc.	∴ E F the same *m* of G L that A B is of C K.

"This is the method by which Simson shows that the Geometrical definition of proportion is a consequence of the Arithmetical definition, and conversely."

"It may, however, be shewn by employing the equation $\frac{a}{b} = \frac{c}{d}$, and taking *ma*, *mc* any equims. of *a* and *c*, the first and third, and *nb*, *nd*, any equims. of *b* and *d*, the second and fourth."—POTTS.

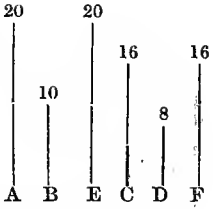
USE.—Prop. C, bk. V., is often made use of, and "is necessary to the 4th and 6th propositions of the 10th Book."

PROP. D.—THEOR.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second, the third is the same multiple or the same part of the fourth.

CON.—Pst. 1, V. DEM.—Cor. 4, V. A, V, B, V.

E. 1	Hyp.	Let $A : B = C : D$; and $A = m\ B$, or $\frac{B}{p}$;
2	Conc.	then $C = m\ D$, or $\frac{D}{p}$.



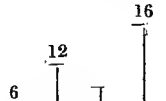
CASE 1.—Let $A = m\ B$; then $C = m\ D$.

C. 1	Pst. 1, V.	Take E = A, and F the same *m* of D that A or E is of B.
D. 1	H.	∵ A : B = C : D,
2	C.	and of B, D, 2nd and 4th, the equims are E and F;
3	Cor. 4, V.	∴ A : E = C : F.
4	C. A, V.	But A = E, ∴ C = E;
5	C.	And F is the same *m* of D that A is of B;
6	Conc.	∴ C ,, *m* of D that A is of B.

Q. E. D.

CASE 2.—Let $A = \frac{B}{p}$, then $C = \frac{D}{p}$.

D. 1	H. & B, V.	∵ A : B = C : D, ∴ *invert.* B : A = D : C;
2	H.	but $A = \frac{B}{p}$, ∴ B is a *m* of A; and ∴ D the same *m* of C,
3	Case 1.	*i. e.*, $C = \frac{D}{p}$, or same *pt.* of
4	Remk.	D, that A is of C.
5	Rec.	Therefore, *if the first be to to the second, &c.*



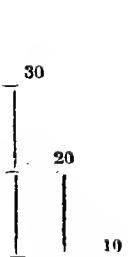
Q. E. D.

N.B.—For sake of preserving the same numbering we insert here as Prop. 7 A, the 7th Prop. of CHAMBERS' *Additional Fifth Book*.

PROP. 7 A.—THEOR.

The ratio of two lines is the same as that of the numbers which express the number of times that any third line is contained in them respectively.

E. 1	Hyp.	Let A=30, and B=20, be two lines, and M = 10 a third;	
2	„	and let $M = \frac{A}{a}$, and $\frac{B}{b}$,	
3	Conc.	then $\frac{A}{B} = \frac{a}{b}$; or A : B = a : b.	



Case.—Let a and b be integers, and $a := 3$, $b = 2$.

D. 1	H.	∵ A = 3M, and B = 2M;
2	Conc.	∴ A contains $1\frac{1}{2}$B, & a, $1\frac{1}{2}b$.
3	H.	and ∴ A : B = a : b, or 30 : 20 = 3 : 2.
4	*Sim.*	So we conclude whatever be the integer values of a and b;
5	Conc.	∴ $\frac{a}{b} = \frac{a\,M}{b\,M} = \frac{A}{B}$, i. e. $\frac{3}{2} = \frac{3 \times 10}{2 \times 10} = \frac{30}{20}$.

If being integers, $a = mq$ and $b = np$, then $\frac{a}{b} = \frac{mq}{np} = \frac{mq\,M}{np\,M}$.

Or, $6 = 3 \times 2$, and $8 = 2 \times 4$;
then $\frac{6}{8} = \frac{3 \times 2}{2 \times 4} = \frac{3 \times 2 \times 10}{2 \times 4 \times 10} = \frac{60}{80}$.

Case 2. Let $a = \frac{m}{n}$, and $b = \frac{p}{q}$; or $3 = \frac{6}{2}$, and $2 = \frac{8}{4}$.

D. 1	Hyp.	∴ M′ is a *subm.* of M by nq, or $M = nq\,M'$, i. e., $10 = 2 \times 4 \times 1\frac{1}{4}$;
2	Case 1.	∴ $\frac{a}{b} = \frac{m}{n} \times \frac{q}{p} = \frac{mq}{np} = \frac{mq\,M'}{np\,M'}$, i. e., $\frac{3}{2} = \frac{6}{2} \times \frac{4}{8} = \frac{24 \times 1\frac{1}{4}}{16 \times 1\frac{1}{4}} = \frac{30}{20}$
3		But $mq\,M' = \frac{mnq\,M'}{n} = \frac{m}{n} \times nq\,M' = \frac{m}{n}M$; i. e. $24 \times 1\frac{1}{4} = \frac{6}{2} \times 10 = 30$.
4	*Sim.*	So $np\,M' = \frac{p}{q}M$, i. e., $2 \times 8 \times 1\frac{1}{4} = \frac{8}{4} \times 10 = 20$;
5	Conc.	∴ $a : b = \frac{m}{n}M : \frac{p}{q}M$, i. e., $3 : 2 = \frac{6}{2} \times 10 : \frac{8}{4} \times 10$; or $a : b = A : B$; $\frac{8}{2} = \frac{30}{20}$.

CASE 3. Let a be interminate and b terminate.

	Sup.	If $\frac{A}{B} \neq \frac{a}{b}$, let $\frac{A}{B} = \frac{a'}{b}$; and let $a' > a$, and a'' be a terminate no. between a & a'.
D. 1	Case 1 & 2.	$\because \frac{a''}{b} = \frac{a''M}{bM}$, and $a''M > aM$, or A;
2		$\therefore \frac{a''M}{bM} > \frac{A}{B}$; $\frac{a''}{b} > \frac{a'}{b}$, and $\therefore a'' > a'$
3	H.	But a'' also $< a'$, an impossibility, $\therefore a' \not> a$.
4	*Sim.*	So $a' \not< a$, $\therefore \frac{A}{B} = \frac{a}{b}$

CASE 4. Let a be terminate and b interminate.

D.	*Sim.*	By a demonstration similar to that of Case 3.

CASE 5. Let a and b be both interminate.

	Sup.	If $\frac{A}{B} \neq \frac{a}{b}$, let $\frac{A}{B} = \frac{a'}{b}$; let $a' > a$ & a'' be a terminate no. between a & a'
D. 1	Case 4.	Then $\frac{a''}{b} = \frac{a''M}{bM}$.
2	*Sim.*	The rest of the dem. as in Case 3.
3		$\therefore \frac{A}{B} = \frac{a}{b}$

PROP. VII.—THEOR.

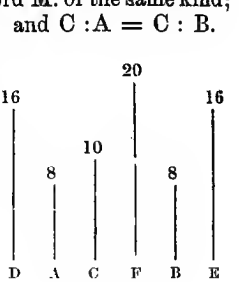
Equal magnitudes have the same ratio to the same magnitude, and conversely, *the same has the same ratio to equal magnitudes.*

CON. Pst. 1, V. A given M. may be so increased that any required m of it may be taken.

DEM. Ax. 1, V. Equims, of the same or of equal Ms. or equims. are equal to one another.

Def. 5, V. The first of four magnitudes is said to have the *same ratio*, &c.

E. 1	Hyp. 1	Let A = B, & C be a 3rd M. of the same kind;
	2 Conc. 1 & 2.	then A : C = B : C; and C : A = C : B.
C. 1	Pst. 1, V.	Of A, B, take any equims. D, E;
	2 Pst. 1, V.	and of C, take any equim. F.



First. A and B shall each have the same ratio to C.

D. 1	C.	∵ D & E are equims. of A & B;
	2 H. Ax. 1, V.	and A = B,
		∴ D = E;
	3	∴ if D > = or > F, E > = or > F;
	4 C. 1 & 2.	but D, E are equims. of A, B, and F is a *m* of C;
	5 Def 5, V.	∴ A : C = B : C.

Second. C shall have the same ratio to A that it has to B.

D. 1	*Sim.*	As before, D = E;
	2	∴ if F > = or > D, F > = or > E;
	3 C. 2 & 1.	but F any *m* of C, and D, E any *m* of A, B;
	4 Def. 4, V.	∴ C : A = C : B.
	5 Rec.	Therefore, *equal magnitudes have the same ratio, &c.* Q. E. D.

Cor.—If a ratio A : C which is compounded of two ratios, A : B, and B : C, be a ratio of equality, one of these must be the inverse, or reciprocal of the other; *i. e.*. A : B is the inverse, or reciprocal of B : C.

Sch.—The second part, C : A = C : B, follows from the corollary of Pr. 4. Bk. V.

PROP. VIII.—THEOR.

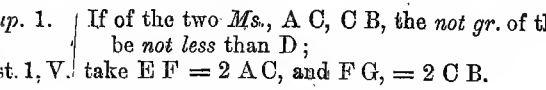
Of two unequal magnitudes, the greater has a greater ratio to another magnitude than the less has; and conversely, *the same magnitude has a greater ratio to the less of two other magnitudes, than it has to the greater.*

CON. Pst. 1, V. DEM. 1, V. DEF. 7, V.—When of the equims. of four Ms., (taken as in the fifth def.) the m of the 1st is $>$ that of the 2nd, but the m of the 3rd $\not>$ than the m of the 4th; then the first is said to have to the 2nd a greater ratio than the 3rd M has to the 4th; and on the contrary, the 3rd is said to have to the fourth a less ratio than the 1st has to the second.

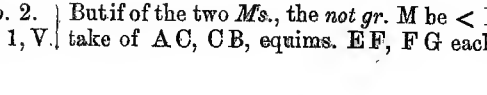
E. 1	Hyp.	Let A B $>$ B C, and D any other M;
2	Conc. 1.	then A B : D $>$ B C : D;
3	,, 2.	and D : B C $>$ D : A B.

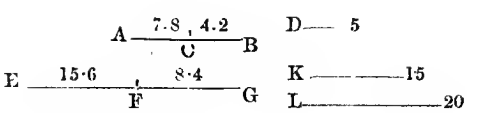
First.—A B : D $>$ B C : D.

C. 1	*Sup.* 1.	If of the two *Ms.*, A C, C B, the *not gr.* of the two be *not less* than D;
2	Pst. 1, V.	take E F $=$ 2 A C, and F G, $=$ 2 C B.



C. 3	*Sup.* 2.	But if of the two *Ms.*, the *not gr.* M be $<$ D;
4	Pst. 1, V.	take of A C, C B, equims. E F, F G each $>$ D.





C. 5	Pst. 1, V.	In all cases take H = 2 D, K = 3 D, &c., till the *m* of D be that which is first > F G.
6	Pst. 1, V.	Let L, a *m* of D, be found > F G;
7	„	and K the *m* of D next less to L; *i. e.*, F G ⋠ K.
D. 1	C.	∵ L is the *m* of D which first is > F G;
2		∴ K, the *m* of D next preceding, is ⋡ F G, *i. e.*, F G ⋠ K.
3	C.	And ∵ E F same *m* of A C, that F G is of C B;
4	1, V.	∴ F G the same *m* of C B that E G is of A B;
5	Remk.	*i. e.*, E G and F G are equims. of A B and C B.
6	D. 2, & C.	And ∵ F G ⋠ K, and E F > D;
7		∴ the whole E G = F G + F E > K + D;
8	C.	but K + D = L, ∴ E G > L.
9	C. D. 5.	But F G ⋡ L, and E G, F G equims of A B, C B;
10	C.	and L is a *m* of D;
11	Def. 7. V.	∴ A B : D > B C : D. Q. E. D.

Second.—D : B C > D : A B.

C.	*Sim.*	As in the *first* part.
D. 1	*Sim.*	As before, L > F G but ⋡ E G;
2	C.	and L a *m* of D;
3	D. 5. pt. 1.	also F G, E G were proved equims of C B, A B;
4	Def. 7, V.	∴ D : B C > D : A B.
5	Rec.	∴ *Of two unequal magnitudes, &c.* Q. E. D.

Otherwise. Let A + B = 5 + 4, and A = 5, be two unequal Ms; and C = 10 a third M; then A + B : C > A : C; *i. e.*, (5 + 4) : 10 > 5 : 10.

Alg. Take mA, mB, each $> C$;
nC the least mult. $> mA + mB$.
Hence $(n-1)\,C < mA + mB$, or $m(A + B)$;
$m(A + B) > (n-1)\,C$, or $nC-C$;
$\therefore n\,C > (mA + mB)$, and $C < mB$,
$\therefore nC - C > mA$, or $mA < (n-1)\,C$.
Hence $m(A + B)$ a m of $A + B > (n-1)\,C$, a m of C.
but mA, *is* a m of $A \not> (n-1)\,C$, a m of C,
$\therefore$ by Def. 7, V. $A+B : C > A : C$.

Arith. 3×5, and 3×4, each > 10;
3×10 least $m > (3 \times 5) + (3 \times 4)$.
Hence $(3-1)\,10 < (3 \times 5) + (3 \times 4)$ or $3\,(5+4)$.
$3\,(5+4) > (3-1)\,10$, or $(3 \times 10) - 10$.
$\because 3 \times 10 > (3 \times 5) + (3 \times 4)$, and $10 < 3 \times 4$.
$\therefore (3 \times 10) - 10 > 3 \times 5$, or $3 \times 5 < (3-1)\,10$.
Hence $3\,(5+4) > (3-1)\,10$.
but $3 \times 5 \not> (3-1)\,10$,
$\therefore 5 + 4 : 10 > 5 : 10$.

PROP. IX.—THEOR.

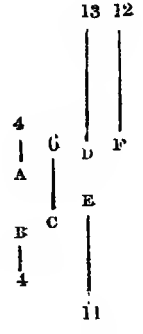
Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

CON. Pst. 1, V. DEM. 8, V.—Def. 7, V.—Def. 5, V.

E. 1	Hyp. 1.	Let $A : C = B : C$;	
2	,, 2.	or $C : A = C : B$;	
3	Conc.	then $A = B$.	

First. Let $A : C = B : C$, then $A = B$.

D. 1	*Sup.*	If $A \neq B$, let $A > B$;	
2	8, V.	then the gr. M, $A : C >$ the less M, $B : C$.	
3	Def. 7, V.	Now equims. of A, B, mA, mB, and a m of C, $(n-1)\,C$, may be taken, so that $mA >$, and $mB \not> (n-1)\,C$;	



4	Pst. 1, V.	Of A & B take equims. D,E; and of C a m, F, so that D $>$ F, but E $\not>$ F;
5	Hyp.	then $\because$ A : C = B : C;
6	D. 4.	and of A & B equims. D, E, and of C a m, F, and D $>$ F;
7	Def. 5, V.	$\therefore$ E $>$ F;
8	D. 4.	But E also $\not>$ F;—two things, of which one is impossible;
9	Conc.	$\therefore$ A not $\neq$ B, *i. e.* A = B.

Second. Let C : A = C : B,—then A = B.

D. 1	*Sup.*	If A $\neq$ B, let A $>$ B.
2	8, V.	then C : B the less M $>$ C : A the gr. M.
3	Def. 7, V.	$\because$ there may be taken a m of C the 1st and 3rd, and equims. of A & B the 2nd and 4th; so that m of C $>$ mB, but $\not>$ m A;
4	Pst. 1, V.	Take F, a m of C, and E, D, equims. of B, A, so that F $>$ E, but $\not>$ D.
5	Hyp.	And $\because$ C : B = C : A,
6	D. 4.	and F, m of C $>$ E, m of B;
7	Def. 5. V.	$\therefore$ F, a m of C $>$ D, a m of A;
8	H. Conc.	but F also $\not>$ D; an impossibility, $\therefore$ A = B.
9	Rec.	$\therefore$ *Magnitudes which have the same ratio, &c.* Q. E. D.

Otherwise. Hyp.—Let A : C = B : C, or C : A = C : B; then A = B; or let 4 : 6 = 4 : 6 and 6 : 4 = 6 : 4, then 4 = 4

CASE I.—If A $\neq$ B, let A $>$ B.
Assume, (8, V.) m A $>$ n C, but m B $\not>$ n C:
$\because$ A : C = B : C, $\therefore$ m A, m B $>$ = or $<$ n C;
but m A $>$ n C, and m B $<$ n C, an absurdity;
$\therefore$ A not $\neq$ B, *i. e.*, A = B.

CASE II.—If C : A = B : A, then A = B.
Invertendo. A : C = B : C; $\therefore$ by Case 1, A = B.

COR.—*A ratio compounded of two ratios, of which one is the reciprocal of the other, is a ratio of equality.*

For in A, B, C, magnitudes of the same kind,—if B : C = B : A, A = C;

i. e., the ratio A : C compounded of A : B and of B : C, one the reciprocal of the other, is a ratio of equality. Note Def. 3, V. Def. A, V.

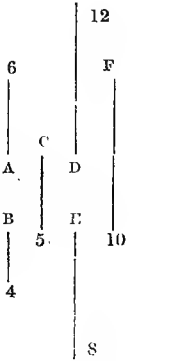
PROP. X.—THEOR.

That magnitude which has a greater ratio than another has unto the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

CON.—Pst. 1, V. DEM. Def. 7, V. AX. 4, V. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

E. 1	Hyp. 1.	Let $A : C > B : C$;
2	Conc. 1.	then $A > B$.
3	Hyp. 2.	Let $C : B > C : A$;
4	Conc. 2.	then $B < A$.

CASE I.—Let $A : C > B : C$, then $A > B$.



D. 1	Hyp. 1.	$\because A : C > B : C$;
2	Def. 7, V.	$\therefore$ of A 1st and B 3rd can be taken some equims., and of C the 2nd and 4th a m so that $m A > m C$ and $m B \not> m C$.
3	Pst. 1, V.	Take D, E, such equims., of A, B, and F a m of C, so that $D > F$ but $E \not> F$;
4		$\therefore D > E$.
5	D. 3.	And $\therefore$ D and E are equims. of A and B,
6	Ax. 4, V.	and if the 1st $<$ the 2nd, the m of the 1st $<$ the same m of the 2nd;
7	D. 4. Conc	and $D > E$; $\therefore A > B$.

CASE II.—Let $C : B > C : A$, then $B < A$.

D. 1	Hyp.	For $\therefore C : B > C : A$;
2		$\therefore$ of C 1st and 3rd can be taken some m, and of B, A, 2nd and 4th equims, so that $n C > m B$, but $\not> m A$.
3	Pst. 1. V.	Of C take such m, F, and of B, A equims. E, D, so that $F > E$ but $\not> D$;
4		$\therefore E < D$.
5	D. 3, 4.	and $\because$ E, D are equims. of B, A, and $E < D$;
6	Ax. 4. V.	$\therefore B < A$.
7	Rec.	Therefore, *that Magnitude which has a greater &c.*

Q. E. D.

Otherwise. Hyp. 1°. Let A : C > B : A, then A > B ; or 6 : 5 > 4 : 5, then 6 > 4.

2°. Let C : B > C : A, then B < A ; or 5 : 4 > 5 : 6, then 4 < 6.

Alg. 1° Def. 7, V. Take m A, > n C, and m B < n C.
∴ m A > m B, and Ax. 4, V. A > B.

2°. Def. 7, V. Take n C > m B, and n C < m A.
∵ m B < n C, and m A > n C ;
∴ m B < m A, and ∴ B < A.

Arith. 1°. 3 × 6 > 3 × 5 ; but 3 × 4 < 3 × 5.
∴ 3 × 6 > 3 × 4, and ∴ 6 > 4.

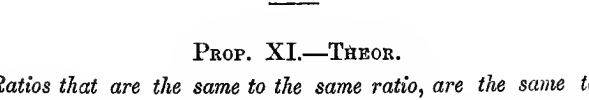
2°. 3 × 5 > 3 × 4, but 3 × 5 < 3 × 6.
∵ 3 × 4 < and 3 × 6 >, 3 × 5.
∴ 3 × 4 < 3 × 6 ; ∴ 4 < 6.

PROP. XI.—THEOR.

Ratios that are the same to the same ratio, are the same to one another.

CON.—Pst. 1, V. DEM. Def. 5, V. Definition of Proportion.

E. 1	Hyp.	Let A : B = C : D, & C : D = E : F;
2	Conc.	then A : B = E : F.



C. 1	Pst. 1, V.	Of A,C,E, take any equims. G,H,K ;
2	,,	and of B,D, E, take any equims. L,M,N.
D. 1	Hyp.	∵ A : B = C ; D,
2	,,	and G,H are equims. of A,C ; L,M equims. of B,D;
3	Def. 5, V.	∴ if G > L, H > M ; if =, = and if less, less.
4	Hyp.	Again, ∵ C : D = E : F,
5	,,	and H, K are equims. of C, E ; M, N, of D, F ;
6	Def. 5, V.	∴ if H > = or < M, K > = or < N.
7	D, 3.	but if G > L, H > M, if =, =, and if less, less;
8		∴ if G > = or > L, K > = or < N.
9	C. 1.	And G, K, are equims. of A, F, and L, N, of B,F;
10	Def. 5, V.	∴ A : B = E : F.
11	Rec.	∵ *Ratios are the same, &c.* Q. E. D

Otherwise. *Hyp.*. If $A : B = C : D$, and $C : D = E : F$; then $A : B = E : F$; or, if $3 : 6 = 1 : 2$; and $1 : 2 = 4 : 8$, then $3 : 6 = 4 : 8$.

Alg. Of antecedents take mA, mC, mE;
and of consequents „ nB, nD, nF.
$\therefore$ A : B = C : D; if m A $>=$or $<$ n B,
$\therefore$ m C $>=$ or $<$ nD.
Again, $\therefore$ C : D = E : F; if m C $>=$ or $<$ n D,
$\therefore$ m E $>=$ or $<$ n F;
Now m A, m E equims. of A, E ; n B, n F, of B, F;
$\therefore$ by Def. 5, V A : B = E : F.

Arith. Take the *ms* $2\times3, 2\times1, 2\times4$;
and $2 \times 6, 2 \times 2, 2 \times 8$;
$\therefore$ $3 : 6 = 1 : 2$, if $2 \times 3 < 2 \times 6$,
$\therefore$ $2\times1 < 2\times2$.
Again, $\therefore$ $1 : 2 = 4 : 8$, if $2 \times 1 < 2 \times 2$,
$\therefore$ $2\times4 < 2\times8$.
Now 2×3, 2×4 are equims; also 2×6 and 2×8;
$\therefore$ $3 : 6 = 4 : 8$.

SCH. This Proposition is to Ratios. what Prop. 30, Bk. 1, is to Parallel lines; Ax. 1, Bk. I, to Magnitudes; and Ax. 1, Bk. V. to Equimultiples.

COR. 1. If $A : B = C : D$, but $C : D >$ or $<$ E : F, then A : B $>$ or $<$ E : F.

For, whatever part of D be contained in C, a greater or less number of times than the like part of F is contained in E, the like part of B must be contained in A the same greater or less number of times.

COR. 2 Thus also, if $A : B >$ or $<$ C : D, and $C : D = E : F$, then A : B $>$ or $<$ E : F.

PROP. XII.—THEOR.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

CON. Pst. 1, V. DEM. Def. 5, V. Definition of Proportion. 1, V. If any no. of *Ms* be equims of as many, each of each, what m soever any one of them is of its *pt.*, the same m shall all the first *Ms* be of all the others.

E. 1	Hyp.	Let A : B = C : D = E : F;
2	Conc.	then A : B = A+C+E : B+D+F.

G — 2	H — 4	K — 6
A — 1	C — 2	E — 3
B — 3	D — 6	F — 9
L — 6	M — 12	N — 18

C. 1	Pst. 1, V.	Of A,C,E, take equims. of G,H,K;
2	,,	and of B,D,F, equims. L,M,N.
D. 1	Hyp.	then ∵ A : B = C : D = E : F,
2	C. 1.	and G,H,K, are equims of A,C,E,
3	C. 2.	and L,M,N, equims. of B,D,F;
4	Def. 5, V.	∴ G > = or < L, H also > = or < M, and K > = or > N;
5		wherefore if G > = or < L,
6		then G + H + K > = or < L + M + N.
7	C. 1.	But G, and G + H + K, are equims. of A, and A + C + E.
8	1, V.	for whatever *m* of A, G is, the same *m* are all, G + H + K of A + C + E;
9	D. 7 & 8.	*i. e.* G, and G + H + K are equims. of A, and A + C + E.
10	*Sim.*	So L, and L + M + N are equims. of B, and B + D + F.
11	Def. 5, V.	∴ A : B = A + C + E : B + D + F.
12	Rec.	Wherefore, *if any number of magnitudes, &c.* Q. E. D.

Otherwise. *Hyp.* A : B = C : D = E : F; then A : B = A + C + E : B + D + F; or 1 : 3 = 2 : 6 = 3 : 9, then 1 : 3 = 1 + 2 + 3 : 3 + 6 + 9, or 1 : 3 = 6 : 18.

Alg. I. Of the Antecs. take *m*A, *m*C, *m*E,
Of the Conseqs. ,, *n*B, *n*D, *n*F;
∵ A : B = C : D, ∴ if *m*A > = or < *n*B, *m*C, > = or < *n*D;
∵ C : D = E : F, ∴ if *m*C > = or < *n*D, *m*E > = or < *n*F:
∴ If *m*A > = or < *n*B, *m*A + *m*C + *m*E > = or < *n*B + *n*D + *n*F.
Now (1, V.) *m*A + *m*C + *m*E = *m*(A + C + E),
so that *m*A, *m*A + *m*C + *m*E are equims. of A, and A + C + E;
and *n*B, *n*B + *n*D, + *n*F are equims. of B, and B + D + F;
∴ A : B = A + C + E : C + D : F.

Arith. Of Antecs. take 3×1, 3×2, 3×3,

Of Conseqs. „ 2×3, 2×6, 2×9;

$\because 1 : 3 = 2 : 6$, $\therefore$ if $3 \times 1 > =$ or $< 2 \times 3$, $3 \times 2 > =$ or $< 2 \times 6$;

and $\therefore 2 : 6 = 3 : 9$, $\therefore$ if $3 \times 2 > =$ or $> 2 \times 6$, $3 \times 3 > =$ or $> 2 \times 9$.

Now $(3 \times 1) + (3 \times 2) + (3 \times 3) = 3(1 + 2 + 3)$,

so that 3×1, $(3 \times 1) + (3 \times 2) + (3 \times 3)$ are equims. of 1 and $(1 + 2 + 3)$

and 2×3, $(2 \times 3) + (2 \times 6) + (2 \times 9)$ equims. of 3 and $(3 + 6 + 9)$;

$\therefore 1 : 3 = (1 + 2 + 3) : (3 + 6 + 9)$, or $1 : 3 = 6 : 18$.

Alg. II. For Antecs. take a, and a, b, c, d, e, &c.

For Conseqs. „ a', and a', b', c', d', e', &c.

By given Hyp. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \frac{e}{e'} =$ &c.

Take product of extremes and means; $a b' = a' b$, $a c' = a' c$, $a d' = a' d$, $a e' = a' e$, &c.

Add Antecs. for numerator, Conseqs. for denominator,

$$\frac{a + b + c + d + e + \text{\&c.}}{a' + b' + c' + d' + e' + \text{\&c.}};$$

Divide by $\frac{a}{a'}$ and we have, $\frac{a a' + a' b + a' c + a' d + a' c \text{ \&c.}}{a a' + a b' + a c' + a' d + a' e \text{ \&c.}}$

Now $\frac{a}{a} = 1$, & $\frac{a a' + a' b + a' c + a' d + a' e \text{ \&c.}}{a a' + a b' + a c' + a d' + a e' \text{ \&c.}} = 1$.

And the *quotients* in each case being *unity*,

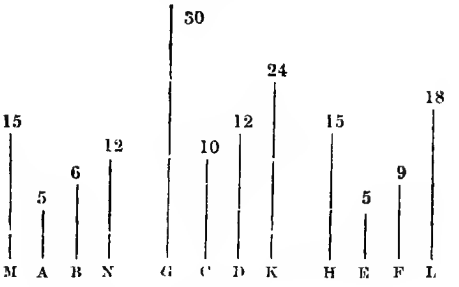
$\therefore a : a' = a + b + c + d + e$ &c. $: a' + b' + c' + d' + e'$ &c.

PROP. XIII.—THEOR.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

CON. Pst. 1, V. DEM. Def. 7, V. Definition of greater and less ratio Def. 5, V. Definition of Proportion.

E. 1	Hyp.	Let A : B = C : D, but C : D > E : F;
	2 Conc.	then A : B > E : F.



C. 1	H.	∵ C : D > E : F,
	2 Def. 7, V.	∴ some equims of C the 1st and E the third, and some of D the 2nd and F the 4th; so that *m* of C > *m* of D. but *m* of E ⩾ *m* of F;
	3 Pst. 1, V.	Take such equims. G & H of C & E, K & L of D & F, so that G > K, but H ⩾ L;
	4 Pst. 1, V.	Whatever *m*, G is of C, take M of A; and whatever *m*, K is of C, take N of B.
D. 1	H&C.3,4.	∵ A : B = C : D, and ∵ M, G, equims. of of A, C; N, K of B, D;
	2 Def. 5, V.	∴ if M > = or < N, G > = or < K.
	3 C. 3.	but G > K ∴ M > N.
	4 C. 3 & 4.	But H ⩾ L, and M, H equims of A, E; — N, L of B, F;
	5 Def, 7, V.	∴ A : B > E : F.
	6 Rec.	Wherefore *if the first has to the second &c.* Q. E. D.

Cor. If A : B > or < B : E, but C : D = E : F, it may also be demonstrated that A : B > or < E F.

Alg. & Arith Hyp. Let A 5 : B 6 = C 10 : D 12; but C 10 : D 12 > E 5 : F 9; then A 5 : B 6 > E 5 : F 9. Let $m = 3$, & $n = 2$.

Alg. ∵ C : D > E : F,
Assume *m* C, *n* D, *m* E, *n* F, so that
if *m* C > *n* D, *m* E < *n* F.

Now $\because$ A : B = C : D,—if m C > n D, m A > nB;
$\therefore$ mA > n B, and mE < nF,
$\therefore$ A : B > E : F.

Arith. $\because$ 10 : 12 > 5 : 9,
Assume 3 × 10 > 2 × 12, but 3 × 5 < 2 × 9.
Now $\because$ 5 : 6 = 10 : 12, if 3 × 10 > 2×12; and 3 × 5 > 2× 6
$\therefore$ 3 × 5 > 2 × 6, and 3 × 5 < 2 × 9;
$\therefore$ 5 : 6 > 5 : 9.

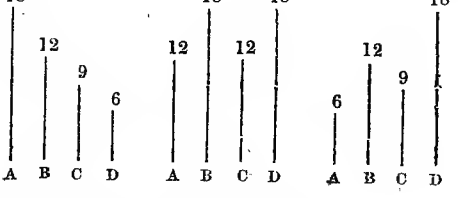
SCH. "This proposition is equivalent to stating; 1°. that if any ratio be greater than another, every ratio which is equal to the former will also be greater than the latter; 2°. Also, that if one ratio be greater than another, every ratio which is greater than the former is also greater than the latter." LARDNER.

PROP. XIV.—THEOR.

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

DEM. 8, V. The gr. M. a gr. ratio. 13, V.—10, V. one M with a gr. ratio to a third. 9, V. Two Ms. each with the same ratio to a third.

E. 1| Hyp. | Let A : B = C : D,
2| Conc. | If A > = or < C, B > = or < D.



CASE I. Let A > C, B will be > D.

D. 1	Hyp.	$\because A : B = C : D$, or $C : D = A : B$;
2	8, V.	and if $A > C$, $A : B > C : B$;
3	13, V.	$\therefore C : B > C : B$.
4	10, V.	But the *M* to which the same has the gr. ratio is the less,
5	Conc.	$\therefore D < B$, *i. e.*, $B > D$.

CASE II. If $A = C$, $B = D$.

D. 1	Hyp. & 9, V.	$\because A : B = C$, *i. e.* $A : D$, $\therefore B = D$.

CASE III. If $A < C$, $B < D$.

D. 1	Hyp.	$\because C > A$, and $C : D = A : B$;
2	Case 1.	$\therefore D > B$, *i. e.* $B < D$.
3	Rec.	Therefore, *if the first has the same ratio, &c.* Q.E.D.

Briefly. 1°. Let $A > C$; then, by 8, V., $A : B > C : B$; but $A : B = C : D$, $\therefore$ by 13, V., $C : D > C : B$, and $\therefore$ by 10, V., $B > D$.

Similarly, 2°. if $A = C$, by 9, V., $B = D$, and 3°. if $A < C$, $B < D$.

COR. Hence also, if $A : B = C : D$, and if the 2nd $B >$ = or $<$ the 4th D, then the 1st $A >$ = or $<$ the 3rd C.

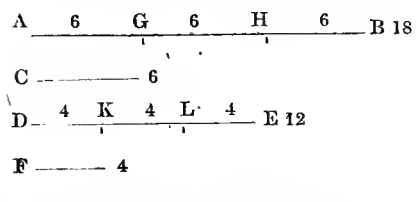
PROP. XV.—THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

CON. Pst. 2, V. Any given *m* of a M may be divided into parts, each of which is equal to that M.

DEM. 7, V. Equal Ms, with the same ratio. 12, V. Antecedents to consequents.

E. 1	Hyp.	Let A B be the same *m* of C, that D E is of F;
2	Conc.	then C : F = A B : D E.



C. 1	Pst. 2, V.	Divide A B into Ms each = C, *i. e.*, A G = G H = H B = C.
2	„ „	& „ D E „ each = F, *i. e.*, D K = K L = L E = F.
D. 1	C.	Now A G, G H, H B in *number* = D K, K L, & L E;
2	C. 1 & 2.	and A G = G H = H B, and D K = K L = L E,
3	7, V.	∴ A G : D K = G H : K L = H B : L E.
4	12, V.	And ∵ A G : D K = A G + G H + H B : D K + K L + L E,
5	C. 1 & 2.	but A G = C, and D K = F;
6	Conc.	∴ C : F = A B : D E.
7	Rec.	Wherefore, *Magnitudes have the same ratio*, &c. Q. E. D.

Otherwise, Take *m* any number, A & B two magnitudes; then A : B = *m* A : *m* B.

D. 1	7, V. & 12, V.	∵ A : B = A : B, ∴ A : B = A + A : B + B, or 2 A = 2 B;
2	D. 1,	and ∵ A : B = 2 A : 2 B,
3	12, V.	∴ A : B = 2 A + A : 2 B + B, or 3 A = 3 B.
4	*Sim.*	And so on for all equims. of A and B;
5	Conc.	∴ A : B = *m* A : *m* B.

Cor. 1. *Magnitudes have the same ratio to one another which their equal submultiples or like parts have.*

thus, $A : B = \frac{A}{3} : \frac{B}{3}$; or, $12 : 18 = \frac{12}{3} : \frac{18}{3}$, *i. e.*, 12 : 18 = 4 : 6.

Cor. 2. In Proportionals the equims. of the 1st and 2nd have the same ratio as the equims. of the 3rd and 4th. If A : B = C : D. then *m* A : *m* B = *n* C : *n* D;

Arith. If $2 : 3 = 4 : 6$, then $2 \times 2 : 2 \times 3 = 3 \times 4 : 3 \times 6$; *i. e.*, $4 : 6 = 12 : 18$.

Cor. 3. In Proportionals also, any like parts of the 1st and 2nd, and also of the 3rd and 4th are proportional; as

$$\frac{A}{2} : \frac{B}{2} = \frac{C}{3} : \frac{D}{3}.$$

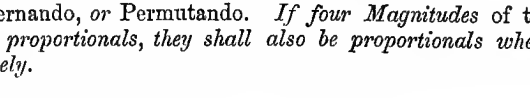
Sch. The Proportion is sometimes explained, "one magnitude shall have the same ratio to another magnitude of the same kind which any multiple of the first has to the same multiple of the second."—Hose.

Prop. XVI.—Theor.

Alternando, *or* Permutando. *If four Magnitudes* of the same kind *be proportionals, they shall also be proportionals when taken alternately.*

Con. Pst. 1, V. Dem. 15, V.—11, V.—14, V.

E. 1	Hyp.	Let A : B = C : D;
2	Conc.	then A : C = B : D.



C.	Pst. 1, V.	Take E, F, equims. of A, B, and G, H, equims. of C, D.
D.1	C.	∵ E same *m* of A, that F is of B,
2	15, V.	∴ A : B = E : F;
3	H. & 11, V.	but ∵ A : B = C : D, ∴ C : D = E : F.
4	C.	Again ∵ G & H are equims. of C & D,
5	15, V.	∵ C : D = G : H.
6	D. 3, & 11, V.	but C : D = E : F, ∴ E : F = G : H.
7	14, V.	In 4 proportionals, if 1st, > = or < 3rd, the 2nd is > = or < 4th;

8		∵ if E > = or < G, F > = or < H;
9	C.	Now E, F, are equims. of A, B; & G, H, of C, D;
10	Def. 5, V.	∴ A : C = B : D.
11	Rec.	*If* then *four magnitudes of the same kind, &c.* Q. E. D.

Or,

C.	Pst. 1, V.	Take m A, m B, n C, n D, equims. of A, B, C, D.
D. 1	15, V. & 11, V,	∵ m A : m B = A : B, ∴ m A : m B = C : D.
2	15, V. & 11, V.	also, ∵ n C : n D = C : D, ∴ m A : m B = nC : nD;
3	14, V.	now m B > = or < n D, as m A > = or < n C;
4	Def. 5, V.	∴ A : C = B : D.

Alg. & Arith. Hyp. Let $\frac{a}{b}$, $\frac{2}{4}$, $= \frac{c}{d}$, $\frac{3}{6}$; then $\frac{a}{c}$, $\frac{2}{3} = \frac{b}{d}$, $\frac{4}{6}$.

Alg. Let $\frac{a}{b} = \frac{c}{d}$, then $a\,d = b\,c$.

(÷ by $c\,d$), and $\frac{ad}{cd} = \frac{bc}{cd}$, or, $\frac{a}{c} = \frac{b}{d}$.

Arith. If $\frac{2}{4} = \frac{3}{6}$, then $2 \times 6 = 4 \times 3$.

(÷ by 3×6,) and $\frac{2 \times 6}{3 \times 6} = \frac{4 \times 3}{3 \times 6}$, or $\frac{2}{4} = \frac{4}{6}$.

USE & APP I. From principles established, especially from Def. 7, V,—and from Props. 7, 11, 14, & 15, bk. V, the following Theorem, *Chambers' Ex. p.* 56, may be demonstrated;

If to the terms of a ratio, A : B, *the same magnitude,* C, *be added, the ratio will be* unchanged, increased, or diminished, *accordiny as it is a ratio* of equality, *of* less inequality, *or of* greater inequality.

CASE I. If A = B, then A + C : B + C = A : B.

D. 1	Ax. 2, I.	∵ A = B, ∴ A + C = B + C;
2	7, V. & 16, V.	∴ A + C : C = B + C : C, and A + C : B + C = C : C.
3	*Sim.* & 11, V.	So A : B = C : C. and ∴ A + C : B + C = A : B.

CASE II. If A < B, then A + C : B + C > A : B.

D. 1.	H.	∵ A < B, if B − A = D; then B = A + D.
2	*Sup.*	Let p C > A, and n D the least m of D exceedin g p A, so that p A < n D;
3	D. 2.	hence p A + A > n D, or = n D;
4	D. 2, 3.	& ∴ p A + p C > n D, or p (A + C) > n D.
5	*Sup.*	Next, let $m = n + p$, or $p = m - n$;
6	D. 2, 5, & 1.	then ∵ p A < n D; $p = m - n$, and D = B − A;
7		∴ $(m-n)$ A < n (B−A); or m A − n A < n B − n A:

8	*Add.*	To both sides add n A, $\therefore m$ A $< n$ B.
9	D. 4.	Again. $\because p$ (A + C) $< n$ D,
10	D. 6.	$\therefore p$ A + p C $> n$ (B−A); or m A−n A + m C−n C $> n$ B−n A.
11	*Add.*	Add n A + n C to each side of the equation;
12	Ax. 4, I.	$\therefore m$ A + m C $> n$ B + n C; or m (A + C) $> n$ (B + C);
13	D. 8. Def. 7, V.	but m A $< n$ B, $\therefore$ A + C : B + C $>$ A : B.

CASE III. If A $>$ B, then A + C : B + C $<$ A : B.

D. 1.	H.	$\because$ A $>$ B, if A−B = D, then A = B + D.
2	*Sup.*	Let p C $>$ B; and n D the least m of D, so that p B $< m$ D;
3	D. 2.	hence p B + B $>$ or $= m$ D;
4	D. 2 & 3,	& $\therefore p$ B + p C $> m$ D, or p (B + C) $> m$ D.
5	*Sup.*	Next, let $n = m + p$; or $p = n - m$;
6	D. 2, 5, & 1.	then $\because p$ B $< m$ D, $p = n - m$, and D = A−B,
7		$\therefore (n-m)$ B $< m$ (A−B); or n B−m B $< m$A−m B;
8	*Add.*	Add m B, $\therefore n$ B $< m$ A; m A $> n$ B.
9	D. 4.	Again $\because p$ (B + C) $> m$ D,
10	D. 6.	$\therefore p$ B + p C $> m$ (A−B); or n B−m B + n C−m C $> m$ A−m B;
11	*Add.*	Add m B + m C to each side of the equation;
12	Ax. 4, I.	$\therefore n$ (B + C) $> m$ (A + C); or m (A + C) $< n$ (B + C).
13	D.8, Def. 7, V.	But m A $> n$ B, $\therefore$ A + C : B + C $<$ A : B.
14	Rec.	Wherefore, *If to the terms of a ratio &c.*

II. Props. 11, 15 & 16, bk. V., also furnish the principles by which to establish the useful Theorem,—*Chambers' Ex. p.* 54;—that,

"If all the terms, or any two homologous terms, or the terms of either of the ratios of a proportion, be multiplied or divided by the same number, the resulting magnitudes will remain proportional.

E. 1 | Hyp. | Let A : B = C : D, and m, n be any two numbers; then,

CASE I. m A : m B = m C : m D;

D. 1	15, V.	$\because$ A : B = mA : mB; and C : D = mC : mD;
2	11, V.	$\therefore m$ A : m B = m C : m D.

CASE II. $\frac{1}{m}$A : $\frac{1}{m}$B = $\frac{1}{m}$C : $\frac{1}{m}$D.

D. 1	H.	$\because$ A & B are mults. of $\frac{1}{m}$A $\times m$, and of $\frac{1}{m}$B $\times m$;
2	15, V.	$\therefore$ A : B = $\frac{1}{m}$A : $\frac{1}{m}$B.
3	*Sim.*	In like manner C : D = $\frac{1}{m}$C : $\frac{1}{m}$D.
4	11, V.	$\therefore \frac{1}{m}$A . $\frac{1}{m}$B = $\frac{1}{m}$C : $\frac{1}{m}$D.

CASE III. $m\,A : B = m\,C : D$.

D. 1	H. & Sch. 3, V.	$\because A : B = C : D$, $\therefore m\,A : B = m\,C : D$;
2	*Sim.*	In like manner, $A : n\,B = C : n\,D$.

CASE IV. $\frac{1}{m}A : B = \frac{1}{m}C : D$.

D. 1	H. & 16, V.	$\because A : B = C : D$, and $A : C = B : D$;
2	Case 2,	$\therefore \frac{1}{m}A : \frac{1}{m}C = B : D$,
3	16, V.	and $\therefore \frac{1}{m}A : B = \frac{1}{m}C : D$.

CASE V. $m\,A : m\,B = C : D$.

D. 1	H. & 15, V.	$\because A : B = C : D$, and $A : B = m\,A : m\,B$;
2	11, V.	$\therefore m\,A : m\,B = C : D$.

CASE VI. $\frac{1}{m}A : \frac{1}{m} = C : D$.

D. 1	H.	$\because A : B = C : D$, & A, B are equims. of $\frac{1}{m}A$, $\frac{1}{m}B$ by m;
2	15, V.	$\therefore A : B = \frac{1}{m}A . \frac{1}{m}B$.
3	11, V.	and $\therefore \frac{1}{m}A : \frac{1}{m}B = C : D$.
4	Rec.	Wherefore, *If all the terms, &c.* Q. E. D.

PROP. XVII.—THEOR.

Dividendo. *If magnitudes taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if two magnitudes together have to one of them, the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.*

N.B. The General Enunciation of this 17th Proposition is variously given;

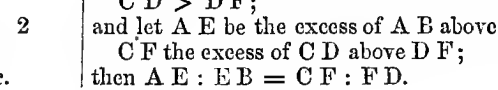
"If magnitudes be proportional, they will also be proportional *by division.*"—EUCLID.

"If four magnitudes, A, B, C, D, be proportionals, they shall also be proportionals, when taken *dividedly*: that is, the difference of the first and second shall be to the second as the difference of the third and fourth to the fourth; or *dividendo*, A ~ B : B = C ~ D : D."—DE MORGAN.

"If four magnitudes be proportional, the first being greater than the second and the third greater than the fourth; then the excess of the first above the second shall be to the second, as the excess of the third above the fourth is to the fourth."—HOSE.

CON. Pst. 1, V. DEM. 1, V.—If any number of *Ms* be equims. of as many, &c., 2, V. If the 1st M be the same *m* of the 2nd, &c. DEF. 5, V. Criterion of the equality of two ratios. Ax. 4, I. If equals be added to unequals the wholes are unequal. Ax. 5, I. If equals be taken from unequals the remainders are unequal.

E 1	Hyp. 1	Let A B, B E, C D, D F, be *Ms* taken jointly and proportionals,
		i. e., A B : B E = C D : D F;—A B > B E and C D > D F;
2	„ 2	and let A E be the excess of A B above B E, and C F the excess of C D above D F;
3	Conc.	then A E : E B = C F : F D.



C. 1	Pst. 1, V.	Of A E, E B, C F, F D take equims. G H, H K, L M, M N;
2	„ „	and of E B, F D any equims. X K, N P.
D. 1	C.	∵ G H, H K are equims. of A E, B E;
2	1, V.	∴ G H same *m* of A E that G K is of A B;
3	C. 1.	But G H same *m* of A E that L M is of C F;
4		∴ G K „ A B „ L M „ C F.
5	C. 1.	Again ∵ L M same *m* of C F that M N is of F D;
6	1, V.	∴ L M „ of C F „ L N „ C D;
7	D. 4.	but L M same *m* of C F that G K is of A B,
8	D. 6 & 7.	∴ G K, L N are equims of A B, C D.
9	C. 1.	Next, ∵ H K, M N are equims of E B, F D;
10	C. 2.	and ∵ K X, N P equims, of E B, F D,
11	2 V.	∴ H K + K X & M N + N P equims. of E B, F D;
		i. e. H X and M P are equims of E B, F D;

12	Hyp.	And ∵ AB : BE = CD : FD;
13	D. 8.	and ∵ GK, LN are equims. of AB, CD,
14	D. 9.	and HX, MP equims. of BE, FD;
15	Def. 5, V.	∴ if GK > = or < HX, then LN > = or < MP;
16	*Sup. Add.* Ax., 5 I.	but if GH > KX, add HK to both, ∴ GK > HX;
17		∴ also LN > MP;
18	*Sub.*	take MN from both; and LM > NP;
19	Def. 5, V.	∴ if GH > = or < KH, LM > = or < NP.
20	C. 1.	But GH, LM are equims. of AE, CF,
21	C. 2.	and KX, NP are equims. of EB, FD;
22	Def. 5, V.	∴ AE : EB = CF : FD.
23	Rec.	Wherefore, *If Magnitudes taken jointly, &c.* Q. E. D.

Otherwise. If A + B : B = C + D : D, then *by division* A , B = C : D;

C. 1	Ps^t. 1, V.	Take *m*A, *n*B, mults. of A, B.
D. 1	*Sup.* 1.	Let *m* A, > *n*B.
2	*Add.* Ax. 4, I,	To both sides add *m*B; ∴ *m* A + *m* B > *m*B + *n*B, or (*m* + *n*) B,
3	H.	but ∵ A + B : B = C + D : D,
4		∴ if *m* (A + B) > (*m* + *n*)B, *m* (C + D) > (*m* + *n*) D;
5		thus *m*C + *m* D > *m*D + *n* D;
6	*Sub.* Ax. 5, I.	from both sides take *m* D, ∴ *m*C > *n*D; *i. e.*, if *m* A > *n* B, *m* C > *n* D.
7	*Sup.* 2. *Sim.*	If *m* A = *n* B, then *m* C = *n* D,
8	*Sup.* 3. *Sim.*	and if *m* A < *n* B, then *m* C < *n*D.
9	Def. 5, V.	∴ A : B = C : D, by division Def. 16, V.
10	Rec.	Wherefore, *if magnitudes, &c.*

Alg. and Arith. Hyp. Take *a*, 8 : *b*, 2 : : *c*, 12 : *d*, 3.

Alg. Here $\frac{a}{b} = \frac{c}{d}$,

Sub 1 from each; $\frac{a}{b} - 1 = \frac{c}{d} - 1$,

i. e. $\frac{a-b}{b} = \frac{c-d}{d}$,

or $a-b : b = c-d : d$.

Arith. $\frac{8}{2} = \frac{12}{3}$,

Sub. 1 $\frac{8}{2} - 1 = \frac{12}{3} - 1$,

i. e. $\frac{8-2}{2} = \frac{12-3}{3}$;

or $8-2 : 2 = 12-3 : 3$, or $6 : 2 = 9 : 3$.

Cor. 1. Convertendo.—*If four Ms*, A,B,C,D be *proportionals* A : B = C : D, *they shall also be proportionals by conversion, i. e.*, the 1st : 1st∼2nd = 3rd : 3rd∼4th,—or A : A∼B = C : C∼D.

For, *invertendo*, Pr. B,V. B : A = D : C;
dividendo, 17, V. B∼A : A = D∼C : C;
invertendo, B, V. A : A∼B = C : C∼D; *or convertendo.*

Cor. 2. *If four Ms of the same kind*, A,B,C,D, *be proportionals*, A : B = C : D, *then the greatest + the least are together greater than the other two.*

D. 1	Sup. 1.	Let A one of the extremes be the greatest; as 12 : 8 = 3 : 2.
2	14, V.	∵ B<A, ∴ D<C; and ∵ C < A, ∴ D < B ∴ D is the least.
3	H. & 17, V.	∴ A : B = C : C, ∴ A—B : B = C—D : D.
4	D. 2,	and ∵ B>D, ∴ A—B > C—D.
5	Add.	To each add B + D; ∴ A+D>C+B; *i. e.* the greatest 12 + the least 2 > the other two 8 + 3.
6	*Sup.* 2.	Let B one of the means be greatest; as 4 : 16 = 3 : 12;
7	Pr. B, V.	*invert.* B : A = D : C.
8	*Sim.*	So, as before, B+C>A+D, *i. e.* the greatest B 16 + the least C 3 > the other two 4 + 12.
9	Rec.	∴ the sum of the greatest and least > the sum of the other two.

N. B.—This Corollary is identical with Prop. 25, Bk. V.

Cor. 3. *In three Proportionals*, A : B : C, as 2 : 4 : 8, or 9 : 6, 4, *the sum of the extremes*, A + C *is greater than twice the mean*, 2 B; *and therefore half the sum*, $\frac{A+C}{2}$ *of the extremes is greater than the mean*, B.

For, if the mean B>C, then B<A; and if B<C, then B>A.
∴ the extremes A & C are the greatest M, and the least.
and as before, in Cor. 2, A + C > B + B, or 2 B;
and ∴ $\frac{A+C}{2}$ > B.

USE & APP. When half the sum of two Ms, $\frac{A + C}{2}$, is as much greater than the one, as it is less than the other, *that half sum* is an *arithmetical mean* between the two; thus, in $\frac{2 + 8}{2} = 5$, 8—5 = 3, & 5—2=3; and 2 : 5 : 8 are in arithmetical progression. Except when the magnitudes are equal, the arithmetical mean between two magnitudes, A and C, is therefore greater than the geometrical mean, *i. e*, the arith. mean, $\frac{A + C}{2} > B$ the geom. mean;

or $\frac{8 + 2}{2}$ *i. e.*, $5 > 4$. *Geom. Plane, Sol. & Sph. pp* 41 *&* 42.

PROP. XVIII.—THEOR.

Componendo. *If magnitudes taken separately, be proportionals, they shall also be proportionals when taken jointly,* by composition; *that is, if the first be to the second, as the third is to the fourth, the first and second together shall be to the second, as the third and fourth together to the fourth.*

"If magnitudes be proportional, they will also be proportional *by composition.*"—EUCLID.

"The terms of an analogy are proportional *by composition.*"—BELL.

CON. Pst. 1, V. DEM. Def. 5, V. Criterion of the equality of ratios.

Ax. 3, V. A *m* of a gr. M is gr. than the same *m* of a less.

5, V. If one M be the same *m* of another which a M taken from the 1st is of a M taken from the other; the rem. is the same *m* of the rem. that the whole is of the whole.

6, V. If two Ms be equims. of two others, and if equims. of these be taken from the first two; the rems. either = these others, or are equims.

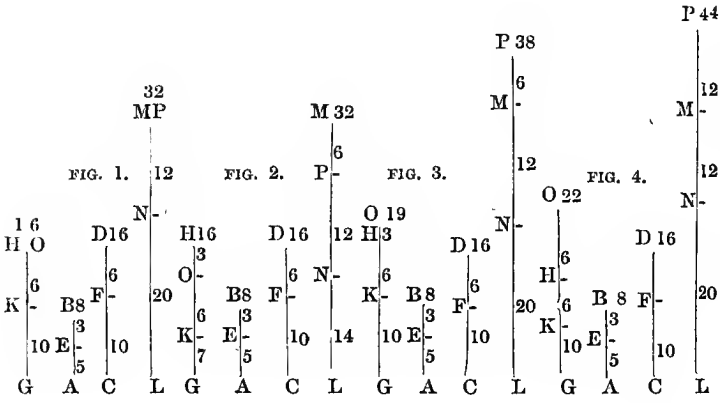
Cor. 4, V. If 1st : 2nd = 3rd : 4th, then any equims. of 1st and 3rd the same ratio to 2nd and 4th; and 1st : 3rd = equim. 2nd : equim 4th.

Pr. A, V. If 1st : 2nd = 3rd : 4th; then if 1st > 2nd, 3rd > 4th, and if =, = ; if <, <.

Ax. 2, V. Those Ms of which the same or equal Ms are equims. are equal.

Ax. 4, V. That M of which a *m* is greater than the same *m* of another is gr. than that other M.

E. 1	Hyp.	Let AE : EB = CF : FD.
2	Conc.	then AE + EB = : BE = CF + FD : DF; *i. e.*, AB : BE = CD : DF.
C. 1	Pst. 1, V.	Of AB, BE, CD, DF take any equims. GH, HK, LM, MN;
2	,,	and of BE, DF any equims. KO, NP.



D. 1	C. 2.	∵ KO, NP are equims of BE, DF, & KH, NM also equims of BE. DF;
2	Def.. 5, V.	∴ if KO > = or < KH, then NP > = or > NM.

CASE I. Let KO ⋗ KH, & ∴ NP ⋗ NM. *Fig.* 1 & 2.

D. 1	C. 1.	∵ GH, HK equims of AB, BE, AB being gr. than BE;
2	Ax. 3, V.	∴ GH > HK,
3	H. Ax. 3, V.	but KO ⋗ KH, ∴ GH > KO.
4	*Sim.*	In the same manner LM > NP;
5	H.	∵ KO ⋗ KH, ∴ GH, a *m* of AB always > KO the same *m* of BE;
		∴ also LM the *m* of CD > NP the *m* of DF.

CASE II. Let KO > KH, and ∴ NP > NM. *Fig.* 3.

D. 1	C. 1.	And ∵ the whole GH is the same *m* of the whole AB as HK of BE,
2	5, V.	∴ rem. GK same *m* of rem. AE as GH of AB, or as LM of CD.
3	*Sim.*	∵ LM the same *m* of CD, as MN of DF,
4	5, V.	∴ rem. LN same *m* of rem. CF as the whole LM of the whole CD.
5	D. 2.	But LM same *m* of CD, as GK of AE;
6	6, V.	∴ GK same *m* of AE, as LN of CF, *i. e.* GK and LN are equims. of AE & CF.
7	C. 2.	And ∵ KO, NP are equims. of BE, DF;
8	Pst. 1, V.	& ∵ from KO, NP may be taken KH, NM also equims. of BE, DF,
9	6, 5.	∴ rems. HO, MP either = rems. BE, DF, or are equims. of BE, DF.

SUBDIVISION 1 Let HO, MP = BE, DF. *Fig.* 3.

D. 1	H & Case 2, D. 6.	∵ AE : EB = CF : FD, & GK, LN are equims. of AE, CF;
2	Cor. 4, V.	∵ GK : EB = LN : FD;
3	H.	But HO = EB & MP = DF, ∴ GK : HO = LN : MP;
4	Pr. A, V.	∴ if GK > = or < HO, LN > = or < MP.

SUBD. 2. Let HO, MP be equims. of EB, FD. *Fig.* 4.

D. 1	H. & Case 2, D. 6.	∵ AE : EB = CF : FD, and of AE, CF, the equims are GK, LN,
2	H.	& of EB, FD the equims are HO, MP;
3	Def. 5, V.	∴ if GK > = or < HO, LN > = or > MP;
4	*Sub.* & Ax. 5, I.	but if GH > KO, from each taking KH, then GK > HO;

5		∴ also LN > MP;
6	*Add.* Ax. 4, I.	to both add NM ∴ LM > NP;
7		∴ if GH > KO, LM > NP.
8	*Sim.*	So, if GH = or < KO, LM = or < NP;
9	Case I. D. 3, 4.	And when KO ⩾ KH, then GH > KO & LM > NP;
10	C.	but GH, LM equims of AB, CD, & KO, NP of BE, DE;
11	Def. 5, V.	∴ A B : BE = CD : DF, *i. e.*, AE + EB : BE = CF + FD : DF,
12	Rec.	Wherefore, *if magnitudes taken separately &c.* Q. E. D.

Otherwise. If A : B = C : D ;— and of B, D any like pts, as the *nth*, be contained in A. C, *m* times exactly, or with like remainders ; *i. e.* if *m* B = A, & *m* D = C,— then those parts will be contained in A + B & C + D, the same number of times exactly, *m* + *n* with the same remainders ;
∴ by Def. 5,V. A + B : B = C + D : D.

Cor, A : A + B = C : C + D.

By *indirect* Proof.

E. 1	Hyp.	If A : B = C : D,
2	Conc.	then A + B : B = C + D : D.
D. 1	*Sup.*	If C + D : D ≠ A + B : B,
2		let *c* + *d* : *d* = A + B : B, and *d* ≠ D.
3	H. 17, V.	∵ A + B : B = C + *d* : *d*, ∴ A : B = C : *d*.
4	H. 11, V.	but A : B = C : D, ∴ C : D = C : *d*.
5	9, V.	∴ D = *d*, contrary to the hypothesis ;
6	Conc.	∴ A + B : B = C + D : D.

Alg. & Arith. Hyp. a 5 : b 3 = c 10 : d 6.

Alg. Let $a : b = c : d$,

$$(+1) \quad \frac{a}{b} + 1 = \frac{c}{d} + 1.$$

$$i.\,e.\ \frac{a+b}{b} = \frac{c+d}{d},$$

or, $a + b : b = c + d : d$.

Arith. Let 5 : 3 = 10 : 6,

$$(+1) \quad \frac{5}{3} + 1 = \frac{10}{6} + 1,$$

$$i.\,e.\ \frac{5+3}{3} = \frac{10+6}{6};$$

or, 8 : 3 = 16 : 6.

USE & APPL. This method of reasoning is often employed. By an extension of it, as indicated in the Corollary, we find that the terms of a proportion are also proportional by addition. For, Let A : B = C : D,—then, *addendo*, A : A + B = C : C + D;

Invertendo, B : A = D : C; *Componendo*, 18, V. A+B : A = C+D : C;
∴ *invert.* (A, V.) A : A + B = C : C + D.

PROP. XIX.—THEOR.

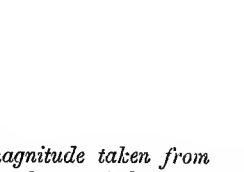
If a whole magnitude be to a whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder shall be to the remainder as the whole to the whole.

"If a whole be to a whole as a part taken away to a part taken away; the part left is to the part left, as the whole is to the whole." EUCLID.

"If four magnitudes, which are all of the same kind, be proportional, the first being greater than the third, and the second than the fourth; then the excess of the first above the third, shall be to that of the second above the fourth as the second is to the first."—HOSE.

DEM. 16, V. *Alternando*;— 17; V. *Dividendo*; 11, V. Ratios the same to the same ratio are the same to one another.

E. 1	Hyp.	Let the whole AB : the whole CD = a pt. AE from AB : CE a pt. from CD;
2	Conc.	then rem. EB : rem. FD = the whole AB : the whole CD.
D. 1	H. & 16, V.	∵ AB : CD = AE : CF, ∴ *alt.* BA : AE = DC : CF;
2	17, V.	& ∵ Ms jointly are propl.,— separately they are also;
3	16, V.	∴ BE : EA = DF : FC, & *alt.* BE : DF = EA : FC;
4	H.	but A E : CF = AB : CD;
5	11, V.	∴ rem. BE : rem. DF = integer AB : integer CD.
6	Rec	∴ *If a whole magnitude, &c.* Q. E. D.



Otherwise,

D. 1	Hyp.	$\because$ A : B = C : D, C being < A & D < B ;
2	16, V. & 17, V.	$\therefore$ *alt.* A : C = B : D, & *div.* A − C : C = B − D : D.
3	16, V.	Again *alt.* A − C : B − D = C : D.
4	Conc.	but A . B = C : D, $\therefore$ A − C : B − D = A : B.

Cor. 1. *Also the remainder shall be to the remainder as the magnitude taken from the first to that taken from the other.*

Or. " the excess of the first above tho third to that of the second above the fourth, as the third to the fourth.—Hose.

$$\because \quad A-C : B-D = A : B; \text{ and } C : D = A : B;$$
$$\therefore A-C : B-D = C : D.$$

Cor. 2. *If any magnitudes,* A, B, C, D &c. *are in Geometrical Progression,* i.e., A : B : C : D &c., *the differences,* A ~ B, : ~ B : C CD &c,— *will form a geom. progression,* A ~ B : B ~ C : C ~ D &c,—*the successive terms of which have the same ratio with the successive terms of the former.*

D. 1	H.	$\because$ B : C = A : B, & C : D = B : C &c.
2	19, V.	$\therefore$ A ~ B : B ~ C = A : B,
		& B ~ C : C ~ D = B : C,
		i. e., as A to B, *i. e.* as A ~ B to B ~ C; and so on.

Cor. 3. *And conversely, any number of Magnitudes,,* A, B, C, D &c. *in geometrical progression,* A : B : C : D &c., *may be considered as the differences of other magnitudes* A, B′ C′ D′ &c., *forming a geometrical progression in which the first term* A′ *is to* A as A to A ~ B, A′ *to* B′ *as* A *to* B &c., *and the successive terms have the same ratio with the successive terms of the former.*

For let a progression be taken in which A′ : A = A : A ~ B, and A′ : B′ = A : B.

D. 1	H. Cor. 1, 17, V.	Then $\because A' : B' = A : B$, $\therefore$ *convert.* $A' : A' \sim B' = A : A \sim B$;
2	H. 9, V.	but $A' : A = A : A \sim B$, $\therefore A' : A' \sim B' = A' : A$.
3	D. 2. 14, V.	and $\because$ the 1st A' is the same with the third A', $\therefore A' \sim B' = A$.
4	Cor. 2, 19, V.	but $A' \sim B'$, $B' \sim C'$, $C' \sim D'$ &c., form a progression in which $A' \sim B' : B' \sim C' = A' : B'$, *i. e.* $= A : B$;
5	14, V.	$\therefore B' \sim C' = B$, $C' \sim D' = C$ &c.
6	Remk.	But the progressions A, B, C, D &c. and $A' \sim B'$, $B' \sim C'$, $C' \sim D'$, $D' \sim E'$ &c. have the same first terms and the same common ratio.
7	Conc.	$\therefore$ those progressions cannot but be identical. *Geom. Pl. Sol. & Sph. p.* 42, 53.

Alg. & Arith. Hyp. Let $a = 15$, $b = 10$ be two Ms; $k = 6$ & $y = 4$ the respective parts.

Alg. By Hyp. $\frac{a}{b} = \frac{x}{y}$;

$\therefore$ $a\,y = b\,x$.

we have to prove that $\frac{a-x}{b-y} = \frac{a}{b}$;

clearing fractions, $ab - bx = ab - by$;

subtract ab, $\therefore bx = ay$ as before.

Because of this identity of products the quantities must be in proportion;

$\therefore$ $a-x : b-y = a : b$.

Cor. Let $\frac{a}{b} = \frac{x}{y}$, $\therefore a\,y = b\,x$ (1).

To prove that $\frac{a}{a-b} = \frac{x}{x-y}$;

clearing fractions $a\,x - a\,y = a\,x - b\,x$;

Subtract $a\,x$, $a\,y = b\,x$, an identity with (1);

$\therefore$ $a : a-b = x : x-y$.

Arith. $\frac{15}{10} = \frac{6}{4}$;

$\therefore$ $15 \times 4 = 10 \times 6$.

to be proved $\frac{15-6}{10-4} = \frac{15}{10}$,

clearing, $(15 \times 10) - (6 \times 10) = (10 \times 15) - (4 \times 15)$,
take away 15×10 $\therefore$ $6 \times 10 = 4 \times 15$.
Now 6×10 & 4×15 give identical products.
$\therefore$ $15 - 6 : 10 - 4 = 15 : 10$.
i. e., $9 : 6 = 15 : 10$.

Let $\frac{15}{10} = \frac{6}{4}$, $\therefore 15 \times 4 = 10 \times 6$ (1).

To prove that $\frac{15}{15-10} = \frac{6}{6-4}$;
clearing, $(16 \times 6) - (15 \times 4) = (15 \times 6) - (10 \times 6)$;
Subtract 15×6, and $15 \times 4 = 10 \times 6$, an identity;
$\therefore$ $15 : 15-10 = 6 : 6-4$. *i. e.* $15 : 5 = 6 : 2$.

Prop. E.—Theor.

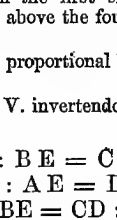
Convertendo. *If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.*

"If four magnitudes be proportional, the first being greater than the second, and the third than the fourth; then the first shall be to its excess above the second, as the third is to its excess above the fourth."—Hose.

"The terms of an analogy are proportional by conversion."—Bell.

Dem. 17, V. dividendo; B, V. invertendo; 18, V. componendo.

E. 1	Hyp.	Let A B : B E = C D : D F;	
2	Conc.	then B A : A E = D C : C F.	
D. 1	H.	$\because$ AB : BE = CD : EF, AB being > BE, & CD > EF,	
2	17, V.	$\therefore$ *div.* AE : EB = CF : FD.	
3	Pr. B, V.	and *invert* BE : EA = DF : CF,	
4	18, V.	wherefore *comp.* BE + AE : AE = FD + CF : CF, *i. e.* BA : AE = DC : CF.	
5	Rec.	*If therefore four magnitudes, &c,.*	Q. E. D.



Or, If $A : B = C : D$, by conversion, $A : A - B = C : C - D$.

D. 1	H. & 17, V.	$\because A : B = C : D$; $\therefore$ *div.* $A - B : B = C - D : D$.
2	Pr. B, V.	and *inv.* $B : A - B = D : C - D$;
3	18, V.	$\therefore$ *comp.* $A : A - B = C : C - D$.

Alg. & Arith. Hyp. Let $a\ 12 : b\ 9 = c\ 8 : d\ 6$, then $a\ 12 \cdot a \pm b\ 12 \pm 9 = c\ 8 : c + d, 8 \pm 6$.

Alg. $\because \frac{a}{b} = \frac{c}{d}$, and by comp. & divid. $\frac{a \pm b}{d} = \frac{c \pm d}{d}$;

by inverting the fractions, $\frac{b}{a \pm b} = \frac{d}{c \pm d}$;

multiply one side by $\frac{a}{b}$, the other by $\frac{c}{d}$;

then, $\frac{a\,b}{ab + b2} = \frac{a}{a \pm b}$ & $\frac{c\,d}{cd \pm d^2} = \frac{c}{c + d}$;

$\therefore a : a \pm b = c : c \pm d$.

Arith. $\because \frac{12}{9} = \frac{8}{6}$; comp. and div. $\frac{12 + 9}{9} = \frac{8 \pm 6}{6}$,

invert. $\frac{9}{12 \pm 9} \pm \frac{6}{8 \pm 6}$,

multiply one side by $\frac{12}{9}$, the other by $\frac{8}{6}$,

then $\frac{12}{12 \pm 9} = \frac{8}{8 \pm 6}$;

$\therefore \quad 12 : 12 \pm 9 = 8 : 8 \pm 6$.

USE & APP. Among other results, Prop. E., bk. V., leads to the following

1. *If any number of Ms be in continued proportion,* as $A : B : C : D : E$, *the difference between the first and second terms,* $A \sim B$, *is to the first* A, *as the difference between the first and last,* $A \sim E$, *is to the sum of all the terms, except the last,* $A + B + C + D$.

D. 1.	H. & Def. 10, V.	$\because A : B = B : C$; $B : C = C : D$ and $C : D = D : E$.
2	12, V.	$\therefore A : B = A + B + C + D : B + C + D + E$.
3	Pr. E, 5.	$\therefore$ *Conv.* $A : A \sim B = A + B + C + D : A \sim E$; B being $>$ A and D $>$ C;
4		And $\because (A + B + C + D) - (B + C + D + E) = A \sim E$,
5	P. B. V.	$\therefore A \sim B : A = A \sim E : A + B + C + D$.

2. *In a series of continued proportionals,* A : B : C : D : E, &c., *the differences of the successive terms* A ~ B, B ~ C, C ~ D, &c., *are also in continued proportion,*— A ~ B : B ~ C : C ~ D, and B ~ C : C ~ D = C ~ D : D ~ E.

CASE 1. *Let* A > B,—*the series is continually decreasing*; then A — B : B — C = B — C : C — D, and B — C : C — D = C — D : D — E.

D. 1.	H. 17, V.	∵ A : B = B : C, ∴ *div.* A — B : B = B — C : C;
2	16, V.	and *altern.* A — B : B — C = B : C,
3	*Sim.*	So, ∵ B : C = C : D, ∴ B—C : C—D = C : D.
4	H. 11, V.	but B : C = C : D, ∴ A — B : B — C = B — C : C — D.
5	*Sim.*	So B — C : C — D = C — D : D — E;
6	Rec.	∴ A — B : B — C : C — D : D — E, are continued proportionals.

CASE II. If A < B, *the series is continually increasing.*

By a like method this case is also proved.

3- *In an infinitely decreasing series of Magnitudes in continued proportion, the first term,* A, *is a mean proportional between its excess above the second,* B, *and the sum of the series.*

Let A, B, denote the 1st and 2nd terms,— Z the last term, and S the sum of the Series, then, as in the last example but one, Use 1, E, V., A—B : A=Z : S—Z. But the last term Z may be less than any magnitude or quantity we fix on, however small; and hence to the values of A—Z and S—Z there will be limits namely A and S.

∴ A—B : A = A : S.

PROP XX—THEOR..

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; then if the first be greater than the third, the fourth shall be greater than the sixth, and if equal, equal; and if less, less.

DEM. 8, V. The gr. M a gr. ratio.

13, V. If 1st : 2nd = 3rd : 4th, but 3rd : 4th > 5th : 6th, the 1st : 2nd > 5th : 6th.

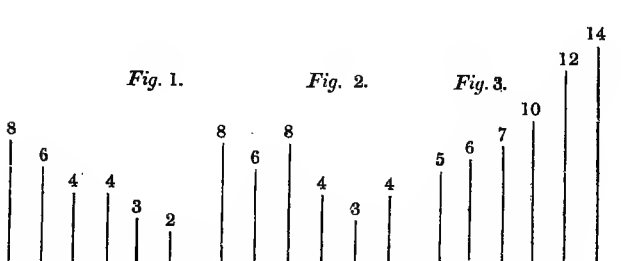
Cor. 13, V. If 1st : 2nd > 3rd : 4th, but 3rd : 4th = 5th : 6th, the 1st : 2nd > 5th : 6th.

10, V. The M with gr. ratio the gr. of two Ms.

7, V. Equal Ms. the same ratio to the same M &c.

11, V. Ratios the same to the same r. the same to one another.
Pr. B, V. *Invertendo.* 9, V. Ms. with the same ratio equal &c.

E. 1	Hyp. 1.	Let there be three Ms. A, B, C, & three other Ms. D, E, F ;
2	Hyp. 2.	let A : B = D : E, & B : C = E : F ;
3	Conc.	if A > = or < C, then D > = or < F.



Case I. *Let* A > C, *then* D > F. *Fig.* 1.

D. 1	H.	∵ A > C, & B is another M;	A. B. C. D. E. F.
2	8, V.	∴ A : B > C : B.	
3	H & 13, V.	but D : E = A : B, ∴ D : E > C : B.	
4	H & B, V.	And ∵ B : C = E : F, ∴ *inv.* C : B = F : E;	
5	D. 3, Cor. 13, V.	and ∵ D : E > C : B, ∴ D : E > F : E;	
6	10, V.	∴ D > F.	

Case II. *Let* A = C, *then* D = F. *Fig.* 2.

D. 1	H. & 7, V.	∵ A = C, ∴ A : B = C : B;
2	H.	but A : B = D : E, & C : B = F : E;
3	11, V. B, V. 9, V.	∴ D : E = F : E, & ∴ D = F.

CASE III. *Let* A < C, then *then* D < F. *Fig.* 3.

D. 1	H. & Case 1.	∵ C > A, and C : B = F : E, & B : A = E : D;
2	Case 1.	∴ F > D, *i. e.*, D < F.
3	Rec.	Therefore, *if there be three magnitudes*, &c. Q. E. D.

Arith. Hyp. Let the Magnitudes be represented by numbers, *Case* 1. by 8, 6, 4, 4, 3. 2 ; *Case* 2. by 8, 6, 8, 4, 3, 4 ; and *Case* 3. by 5, 6, 7, 10, 12, 14.

Case 1. If $\frac{8}{6} = \frac{4}{3}$, and $\frac{6}{4} = \frac{3}{2}$; then the 1st, 8 > 3rd, 4, & 5th, 4 > 6th 2

Case 2. If $\frac{8}{6} = \frac{4}{3}$, and $\frac{6}{8} = \frac{3}{4}$; then the 1st, 8=3rd 8, & 4th 4=6th 4.

Case 3. If $\frac{5}{6} = \frac{10}{12}$, and $\frac{6}{7} = \frac{12}{14}$; then the 1st 5 < 3rd, 7, & 4th 10 < 6th 14.

SCH. The Proposition is also enunciated, with reference to the formula annexed, "If the first magnitude, A, be to the second B, as the third, C, is to the fourth, D ; and, if the second B, be to the fifth E, as the fourth, D, is to the sixth, F ; then the third, C, shall be greater than, equal to, or less than the sixth, F, according as the first, A, is greater than, equal to, or less than the fifth E."—HOSE.

A.	C.
B.	D.
E.	F.

There are of course *three* Cases, A > E, A = E, and A < E, to he proved as in the foregoing proposition from SIMSON'S text.

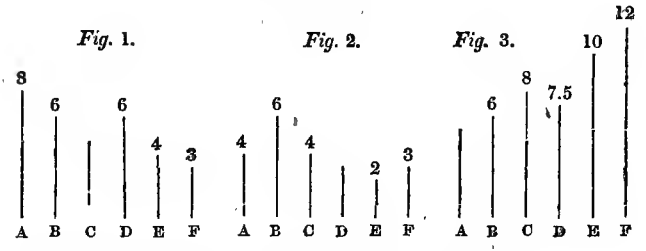
PROP. XXI.—THEOR.

If there be three magnitudes and other three which have the same ratio taken two and two, but in a cross order, i. e. in proportione perturbatâ, *in disturbed proportion ; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth* ; *and if equal, equal ; and if less, less.*

DEM. 8, V. 13, V. Pr. B, V. *invertendo* ; Cor. 13, V. 10, V. 7, V. 11, V. and 9, V.

E.			
1	Hyp. 1	Let there be two series of Ms, A, B, C and D, E, F:	
2	2.	and let A : B = E : F, and B : C = D : E;	
3	Conc.	then, if A >, =, or < C, D >, =, or < F.	

A.	B.	C.
D.	E.	F.



CASE. I. *Let* A > C, *then* D > F. *Fig.* 1.

D.		
1	H. & 8, V.	∴ A > C, and B another M, ∴ A : B > C : B;
2	H. & 13, V.	but E : F = A : B, ∴ E : F > C : B;
3	H. Pr. B, V.	& ∵ B : C = D : E, ∴ *inv.* C : B = E : D;
4	D. 2. Cor. 13, V.	& ∵ E : F > C : B, ∴ E : F > E : D;
5	10, V.	∴ F < D, *i.e.*, D > F.

CASE II. *Let* A = C, *then* D = F. *Fig.* 2.

D.		
1	H & 7, V.	∵ A = C, ∴ A : B = C : B;
2	H. D. 3, Case I.	but A : B = E : F, & C : B = E : D,
3	11, V.	∴ E : F = E : D;
4	9, V.	and ∴ D = F.

CASE III. *Let* A < C, *then* D < F. *Fig.* 3.

D.		
1	H & D. 3, Case I.	∵ C > A, ∴ C : B = E : D,
2		and B : A = F : E;
3	Case I.	∴ F > D, *i. e.* D < F.
4	Rec.	Therefore, *if there be three magnitudes, &c.* Q. E. D.

Arith. Hyp. Let the two series of three magnitudes each be represented by numbers,—*Case* 1. by 8, 6, 4, 6. 4, 3; *Case* 2. by 4, 6, 4, 3, 2, 3: and *Case* 3. by 5, 6, 8, 7.5, 10, 12.

Case 1. If $\frac{8}{6} = \frac{4}{3}$, & $\frac{6}{4} = \frac{6}{4}$; then if 1st 8 > 3rd 4, the 4th 6 > 6th, 3.

Case 2. If $\frac{4}{6} = \frac{2}{3}$ & $\frac{6}{4} = \frac{3}{2}$; then if 1st 4 = 3rd 4, the 4th 3 = 6th, 3.

Case 3. If $\frac{5}{6} = \frac{10}{12}$, & $\frac{6}{8} = \frac{7.5}{10}$; then if 1st 5 < 3rd 8, the 4th 7.5 < 6th 8.

Sch. The following is a variation of Prop. 21, having reference to the annexed formula; "If the first magnitude, A, be to the second B, as the third C is to the fourth D; and if the second, B, be to the fifth, E; as the sixth, F, to the third, C; then the sixth, F, shall be greater than, equal to, or less than the fourth, D, according as the first, A, is greater than, eqnal to, or less than the fifth E."—Hose.

A.	C.
B.	D.
E.	F.

Again, there are *three* Cases, A > E, A = E, & A < E; and the demonstration depends on the same principles as those employed in Simson's Text.

Prop. XXII.—Theor.

Ex æquali, *or* ex æquo, by equality. *If there be any number of magnitudes and as many others, which taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first has to the last of the others.*

'If there be any number of magnitudes, and as many others, which taken two and two, have the same ratio, they shall have the same ratio by equality."—Euclid.

Con. Pst. 1, V. Dem. 4, V. In an analogy equims. of 1st and 3rd have the same ratio to any equims. of 2nd and 4th.
20, V. In the two series of 3 Ms, which taken two and two have the same ratio, of the 1st > = or < 3rd, the 4th > = or < 6th.
Def. 5, V. Criterion of the equality of ratios.

CASE I. *Let there be two series of three Ms, A, B, C, and D, E, F.*

E. 1 | Hyp. | Let A : B = D : E, & B : C = E : F;
2 | Conc. | then A : C = D : F.

16 ——— G 8 ——— A D — 4 H ——— 8
12 ——— K 6 ——— B E — 3 L ——— 6
8 ——— M 4 — C F — 2 N — 4

C. 1. | Pst. 1, V. | Of A,D take equims G,H; of B,E equims K,L; and of C,F, equims. M,N.
D. 1 | H & C. | ∵ A : B = D : E; & G,H equims of A,D, and K,L, equims of B,E;
2 | 4, V | ∴ G : K = H : L; & K : M = L : N;
3 | D. 2. | and ∵ G ,K,M are 3 mags.. & H,L,N, 3 others, which taken two and two have the same ratio;
4 | 20, V. | ∴ if G >, = or < M, H >, =, or < N.
5 | C. | But G,H are equims. of A,D; M,N equims of C,F;
7 | Def. 5, V. | ∴ A : C = D : F.

CASE II, *Let there be two series of four Ms each*, A,B,C,D *and* E,F,G,H.

E 1. | Hyp. | Let A : B = E : F; B : C = F : G, and C : D = G : H;
2 | Conc. | then A : D = E : H.

A. 12 — B. 6 — C. 18 — D. 36.
E. 6 — F. 3 — G 9. — H. 18.
Then A 12 : D 36 = E 6 : H 18.

D. 1 | H. | ∵ A, B, C, are 3 Ms, and E, F, G, three others, which taken two and two have the same ratio;
2 | Case 1, | ∴ A : C = E : G;
3 | H. & Case 1, | but C : D = G : H, ∴ A : D = E : H.
4 | *Sim*, | And in a similar way, whatever be the number of Ms.
5 | Rec. | Therefore, *if there be any number &c.*

Q. E. D.

Alg. & Arith. Hyp. Take the two series of quantities, a 12, b 6, c 18, and a' 6, b' 3, c' 9.

Alg.

By Hyp. $\frac{a}{b} = \frac{a'}{b'}$, and $\frac{b}{c} = \frac{b'}{c'}$;

Multiply. $\frac{a}{b} \times \frac{b}{c}$, and $\frac{a'}{b'} \times \frac{b'}{c'}$;

We have $\frac{a\,b}{b\,c}$, or $\frac{a}{c} = \frac{a'\,b'}{b'\,c'}$, or $\frac{a'}{c'}$;

i. e. $a : c = a' : c'$.

Arith.

By H. $\frac{12}{6} = \frac{6}{3}$, & $\frac{6}{18} = \frac{3}{9}$;

Mult. $\frac{12}{6} \times \frac{6}{18}$, & $\frac{6}{3} \times \frac{3}{9}$;

We have $\frac{72}{108}$, or $\frac{12}{18}$, & $\frac{18}{27}$, or $\frac{6}{9}$.

i. e. $12 : 18 = 6 : 9$.

Cor. Whatever be the number of Analogies,— for instance, three, *i. e.* A : B = C : D;, B : E = D : F, and E : G = F : H; if they are so constituted that the second and fourth terms of each, as B & D, form respectively the first and third of the next, as B & D; then A, the first term of the next proportion shall be to the second of the last, as the third C of the first proportion to the fourth of the last.

A2 : B3	= C4 : D6
B3 : E5	= D6 : F10
E5 : G4	= F10 : H8
G4 : I12	= H8 : K24
I12 : L9	= K24 : M18
A2 : L9	= C4 : M18

In other words,—"*Ratios compounded of any number of equal ratios in the same order, are equal to one another.*"

For, by Hyp. ∵ A : B = C : D, and B : E = D : F,

∴ A : E = C : F.

And again, ∵ A : E = C : F, and E : G = F : H;

∴ A : G = C : H.

And so on, whatever be the number of proportions.

SCH. The 22nd Prop. may be thus varied ; "If the first Mag. A, be to the second B, as the third C, is to the fourth D ; and if the second Mag. B, be to the fifth E, as the fourth D is to the sixth F ; then the first A shall be to the fifth E, as the third C, is to the sixth F."—HOSE.

A similar process of reasoning is to be followed. as in Case 1 and 2, Pr. 32

G.	A.	C.	H.
K.	B.	D.	L.
M.	E.	F.	N.

USE AND APP. By combining the principles contained in Props. 18, 17 and 22, *i. e. componendo.* and *ex æquo*, we arrive at the further truth, that

Proportionals remain proportional miscendo, *by mixing*, or as it is sometimes named,—by using the sum and difference.

E. 1	Hyp.	Let $A : B = C : D$;
2	Conc.	then $A + B : A \sim B = C + D : C \sim D$;
		adopting $A - B$, or $B - A$ and $C - D$ or $D - C$,
		as $B <$ or $>$ A, and $C <$ or $> D$.
D. 1	18, V.	*Comp.* $A + B : B = C + D : D$;
2	17, V.	*Div.* $A - B : B = C - D : D$;
3	Pr. B, V.	*Inv.* $B : A - B = D : C - D$;
4	D. 1.	but $A + B : B = C + D : D$;
5	22, V.	*Ex æquo* $A + B : A - B = C + D : C - D$.

PROP. XXIII.—THEOR.

Ex æquo perturbato. *If there be any number of magnitudes, and as many others, which taken two and two in a cross order, have the same ratio ; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others.*

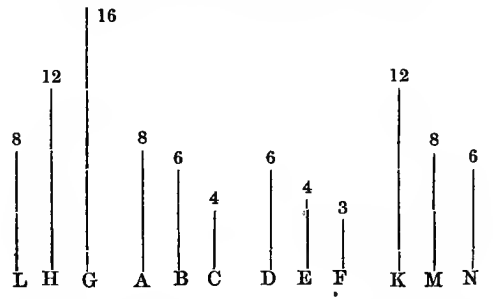
"If there be any number of magnitudes, and as many others, which taken, two and two, have the same ratio, and their proportion be disturbed, they shall be in the same ratio by equality."—EUCLID.

N. B. This Proposition is usually cited by the words, "*ex æquali in proportione perturbatâ*, by equality in perturbate proportion; or "*ex æquo perturbato*, by perturbate equality.

Con. Pst. 1, V. Dem. 15, V. *Ms* have the same ratio as their equims.
11, V. Ratios the same to the same ratio, the same to one another.
4, V. Equims of the 1st and 3rd have the same ratio to equims of 2nd and 4th,
21, V, Two series of 3 Ms. in each having the same ratio but in a cross order.
Def. 5, V. Criterion of the equality of ratios.

Case I. *Let there be two series of* 3 *Ms, each,* A,B,C,D,E,F, *which taken two and two in cross order have the same ratio.*

E. 1	Hyp.	Let A : B = E : F, & B : C = D : E;
2	Conc.	then A : C = D : F.
C. 1	Pst. 1, V.	Take of A,B, D any equims. G,H,K;
2	,,	and of C,E,F any equims. L,M,N.



D. 1	C. & 15, V.	∵ G,H are equims. of A,B; ∴ A : B = G : H,
2	*Sim.*	∵ M,N are equims of E,F, ∴ E : F = M : N;
3	H & 11, V.	but A : B = E : F, ∴ G : H = M : N;
4	H.	And ∵ B : C = D : E, & H,K are equims of B,D, & L,M equims. of C,E;
5	4, V.	∴ H : L = K : M;
6	D. 3.	now G : H = M : N, and G,H,L — K,M,N, are two series of 3 Ms. each, and taken in cross order two and two, they have the same ratio;
7	21, V.	∴ if G >, =, or < L, K >, = or < N;
8	C. 1 & 2.	but G,K, any equims. of A,D. & L,N any equims. of C, F;
9	Def. 5, V.	∴ A : C = D : F.

CASE II. Next, *let there be two series of four Ms. each*, A, B, C, D, and E, F, G, H *which taken two and two in a cross order also have the same ratio*;

E. 2	Hyp.	And let A : B = G : H; B : C = F : G; and C : D = E : F;
2	Conc.	then A : D = E : H.

A 2	B 3	C 4	D 12
E 5	F 15	G 20	H 30

A 2	: B 3 = G 20	: H 30
B 3	: C 4 = F 15	: G 20
C 4	: D 12 = E 5	: F 15
A 2	: D 12 = E 5	: H 30.

D. 1	H.	∵ A, B, C & F, G, H, are two series of 3 Ms each, and taken 2 & 2 in a cross order have the same ratio.
2	Case 1.	∴ A : C = F : H;
3	H. & Case 1.	but C : D = E : F, ∴ A : D = E : H.
4	*Sim.*	And so on, whatever be the no. of *Ms.*
5	Rec.	Therefore, *if there be any number*, &c. Q. E. D.

Alg. & Arith. Hyp. Let $a = 8$, $b = 6$, $c = 4$; $a' = 6$, $b' = 4$, $c' = 3$.

Alg. By Hyp. $\frac{a}{b} = \frac{b'}{c'}$, and $\frac{b}{c} = \frac{a'}{b'}$;

Multiply; $\frac{a\,b}{b\,c} = \frac{b'\,a'}{c'\,b'}$, or $\frac{a}{c} = \frac{a'}{b'}$.

i. e. $a = c = a' : c'$.

Arith. $\frac{8}{6} = \frac{4}{3}$, & $\frac{6}{4} = \frac{6}{4}$;

$\frac{8 \times 6}{6 \times 4} = \frac{4 \times 6}{3 \times 4}$, or $\frac{8}{4} = \frac{6}{3}$;

i. e. 8 : 4 = 6 : 3.

Illustration. If A 12 : B 6 :: C 3 : D 9,
E 4 : F 12 :: G 6 : H 3,

then, A 12 : D 9 :: E 4 : H 3.

Sch. There are different ways of announcing Prop. 23.

1. Ratios, compounded of any number of equal ratios, but in reverse order, are equal to one another;

For let there be the two series, A,B,C,D and E,F,G,H;
As before, Pr. 23 Case 2, if $A : B = G : H$,
$B : C = F : G$,
and $C : D = E : F$;
then, *ex æquo perturbato*, $A : D = E : H$.

2. In any two Series of Magnitudes, A, B, C & D, E, F, which taken two and two in a cross order, have the same ratio, if A, the first M, be to B, the second, as C the third is to D the fourth; and if B, the second, be to E the fifth, as F the sixth, to C the third; then the first A is to the fifth E, as the sixth F to the fourth D.

And if there be any number of Analogies; for instance three: and if the second and third terms of each Analogy form respectively the first and fourth terms of the next; then the first term, A of the first Analogy, shall be to the second, G, of the last, as the third term H, of the last, is to D the fourth of the first.

A	12	:	B	6	=	C	6	: D 3
B	6	:	E	4	=	F	9	: C 6
E	4	:	G	2	=	H	18	: F 9
H	12	:	G	2	=	H	18	: D 3

The same conclusion would follow were the number of Analogies four or more.

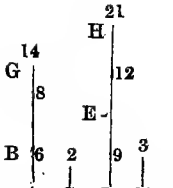
Prop. XXIV.—Theor.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Dem. Pr. B. V. Four Ms being proportionals are proportionals *inversely*. 22, V. *Ex æquo*. 18, V. *Componendo*.

Pr, A, V. If 1st : 2nd = 3rd 4th,—then if 1st $>$ = or $<$ 2nd, the 3rd $>$ = or $<$ 4th. 17, V. *Dividendo*.

E. 1	Hyp.	Let AB : C=DE : F, and BG : C=EH : F;
2	Conc.	then AB+BC, *i. e.*, AG : C=DE+EH, *i. e.* DH : F.
D. 1	H. & Pr. B, V.	∵ BG : C=EH : F, and *inv.* C : BG=F : EH;
2	H. & D. 1.	and ∵ AB : C=DE : F, and C : BG=F : EH;
3	22, V.	∴ *ex æq.* AB : BG = DE : EH;
4	18, V.	∴ *comp.* AG : BG=DH : HE;
5	H. & 22, V.	but GB : C=HE : F; ∴ *ex æq.* AG : C = DH : F.
6	Rec.	∴ If *the first has to the second, &c.* Q. E. D.

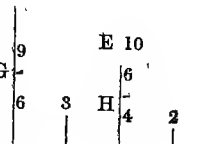


Arith. Illust. If A B 6 D E 9.
C 2, F 3.
B G 8. E F 12.

Then A B + B G : C = D E + E H : F;
Or, A G 14 : C 2 = D H 21 : F 3.

Cor. 1. On the same Hypothesis being made, the first however, being greater than the fifth and the third than the sixth, it follows, that "*the excess*, A G, *of the first*, A B, *above the fifth*, BG, *shall be to the second* C, *as the excess*, DH, *of the third* D E, *above the sixth*, E H, *is the fourth* F; or, in other words, the difference between the first and fifth shall be to the second, as the difference between the third and sixth is to the fourth.

E. 1	Hyp.	Let AB : C = DE : F, and BG : C = EH : F, AB being > BG;
2	D. 3, Pr. A. V.	and ∴ AB : BG = DE : EH, & DE > EH;



	3 Conc.	then AG, *i. e.* AB—BG : C = DH, *i. e.* DE—EH : F.
D.	1 H.	∵ AB : BG = DE : EH, AB being > BG and DE > EH;
	2 17, V.	∴ *div.* AG : BG = DH ; EH;
	3 H. & 22, V.	and ∵ BG : C = EH : F; ∴ *ex. æq.* AG: C = DH : F.
	4 Rec.	∴ *the excess of the first, &c.* Q. E. D.

Cor. 2. The Proposition holds true of two ranks of magnitudes whatever be their number, *of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to the fourth magnitude*; Or, in any number of Proportions, if the second term, B, is the same throughout, and also the fourth term, D, the same; then the sum of all the first terms, A + E + G, is to the common second term, B, as the sum of all the third terms, C + F + H, is to the common fourth term, D.

A 4 : B 2 = C 6 : D 3,

E 6 : B 2 = F 12 : D 3,

G 10 : B 2 = H 15 : D 3.

A+E+G 22 : B 2 = C+F+H 33 : D 3.

E.	1 Hyp.	Suppose three proportions so constituted, that A : B = C : D; E : B = F : D, and G : B= H : D;
	2 Conc.	then A + E + G : B = C + F + H : D.
D.	1 H.	∵ A : B = C : D, and E : B = F : D;
	2 18, V.	∴ A + E : B = C + F : D.
	3 D. 2, & H.	And ∵ A + E : B = C + F : D, and G : B= H : D;
	4 18, V.	∴ A + E + G : B = C + F + H : D.
	5 *Sim.*	In the same way for any number of proportions.

Sch. Prop. 24 may also he expressed,—"*If two series of Proportionals have the same consequents*, B, D, *the sum of the first antecedents*, A + E, *shall be to their common consequent*, B, *as the sum of the second antecedents*, C + F, *to their common consequent*, D.

E. 1	Hyp.	Let A : B = C : D, and E : B = F : D;
2	Conc.	Then A + E : B = C + F : D.
D. 1	H.	∵ E : B = F : D, ∴ *inv.* B : E = D : F;
2	H. & 22, V.	but A : B = C : D, ∴ *ex. æq.* A : E = C : F;
3	18, V.	and *comp.* A + E : E = C + F : F.
4	H.	Again, E : B = F : D;
5	22, V.	∴ *ex æquo* A + E : B = C + F : D.

Cor. 1. Thus also by *dividendo* instead of *componendo*,—"*If two proportions have the same consequents*, the difference *of the first antecedents shall be to their common consequent, as* the difference *of the second antecedents to their common consequent.*

E. 1	Hyp.	Let A : B = C : D, and E : B = F : D;
2	Conc.	then A − E : B = C − F : D.

Cor. 2. *And if four magnitudes form a proportion*, A 10 : B 5, = C 6 : D3, *then miscendo*, using the sum and difference, as in Use & App. Pr. 22, V., *the sum of the first and second*, A + B, *is to their difference*, A ∼ B, *as the sum of the third and fourth*, C + D *to their difference* C — D.

D. 1	18. V.	*comp.* A + B : B = C + D : D;
2	17, V.	*div.* A − B : B = C − D : D;
3	B. V.	*inv.* B : A − B = D : C − D;
4	22, V.	*ex æq.* A + B : A − B = C + D : C−D.

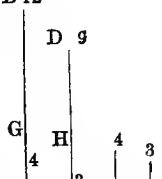
Cor. 3. In any number of magnitudes of the same kind forming two series, one A, B, C, D, E, F, &c., and the other as many, G, H, I, K, L, M, &c. —*if the ratios of the first to the second*, A : B, or G : H; *of the second to the third*, B : C, or H : I; *of the third to the fourth*, C : D, or I : K; *and so on, be the same in the two series; then, any two combinations whatever*, miscendo, i. e., *by the sum and difference of the magnitudes of the first series*, A + C — E, and B — C + D, *shall be to one another as two similar combinations of the corresponding magnitudes of the second series*, G + I — L, and H — I + K. See *Geom. Plane, Sol. & Sph. p.* 55.

Prop. XXV.—Theor.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

Con. 2 & 3, I. To draw a line equal to a given st. line, and from a gr. line to cut off a part equal to the less.
Dem. 11, V. Ratios the same to the same ratio are the same to one another.
7, V. Equal Ms the same ratio to the same M, and conversely.
19, V. If a M : a M = a pt. : a pt.—the rem : the rem. = the M : the M.
A, V. If 1st : 2nd = 3rd : 4th, then, if 1st > = or < 2nd, 3rd, > = or < 4th.
Ax. 2. I. If equals be added to equals, the wholes are equal.
Ax. 4, I, If equals be added to unequals the wholes are unequal.

E. 1	Hyp.	Let AB : CD = E : F;
2	14, V. & A, V.	and let A B be the greatest, consequently F the least;
3	Conc.	then AB + F > CD + E.
C.	2 & 3, I.	Take in AB, AG = E, and in CD, CH = F.
D. 1	H. & C.	∵ AB : CD = E : F, and AG = E, & CH = F;
2	11 & 7, V.	∴ AB : CD = AG : CH.
3	D. 2.	but ∵ the whole AB : the whole CD = pt. AG : pt. : CH;
4	19, V.	the rem. GB : rem. HD = AB : CD.
5	H. & A, V.	but AB > CD, ∴ GB > HD.
6	C.	And ∵ AG = E, and CH = F;
7	Ax. 2, I.	∴ AG + F = CH + E.
8	*Add.*	To the unequal Ms, BG, HD, BG being > HD, add AG + F, and CH + E respectively.
9	Ax. 4, I.	∴ BG + AG + F > HD + CH + E, *i. e.*, AB + F > CD + E.
10	Rec.	Therefore, *if four magnitudes of the same kind, &c.* Q. E. D.



Arith. Illust. If AB 12 : CD 9 = E 4 : F 3;
Then A + F, 12 + 3 > CD + E, 9 + 4.

N.B. This Proposition has been inserted as Corollary 2 to Pr. 17, bk. V.

COR. *If three magnitudes be proportionals*, A : B: C, *the sum of the extremes* A + C, *will be greater than twice the mean*, 2 B, *and therefore half the sum* $\frac{A+C}{2}$ *greater than the mean* B.

For the proof see Cor. 3, Pr. 17, bk. V.

Thus the *arithmetical mean* between two magnitudes *is greater* than the *geometrical mean* between them, the case excepted in which the magnitudes are equal to one another.

SUPPLEMENTARY PROPOSITIONS.

It has been customary from very early times, perhaps from the days of EUCLID himself, to add several Propositions to this Fifth Book. The first English Edition, which added *Nine*, announces them in this way,—"Here follow certayne propositions added by *Campane* which are not to be contemned, and are cited even of the best learned, namely of *Johannes Regio Montanus*, in the Epitome which he writeth vpon Ptolome."—BILLINGSLEY'S Euclid, fol. 150.

The SUPPLEMENTARY PROPOSITIONS here given are chiefly from SIMSON with some from other sources.

PROP. F.—THEOR.

Ratios which are compounded of the same ratios are the same to one another.

DEM. 22. V. *Ex æquo* by equality. 23, V. *Ex æquo perturbato* by perturbate equality.

E. 1	Hyp.	Let A : B = D : E & B : C = E : F;
	2 Conc.	then the ratio compounded of A : B & B : C,— (by Def. A, V.) the ratio A : C,— shall be the same with the ratio D : F, which (by Def. A, V.) is compounded of D : E & E ; F.
D. 1	H.	∵ in the 2 series of Ms, A,B,C, & D,E,F, A : B = D : E, & B : C = E : F ;

	A 6. B 4. C. 10. D 3. E 2. F. 5.	A 8. B 6. C 8. D 3. E 4. F 3.
	A 6 : C 10=D 3 : F 5	A 8 : C : 8=D 3 : F3
2	22, V.	∴ *ex æq.* A : C = D : F.
3	H.	Next, ∵ A : B = E : F, and B : C = D : E;
4	23, V.	∴ *ex æq. pert.* A : C = D : F. *i. e.*, ratio A : C compd. of A : B & B : C is the same with ratio D : F ,, D : E & E : F.
5	*Sim.*	In like manner for any number of ratios.
6	Rec.	Therefore, *Ratios which are compounded*, &c.

Sch. Two cases only are demonstrated in the above proposition ; *one*, of ratios compounded of the same ratios in the same order, as in Pr. 22, V: the *other* of ratios compounded of the same ratios in a reverse order as in 23, V. There remains the Case of *Ratios compounded in any other order* ; which may be demonstrated in a similar way.

"For if K,L,M represent the three ratios in one order, in whatever other order they may be arranged, two of them will be found which are contiguous in both arrangements ; commencing with which two, the demonstration will differ little from the above."

Also, "*ratios which are compounded of the same four ratios*, K,L,M,N *in whatsoever orders, are the same with one another* ; as for instance, in the orders K,L,M,N, and M,K,N,L ;—for the latter ratio is the same with the ratio which is compounded of the same ratios in the order M,K,L,N, because the ratio which is compounded of K,N,L, is the same with that which is compounded of K,L,N ; and for a similar reason, the ratio which is compounded of M,K,L,N, is the same with that which is compounded of K,L,M,N."

"And the same reasoning may be extended to five, six, or any other number of ratios."—Geom. Plane, Sol. & Sph. p. 55, 56.

PROP. G.—THEOR.

If several ratios be the same to several ratios, each to each; the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

DEM. Def. A, V. In any number of Ms of the same kind, the 1st has to the last the ratio compd. of the ratio which the 1st has to the 2nd, and of the ratio which the 2nd has to the 3rd, and of the ratio which the 3rd has to the 4th, and so on unto the last magnitude.

22; V. *Ex æquo* by equality.

E. 1	Hyp. 1.	Let A : B = E : F, & C : D = G : H; also A : B = K : L, & C : D = L : M;
2	Conc. 1. by Def. A, V.	then, K : M is compounded of K; L & L : M; and K : L & L : M the same with A : B & C : D;
3	Hyp. 2.	Again as E : F let N : O, & as G : H let O : P;
4	Conc. 2, by Def. A, V.	then ratio N : P is compd. of ratios, N : O & O : P; and N : O, & O : P are the same with E : F and G : H;
5	Conc. 3.	And it is to be shewn that K : M = N : P.

A 6.	B 4.	C 8.	D 6.	K 12.	L 8.	M 6.					
E 3.	F 2,	G 4.	H 3.	N 18.	O 12.	P 9.					
K 12 :	M 6	= 18 :	P 9.								

D. 1	H. 1 & 2.	∵ K : L, as (A : B, as E : F as) N : O;
2	H. 1 & 2.	and ∵ as L : M so is (C : D, and so is G : H, and so is) O : P;
3	22, V.	∴ *ex æquali*. K : M = N : P;
4	Rec.	Therefore, *if several ratios be the same, &c.* Q. E. D.

Otherwise

E. 1	Hyp. 1.	In two series of Ms, A,B,C,D; and A′, B′, C′, D′, let A : B = A′ : B′; and C : D = C′ : D′;
2	Hyp. 2.	Also in two other series, K, L, M, and K′, L′, M′, let K : L = A : B; and L : M = C : D;
3		Also K′ : L′ = A′ : B′, and L′ : M′ = C′ : D′;
4	Conc. 1.	Then, (by Def. A, V.) ratio K : M is compd. of K : L and of L : M; and ratios K : L and L : M are equal to ratios A : B and C : D.
5	Conc. 2.	Again, (by Def. A, V.) ratio K′ : M′ and of K′ : L and of L′ : M′; and ratios K′ : L′ and L′ : M′ are equal to ratios A′ : B′ and C′ : D′;
6	Conc.	It is then to be proved that ratio K : M = ratio K′ : M′,
D. 1	H. 2, 1.	∵ K : L = A : B, and A : B = A′ : B′ = K′ : L′;
2	11, V.	∴ K : L = K′ : L′.
3	H. 2 & 3.	Again, ∵ L : M = C : D, and C : D = C′ : D′, and C′ : D′ = L′ : M′;
4	11, V.	∵ L : M = L′ : M′.
5	22. V..	∴ in series K, L, M, and K′, L′, M′, *ex æq.* K : M = K′ : M′.

Prop. H.—Theor.

If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them;—then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

Dem. Pr. B, V. *Invertendo.* 22, V. Ex *æquo.*

E. 1	Hyp. 1.	Let the first ratios be A : B, B : C, C : D, D : E E : F,
2	„ 2.	and the other ratios G : H, H : K, K : L, L : M;

3	„ 3.	Also let A : F, compd. of the 1st ratios, = G : M, compd. of the other ratios;
4	„ 4.	and let A : D, compd. of A : B, B : C, C : D, be the same with the ratio G : K, compd.
5	Conc.	of G : H, and H : K, then the ratio D : F, compd. of D : E, and E : F shall equal K : M, compd. of K : L, and L : M.

A 12	B 8.	C 6.	D 4.	E 3.	F 2.	
G 18.	H 12.	K 6.	L 6.	M 3.		
D 4	:	F 2	=	K 6	:	M 3.

D. 1	H. Pr. B, V.	$\because$ A : D = G : K; $\therefore$ (*inv.*) D : A = K : G.
2	H. & 22, V.	and $\because$ A : F = G : M; $\therefore$ *ex æquo.* D : F = K : M.
3	Rec.	Therefore, *if a ratio, which is compounded, &c.* Q. E. D.

PROP. K.—THEOR.

If there be any number of ratios, and any number of other ratios such, that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be

the same to one of the last ratios, or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios; then, the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same, each to each, to the remaining ratios of the first, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are the same each to each, to these remaining ratios.

DEM. Def. A, 5. Compound Ratio. 22, V. *Ex æquo.*

11, V. Ratios the same to the same ratio are the same to one another.

Pr. B, V. *Invertendo.*

E, 1	Hyp. 1.	Let the *first* ratios be A : B, C : D, E : F;
2	,, 2.	& the *others*, G : H, K : L, M : N, O : P, Q : R.
3	,, 3.	and let A : B = S : T; C : D = T : V; and E : F = V : X;
4	Conc.	then (Def. A, V.) S : X is compd. of S : T, T : V, and V : X, which are the same to A : B, C : D, and E : F, each to each.
5	Hyp. 4.	Also let G : H = Y : Z, and K : L = Z : a; M : N = a : b; O : P = b : c, and Q : R = c : d;
6	Conc. 2.	then, (Def. A, V.) Y : d is compd. of Y : Z, Z : a, a : b, b : c, and c : d; which are the same, each to each, to the ratios of G : H, K : L, M : N, O : P, and Q : R;
7	Conc. 3.	∴ S : X = Y : d.
8	Hyp. 5.	Also, let A : B, *i. e.*, S : T = e : g, compd. of e : f, and f : g, the same as G : H and K : L;
9	Hyp. 6.	and let h : l be compd. of h : k and k : l; which are the same to the remaining *first* ratios, namely, to those of C : D and E : F,
10	Hyp.	Also let m : p be compd. of m : n, n : o, & o : p; which are the same, each to each, to the remaining *other* ratios, of M : N, O : P and Q : R.
11	Conc. 4.	*then* h : l, = m : p.

		$h,\ k,\ l.$					
		4, 14, 10.					
	A, B;	C, D;	E, F.		S, T, V. X.		
	18, 9.	8, 28.	21, 15.		4, 2, 7. 5.		
G, H;	K, L;	M, N;	O, P;	Q, R.	Y, Z, a, b, c, d.		
9, 6.	12, 9.	6, 9.	15, 20.	24, 30.	12, 8, 6, 9, 12, 15.		
$e, f, g,$		$m,\ n,\ o,\ p.$			$\therefore h : l = m : p,$		
6, 4, 3,		8, 12, 16, 20.			$4 : 10 = 8 : 20.$		

D. 1	E. 8 & 5.	$\because e : f =$ G : H = Y : Z;
3	E. 8 & 5.	and $f : g =$ K : L = Z : a;
2	22, V.	$\therefore$ *ex æq.* $e : g =$ Y : a.
4	H. 3 & 5.	And $\because$ A : B $\therefore$ S : T = $e : g$;
5	11, V.	$\therefore$ S : T = Y : a;
6	Pr. B, V.	and $\therefore$ *inv.* T : S = a : Y.
7	E. 7 & 22, V.	But S : X = Y : d, $\therefore$ *ex æq.* T : X = $a : d$.
8	E. 8 & 3.	Also $\because h : k =$ C : D = T : V;
9	E. 8 & 3.	and $k : l =$ E : F = V : X;
10	22, V.	$\therefore$ *ex æq.* $h : l =$ T : X.
11	*Sim.*	So $m : p = a : d$, and T : X = $a : d$;
12	Con. 11, V.	$\therefore h : l = m : p$;
13	Rec.	Therefore, *if there be any number of ratios, &c.*

Q. E. D.

Sch. Propositions F, G, H, and K "are annexed to the 5th book," says Simson in his Notes," because they are frequently made use of by both ancient and modern geometers. And in many cases, compound ratios cannot be brought into demonstration, without making use of them."

And Simson adds, "Whoever desires to see the doctrine of ratios delivered in this 5th book solidly defended, and the argument brought against it by And. Tacquet, Alph. Borellus, and others, fully refuted, may read Dr. Barrow's mathematical lectures, viz., the 7th and 8th of the year 1666."

Fuller information, if desired, may be obtained from Geometry, Plane, Solid and Spherical, Bk. II., pp. 31—78. De Morgan's *Study and Difficulties of Mathematics,*—On Proportion, ch. XVI, pp. 79—86. *Arithmetic and Algebra*, pp. 33—39. *Connection of Number and Magnitude* and *Proportion* and *Ratio.* *Penny Cyclopædia,* Vol. XIX., p. 49 & 307.

Prop. L.—Theor.

A compound Ratio is equal to the product of its component simple ratios.

Dem. Def. A, V. In any number of magnitudes of the same kind, the first to the last has the ratio compounded of the ratio of the 1st to the 2nd, of the 2nd to the 3rd, of the 3rd to the 4th, and so on to the last magnitude.

Pr. F, V. Ratios compounded of the same ratios are the same to one another.

7 A, V. The ratio of two lines is the same as that of the numbers which express the number of times that any third line is contained in them respectively.

15, V. Magnitudes have the same ratio to one another which their equimultiples have.

E. 1	Hyp.	Let the ratio A : B = the ratios compounded of C : D and of E : F;
2	Conc.	then $\frac{A}{B} = \frac{C}{D} \cdot \frac{E}{F}$.
C.	Sup.	Let ratio C : D = ratio G : H, and ratio E : F = ratio H : K.
D. 1	Def. A, V.	∵ G : K is compd. of G : H and H : K;
2	C.	and ∵ G : H and H : K = C : D and E : F;
3	H.	and ∵ also the ratio compounded of C : D and E : F = A : B;
4	Pr. F, V.	∴ the ratio G : K = ratio A : B.
5	Pr. 7 A, V. & 15, V.	But $\frac{G}{K} = \frac{g}{k} = \frac{m\,g}{m\,k} = \frac{g}{m} \cdot \frac{m}{k} = \frac{c}{d} \cdot \frac{e}{f}$;
6	D. 4.	and ratio $\frac{A}{B}$ = ratio $\frac{G}{K}$;
7	Conc.	∴ the comp. ratio $\frac{A}{B} = \frac{C}{D} \times \frac{E}{F}$.

Q. E. D.

Arith. Illustration. Let $\frac{16}{64}$ be compounded of $\frac{2}{4}$ and $\frac{8}{16}$;

then $\frac{16}{64} = \frac{2}{4} \times \frac{8}{16}$.

Take $\frac{2}{4} = \frac{3}{6}$, and $\frac{8}{16} = \frac{6}{12}$;

$\because \frac{3}{12}$ is compounded of $\frac{3}{6}$ and $\frac{6}{12}$, or of $\frac{2}{4}$ and $\frac{8}{16}$

$\therefore \frac{3}{12} = \frac{16}{64}$;

But $\frac{3}{12} = \frac{1}{4} = \frac{3 \times 1}{3 \times 4} = \frac{1 \times 3}{3 \times 4} = \frac{1}{2} \times \frac{4}{8}$;

And $\frac{16}{64} = \frac{3}{12}$; $\therefore \frac{16}{64} = \frac{2 \times 8}{4 \times 16}$.

PROP. M.—THEOR.

If there be two fixed magnitudes, A and B, which are the limits of two others, P and Q, (that is, to which P and Q, by increasing together, or by diminishing together, may be made to approach more nearly than by any the same given difference), and if P be to Q always in the same given ratio of C to D; then A shall be to B in the same ratio.

CON. N. B. In the *first* case, that of *commensurable* proportion, the obvious principle is assumed, that to two given magnitudes of the same kind, and a third there is some magnitude which is a fourth proportional; but in the *second* or *other* case, that of *incommensurable* proportion, we can only approximate to the fourth proportional, as we approximate to the ratio of the two magnitudes numerically; since, however, such approximation may be contained without limit, it is *presumed*, that there is some magnitude between them, which is to the given third magnitude in the same ratio which the second has to the first; that is, some magnitude which is a fourth proportional to the three.

DEM. 11, V. Ratios that are the same to the same ratio, are the same to one another; *or* Magnitudes A, B and C, D, which have the same ratio with the same magnitudes P, Q, have the same ratio with one another.

14, V. If four magnitudes of the same kind be proportionals, then if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

First. Let P and Q, *by a continual increase*, approach to A and B, respectively, so that P and Q can never equal, much less exceed, A and B, but may be made to approach A and B more nearly than by any the same difference.

C.	As above	Take a magnitude B′, such that A : B′ = C : D

Sup. If B′ ≠ B′, then B′ either < B, or > B.
1°. *Let B′ be < B, or B′ = B—b.*

D.			
1	H. & C.	∵ P : Q, = C : D, and A : B′ = C : D;
2	11, V.	∴ A : B′ = P : Q;
3	14, V.	But A always > P, ∴ B′ always > Q.
4	D. 3, & H.	Now ∵ Q < B′ and B′ < B by the difference *b*,
5	Conc.	∴ Q cannot approach B within the difference *b*;
6	Remk.	,— but this is contrary to the hypothesis;
7	Conc.	∴ B′ cannot be < B.

2°. *Let B′ be > B; and take A′ such that A′ : B = A : B′.*

D.			
1	14, V.	Then ∴ B < B′, A′ is < A, as by the difference *a*;
2	H.	And ∵ A′ : B=A : B′ & P : Q =A : B;
3	11, V.	∴ A′ : B′ = P : Q.
4	18, V.	but B always > Q, ∴ A′ always > P.
5	D. 4.	Wherefore, ∵ P is always < A, and A′ < A by *a*;
6	Conc.	∴ P cannot approach A within the difference *a*,
7	Remk.	but this is contrary to the hypothesis;
8	Conc.	∴ B′ cannot be > B;
9	D. 7, I. & D. 8.	And ∵ B′ neither < nor > B;
10	Conc.	∴ B′ = B, *i. e.*, A : B = C : D.

Second. Let P and Q approach to A and B respectively by a *continual decrease.*

D. 1	*Sim.*	In the same manner, by substituting "greater" for "less," and "less" for "greater," we demonstrate,
2		that B′ also is neither > nor < B;
3	Conc.	again ∴ B′ = B, *i. e.*, A : B = C : D.
4	Rec.	Therefore, *If there be two fixed magnitudes, &c.* Q. E. D.

USE & APPL. The Author of *Geometry, Plane, Sol. and Spher.*, p. 46, says of this Proposition,—"it will be found of very extensive application in Geometry. By help of it, the lengths of plane curves, and the areas bounded by them, the curved surfaces of solids, and the contents they envelope, may in many instances be brought into comparison with little greater difficulty than right lines, rectilineal areas, and solids bounded by planes." "But the use of the proposition is by no means confined to these. It may be regarded as one of the first steps to what is called the higher Geometry, and in this view likewise, is well worth the attention of the student."

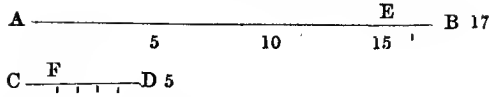
PROP. N.—PROB.

To find a common measure of two lines.

SOL. & DEM. Def. 1, V.—*Note.* One magnitude *measures* another when it is contained in that other magnitude an exact number of times.

And a magnitude which is a measure of two or more magnitudes is named the *common measure* of those Magnitudes.

E. 1	Dat.	Let A B and C D be two lines, or magnitudes;
2	Quaes.	to find a line which will be contained exactly both in A B and in C D;
S. 1	SUP. 1.	If C D is contained exactly in A B, then C D measures A B;—and then also any aliquot part of C D will be the common measure both of C D and A B; *or*
2	Def. 1, V.	∵ *m* C D = A B, ∴ C D is a measure of A B;
3	,,	and ∵ *m* C F = C D, ∴ C F a com. meas. of C D and A B,



4	Sup. 2	But if m CD = AB — EB, and n E B = C D;
5	Def. 1, V.	then n E B also measures A E, a mult. of C D,
6	Conc.	and ∴ E B will measure A B, and ∴ is a com. measure of A B and C D.
7	Sup. 3.	But if E B does not measure C D,
8	Hyp.	let 2 E B = CD — C F, *i. e.*, = D F.
9	S. 8.	Then ∵ E B measures D F, if C F measures E B, C F also will measure D F;
10	S. 8	and D F is a mult. of E B;
11	Conc.	∴ C F a measure of C D, and of A E and also of A B.
12	Sup. 4.	Let 2 C F = E B;
13		then C D = 2 E B + C F = 4 C F + C F = 5 C F;
14		and A B = 3 C D + EB = 15 C F + 2 C F = 17 C F.
15	Conc.	∴ C F is contained in C D *five* times, and in A B *seventeen* times; and C F is the *com. meas.* of C D and A B.
16	*Sim.*	Thus may be found the *com. meas.* of any other two commensurable lines.

Cor. 1. "*The greatest common measure of the remainder and lesser magnitude is also the greatest common measure of the two magnitudes,*" For

∵ every com. meas. of A and B is also a com. meas. of B and R—the remainder;

∴ the greatest com. meas. of A and R will be found among the com. measures of R and R.

Now every one of the latter measures both A and B;

∴ the greatest among them is the greatest com. meas. of A and B.

COR. 2. *Any aliquot part or submultiple of a common measure, is also a common measure.*

COR. 3. *By repeating the process with the remainder and lesser magnitude, and again with the new remainder (if there be one) and the preceding, and so on,* the greatest common measure *of two given commensurable magnitudes, A and B, may be found.*

S. 1	Hyp.	Let $2\,B = A - R$; $3\,R = B - R_2 = R - R_3$ and $5\,R_3 = R_2$ exactly.
2	Cor. 1, N, V.	Then ∵ the greatest *com. meas.* of A & B, is also the greatest for B & R;
3	,,	and the greatest for B & R, the greatest for R & R_2 and so on;
4	Hyp.	and ∵ R_3 is contained in itself and also in R_2 exactly;
5		∴ R_3 is the greatest common measure of R_2 & R_3,—
6		and ∴ also the greatest common measure of A & B.

COR. 4. *Any two commensurable lines are to one another as the numbers denoting the number of times that they respectively contain their common measure; thus, if the com. meas. of* A B *be* E F, *taken* 5 *times; and that of* C D, *the same* E F, *taken* 7 *times, then the ratio* A B : C D = *the ratio* 5 : 7.

SCH. If, on continuing the process, we never arrive at a remainder which exactly measures the preceding remainder, the magnitudes are *incommensurable.*

USE & APPL. The Process *for finding the greatest common measure of two numbers* is included in this Prop. N, Bk. V:—for, by dividing the greater number by the less, and finding the remainder;—then, by dividing the less by the remainder, and finding the second remainder, if there be one; and by dividing the first remainder by the second, and finding the third remainder; and so on, until a remainder be found which exactly measures the last preceding remainder; this final remainder that exactly measures the last preceding will be the greatest common measure.

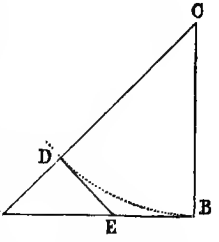
Prop. O.—Theor.

The diagonal and side of a square are incommensurable.

Con. Pst. 3, I. A circle may be drawn at any distance from the centre.
11, I. To draw a perpendicular from a given point.

Dem. 32, I. The int. ∠s of every △ are together equal to two rt. ∠s.
20, I. Any two sides of a △ are together greater than the third side.
Def. 15, I. Every point in the ⊙ce of a circle is at an equal distance from the centre.
Cor. 3, 16, III. Tangents to a ⊙ from the same · are eqnal.
37, III. If from a point without a circle there be drawn two lines, one of which cuts the circle and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets it shall touch the circle.
5, I. The ∠s at the base of an isosc. △ are equal.
6, I. If the ∠s at the base of a triangle, are equal the sides opposite are eqnal.

E. 1	Hyp.	Let A B C be the half of a square; A C the diagonal, and B A, B C two sides conterm. in B.
2	Concl.	Then the side B A, or B C, does not measure A C; nor have B A and A C any common measure.
C. 1	Pst. 3.	From C, the extr. of A C, with rad. C B draw arc B D;
2	11, I.	and from D draw D E ⊥ A C.
D. 1	H. & 32, I,	∵ ∠ B is a rt. ∠, and ∠s A, C each < a rt. ∠;
2		∴ A B < A C.
3	20, I.	Again ∵ AB+BC>AC, or 2 AB>A C;
4	D. 3 & 2.	∴ A C > A B, but < 2 A B.
5	*Sim.*	And ∵ the same is true of every square;



6		$\therefore$ AC — AB = AD, and AD is < AB.
7	Def. 15, I & H	Also CD = CB = AB, and AD < AB;
8	Cor. 16, III.	and $\because$ ED and EB are tangts. from the same · E;
9	37, III.	$\therefore$ E D = E B.
10	C.	And $\because$ A D E is a $\triangle$, and $\angle$ A D E a rt. $\angle$; and $\angle$ A = $\frac{1}{2}$ a rt. $\angle$;
11	5, I. & 6, I.	$\therefore$ $\angle$ DEA = $\frac{1}{2}$ a rt. $\angle$, and $\therefore$ AD = DE = EB.
12	Remk.	Now when A D the first rem., or its equal E B is taken from A B,
13		then the rem. A E is the diag. of a sq., of which A D, D E are the sides.
14	*Sim.*	The same process as before will then have to be followed out;
15		and when A D as side has been taken from A E as diag.;
16		then the rem. lines will again be side and diag.
17	D. 12—16	But $\because$ *a diag. — a side* always leaves a *remainder.*
18	Conc.	$\therefore$ in this process there will *ever* be a rem.;
19		$\therefore$ the process will never terminate;
20	Conc.	and $\therefore$ A C the diag. of a sq., and C B a side, are *incommensurable.* Q. E. D.

PROP. P.—THEOR.

If four straight lines, A, B, C, D, *be proportionals, (whether commensurable, or incommensurable,) the rectangle under the extremes* A·D *will be equal to the rectangle under the means* B. C.

Book II. p. 145. The numerical area of a rectangle is obtained by supposing the two sides containing the rectangle to be divided into a number of linear units of the same kind, as inches, feet, &c., and then multiplying the units on one side by the units on the other: the product represents the area or enclosed space.

Cor. 1. Pr. 29, I. § 4. p. 18. *Geom. Plane, Sol. & Spher.* "If there be two st, lines, one of which is contained an exact number of times in one side of a rectangle, and the other an exact number of times in the side adjoining it; then, the rectangle under those two st. lines shall be contained as often in the given rectangle, as is denoted by the product of the two numbers which denote how often the lines themselves are contained in the two sides."

Def. 5, V. Criterion of the Equality of Ratios. Pr. M, Bk. V.

First. *Let* A & B *be commensurable, and* ∴ *also* C & D.

C.	1	*Assum.*	Take any com. ratio whatever, as 7 : 5;
	2		and for com. measures, M and N;
	3	Pst. 2, V.	let M be contained in A *seven* times, in B *five* times; & N be contained in C *seven* times, in D *five* times.
D.	1	C. 3	∵ A = 7 *m*, and D = 5*n*,
	2		∴ rect. A · D = 7 × 5 (*m.n*) = 35 *m* . *n*.
	3	*Sim.*	So, rect. B · C = 5 × 7 (*n.m*) = 35 *m* . *n*;
	4	Ax. 1, I.	∴ rect. A · D = rect. B · C

Second. *Let* A & B *be incommensurable, and* ∴ C & D.

C.	1	Pr. M, V.	Find st. lines P & Q which approach nearer A & C than any assigned difference,
	2		and let P & Q contain like parts of B & D, so that P . D = Q . B.
D.	1	Pr. M, V.	Now, ∵ by taking like parts of B & D, continually less and less, P & Q, increase towards A & C *within* any assigned difference;
	2		∴ P . D and Q . B, by increasing together, approach more nearly A . D and C . B than any assigned difference;
	3	Pr. M, V.	∴ rect. A . D = rect. C . B or B . C.
	4	Rec.	Therefore, *If four st. lines, &c.,* Q. E. D.

USE & APPL. The Theory of Proportion in Arithmetic and Algebra is founded on a similar truth; namely,

If four magnitudes be proportionals, and if A, B, C, D, *represent those magnitudes numerically,* i.e., *if* A *and* B *represent the numbers of times, the unit of their kind is contained in the two first, and if* C *and* D *represent the numbers of times, the unit of their kind is contained in the two last, then the quotient or fraction* $\frac{A}{B}$ *shall be equal to* $\frac{C}{D}$; *and conversely,*" See *Geom. Plane, Sol. and Spher. p.* 46 & 47.

EXAMPLES OF REASONING BY PROPORTION.

1°. If A 2 : B 4 : C 8;
then A 2 : C 8 = A^2 4 : B^2 16, by Def. 10, V.

2° If A 2 : B 4 : C 8 ; D 16;
then A 2 : D 16 = A^3 8 : B^4 64, by Def. 11, V.

3°. If A 6 : B 9 = C 8 : D 12;
invertendo, B 9 : A 6 = D 12 : C 8;

4° If A 9 : B 6 = C 12 : D 8;
alternando, A 9 : C 12 = B 6 : D 8, by 16, V.

5°. If A 5 : B 4 = C 10 : D 8;
dividendo, A 5 —B 4 : B 4 = C 10—D 8 : D 8, by 17, V.

6° If A 5 : B 3 = C 10 : D 6;
componendo, A 5 + B 3 : B 3 = C 10 + D 6 : D6, by 18, V.

7° If A 5 : B 4 = C 10 : D 8;
convertendo, A 5 : A 5—B 4 = C 10 : C 10—D 8. by Pr. E, V.

8°. If A 12 : B 6 :: C 18 : D 36;
E 6 : F 3 :: G 9 : H 18

ex æquali, A 12 : D 36 = E 6 : H 18. by 22, V.

9°. If A 12 : B 6 :: C 3 : D 9;
E 4 : F 12 :: G 6 : H 3

ex. æq. perturbato, A 12 : D 9 = E 4 : H 3. by 23, V.

10°. If A 12 : B 6 = C 6 : D 3;

miscendo. A 12 + B 6 : A 12 — B 6 = C 6 + D 3 : C 6 — D 3. Use and App. 22, V.

11°. If A 5 : B 4 = C 10 : D 8,

permutando, A 5 : C 10 = B 4 : D 8, by 16, V.

invertendo, C 10 : A 5 = D 8 : B 4, by B, V.

componendo, C 10 + A 5 : A 5 = D 8 + B 4 : B 4, by 18, V.

REMARKS ON BOOK V.

To the Notes and Observations gathered from various sources we simply add the commendation of BILLINGSLEY, fol. 126.

"THIS FIFTH BOOKE of EUCLIDE is of very great commoditie and vse in all Geometry, and much diligence ought to be bestowed therin. It ought of all other to be throughly and most perfectly and readily knowne. For nothyng in the bookes followyng can be vnderstood without it: the knowledge of them all depende of it.

And not onely they and other writinges of Geometry, but all other Sciences and also artes : as *Musike*, *Astronomy*, *Perspective*, *Arithmetique*, the arte of accomptes and reckoning, with other such like. This booke therefore is as it were a chiefe treasure, and a peculiar iuell much to be accompted of. It entreateth of proportion and Analogie, or proportionalitie, which pertayneth not onely vnto lines, figures, and bodies in Geometry; but also vnto soundes & voyces, of which Musike entreateth, as witnesseth *Boetius* and others, which write of Musike. Also the whole arte of Astronomy teacheth to measure proportions of tymes and mouings. *Archimedes* and *Iordan*, with other, writing of waightes, affirme, that there is proportion betwene waight and waight, and also betwene place and place. Ye see therefore how large is the vse of this fift booke. Wherfore the definitions also thereof are common, although here, of *Euclide* they be accommodate and applied onely to Geometry. The first author of this booke was, as it is affirmed of many, one *Eudoxus*, who was *Platos* scholer, but it was afterwards framed and put in order by *Euclide*."

GRADATIONS IN EUCLID.

BOOK VI.

THE THEORY OF PROPORTION APPLIED, FOR COMPARING THE SIDES AND AREAS OF PLANE RECTILINEAL FIGURES.

"THIS SIXTH BOOKE is for vse and practise a most speciall booke. In it are taught the proportions of one figure to an other figure, and of their sides the one to the other, and of the sides of one to the sides of an other, likwise of the angles of one to the angles of the other. Moreover it teacheth the description of figures like to figures geuen and marueilous applications of figures to lines, euenly, or with decrease or excesse, with many other theoremes, not onely of the Proportions of right lined figures, but also of sectors of circles, with their angles. On the Theoremes and Problemes of this Booke depend for the most part the compositions of all instrumentes of measuring length, breadth, or deepenes, and also the reason of the vse of the same instrumentes, as of the Geometrical square, the Scale of the Astrolabe, the quadrant, the staffe, and

such others. The vse of which instrumentes, besides all other mechanicall instrumentes of raysing up, of mouing, and drawing huge things incredible to the ignorant, and infinite other ginnes (which likewise haue their groundes out of this Booke) are of wonderfull and vnspeakeable profite, besides the inestimable pleasure which is in them."—BILLINGSLEY, fol. 153.

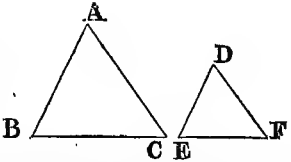
The Theory of Proportion, exhibited in the fifth book, is in the sixth applied to determining the proportions which exist between both the sides and the areas of similar plane rectilineal figures. The basis of the comparisons instituted is not identity of size, but identity of form; and when this cannot be predicated, or clearly inferred, no true Geometrical proportion can be established. The sixth book however advances further than this, and enables us to construct a figure, which shall possess the form of a first given figure and the size of a second. By the second book we may describe a square equal to a given rectilineal figure;—by the sixth we may make any right-lined figure which we choose, equal in size to a given rectilineal figure.

We are also empowered, to find Lines and to draw rectilineal figures in proportion the one to the other; and to increase or diminish any figure according to a given Ratio. From this book we derive the principles of what is termed the Rule of Three, and the geometrical form for the solution of a quadratic equation; it extends also to a much wider application the fertile truth, that the square of the hypotenuse equals the sum of the squares of the sides of a right-angled triangle; and it supplies the easiest and most certain rules by which to conduct Measurements of all kinds. These will be seen when we show the Uses of various Propositions.

In general terms it may be said that the sixt book establishes; 1st, the proportion between the sides of similar triangles; and, 2nd, the proportion existing between the areas of similar rectilineal figures: it also lays down the methods, either of finding magnitudes proportional to other magnitudes, or of describing figures similar to other figures, or equal to them.

DEFINITIONS.

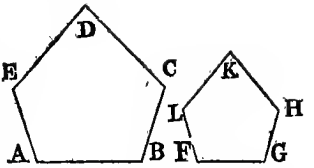
I. Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals: thus the $\triangle$s ABC, DEF, are similar, if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and if AB : AC, = DE : DF, and AB : BC = DE : EF.



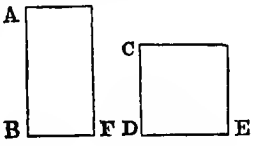
"This definition, like some others to be found in the Elements, is excessive. To contain no more than is strictly necessary (or, indeed, than as yet has appeared to be probable), it should be modified as follows:—*Two rectilineal figures are said to be similar when the first has all its sides but one proportional to the sides of the other, and* the angles included by those sides equal to the angles included by the corresponding sides of the other." *Geom. Pl., Sol. & Sph.* p. 57.

According to the definition, for one rectilineal figure to be similar to another, the conditions to be fulfilled are equal to twice the number of

sides, or rather to the sum of the number of sides and of the number of angles. Thus in the pentagons A B C D E, F G H K L; 1°. ∠ A = ∠ F; 2°. ∠ B = ∠ G; 3°. ∠ C = ∠ H; 4°. ∠ D = ∠ K; and 5°. ∠ E = ∠ L. Also 6°. E A : A B = L F : F G; 7°. A B : B C = F G : G H; 8°. B C : C D = G H : H K; 9°. C D : D E = H K : K L; and 10°. D E : E A = K L : L F. See Hosr's *Euclid*, p. 198.



II. "Reciprocal figures, viz., triangles and parallelograms, are such as have their sides about two of their angles proportionals in such a manner, that a side of the first figure is to a side of the other, as the remaining side of this other is to the remaining side of the first;" thus, A B : C D = D E : B F;—the analogy beginning in one figure and ending in the same.



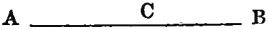
"Figures are reciprocal when the antecedents and the consequents of ratios are in each of the figures." EUCLID.

Another way of putting the definition is:—"The sides of two figures, ABF, CDE, are *reciprocally* proportional, when the *extremes* of the proportion are sides of one figure, and the *means* are sides of the other;" as AB . BF = CD . DE.

The sides are *directly* proportional, when in each figure the two sides compared are one an extreme and the other a mean: thus, if AB : BF = CD : EF, the proportion is direct.

III. A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less; thus in the line A B and its parts,

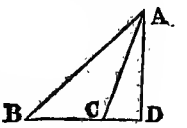
A B : A C = A C : C B.



For EUCLID's definition, LARDNER substitutes,—"when the whole line is to one segment as that segment to the remaining one."

A line thus divided is also said to be divided *medially*; and the ratio of its segments is named the *medial ratio*.—Prop. 30, VI. is the problem by which the segments are made, and is but another form of Prop. 11, II.; to divide a line so that the rectangle under the whole line and one segment shall equal the square of the other segment.

IV. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base; thus A D is the altitude of the triangle A B C.



Whichever side of a figure is assumed as the base, the altitude is the perpendicular distance from the base, or the base produced, to the point or line most distant from the base. The altitude of the same figure may vary with the change in position of its base.

SUBSIDIARY DEFINITIONS.

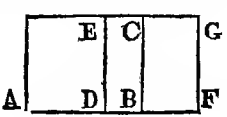
DEF. A. A straight line, O C, divided into three parts, is said to be *harmonically* divided, when the whole line C O is to one of its extreme segments O E, as the other extreme C D is to the middle part D E; *i. e.*, O C : O E = C D : D E.

Four st. lines are said to be *harmonicals*, when they pass through the same point, and divide any one st. line harmonically.

DEF. B. "A figure is given in *species*, when its several angles and the ratios of the sides about them are given."

DEF. C. "A figure is given in *magnitude*, when its area, or any figure equal to it in area, is given."

DEF. D. "A parallelogram is said to be applied to a straight line, when it is described upon it as one of its sides; *ex. gr.* the parallelogram A C is applied to the straight line A B.



DEF. E. "But a parallelogram A E, is said to be applied to a straight line A B, *deficient* by a parallelogram, when A D, the base of the parallelogram A E, is less than A B, the base of parallelogram A C; and because the ▱ A E is less than the ▱ A C, (described upon A B, with the same ∠ A, and between the same parallels A B, E C,) by the ▱ D C;—therefore the ▱ D C is called the *defect* of ▱ A E.

DEF. F. And a parallelogram A G is said to be applied to a straight line A B *exceeding* by a parallelogram, when A F, the base of ▱ A G is greater than the base A B of ▱ A C; and because the ▱ A G exceeds the ▱ A C, (described upon A B, with the same ∠ A, and between the same parallels A F, E G,) by the ▱ B G; therefore the ▱ B G is named the *excess* of ▱ A C.

PROPOSITIONS.

PROP. 1.—THEOR.

Triangles and parallelograms of the same altitude are one to another as their bases.

CON. Pst 1, I. A st. L. may be drawn from one • to any other point.
Pst. 2, I. A terminated st. L. may be produced in a st. L.
3, I. From the gr. of two st. Ls. to cut off a pt. = the less.

DEM. 38, I. Triangles upon equal bases and between the same parallels are equal to one another. 1, V.
Def. 5, V. The 1st of four Ms is said to have the same ratio to the 2nd which the 3rd has to the 4th, when any equims. whatsoever of the 1st and 3rd being taken, and any equims whatsoever of the 2nd and 4th; if the *m* of 1st be $<$ = or $>$ that of the 2nd, the *m* of the 3rd is also $<$ = or $>$ that of the 4th;
41, I. If a ▱ and a △ be upon the same base and between the same ‖s, the ▱ shall be double of the △.
15, V. Ms have the same ratio to one another which their equims. have.
11. V. Ratios that are the same to the same ratio, are the same to one another.

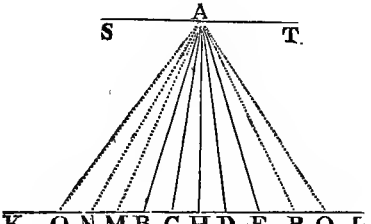
28, I. If a st. line falling upon two other st. lines makes the ext. ∠ = the int. and opp. ∠ upon the same side of the line; or makes the ∠s upon the same side together = 2 rt. ∠s; the two st. lines shall be parallel.

33, I. The st. lines which join the extrs. of two eq. and parallel st. lines towards the same parts, are also = and ∥.

36, I. ▱s on eq. bases and between the same ∥s are equal to one another.

CASE I. *Triangles of the same alt. are to one another as their bases.*

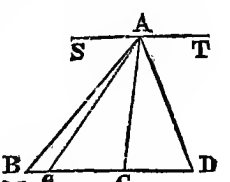
E. 1	Hyp.	Let △s ABC & ADE have the same alt. AH; *i. e.*, let BE ∥ ST;
2	Conc.	then BC : DE = △ABC : △ADE.



C. 1	Pst. 2, I.	Produce BE indef. to K & L;
2	3, I.	from BK cut off, BM, MN, NO, each = BC;
3	3, I.	and from EL cut off EP, PQ, each = DE;
4	Pst. 1, I.	join AM, AN, AO; AP, AQ.
D. 1	C. 2 & H.	Then ∵ BC = BM = MN = NO, & KL ∥ ST;
2	38, I.	∴ △s ABC, ABM, AMN & ANO are equal;
3	1, V.	∴ the *m* which CO is of CB, △ACO is of △ACB;
4	Remk.	*i. e.* OC & △ACO are equims. of BC & △ABC.
5	*Sim.*	So, DQ & △ADQ are equims. of DE & △ADE;
6	38, I.	and if OC > = or < DQ, △ACO > = or < △ADQ.
7	D. 4.	Now, ∵ CO & △ACO are equims. of CB & △ACB;

8	D. 5.	& ∵ DQ & △ ADQ are equims. of DE & △ ADE;
9	D. 6.	& ∵ △ AOC, the *m* of △ ACB > = or < △ ADQ, the *m* of △ ADE, as OC, the *m* of BC, is > = or < DQ, the *m* of DE;
10	Def. 5, V.	∴ BC : DE = △ ABC : △ ADE. Q. E. D.

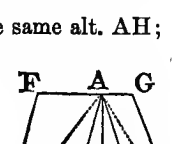
Otherwise. Let B C contain the *subm.* M. 4 times, and let C D contain it 3 times; and let B*e* = M; then



D. 1	38, I.	△ A B C = 4 △ A B*e* and △ A C D = 3 △ A B *e*;
2	15, V.	hence △ A B C : △ A C D = 4 : 3,
3	H.	But B C : C D = 4 : 3;
4	11, V.	∴ △ A B C : △ A C D = B C : C D.

Case II. *Parallelograms of the same alt. are also to one another as their bases.*

E. 1	Hyp.	Let ▱s FC & GD have the same alt. AH;
2	Conc.	*i. e.*, let FAG ∥ BHE; then BC : DE : : ▱ FC : ▱ GD.
C.	Pst. 1, I.	Join AB & AE.
D. 1	C. & 41, I.	∵ ▱ FC = 2 △ ABC, & ▱ GD = 2 △ ADE;
2	15, V.	& ∵ Ms have the same R. as their equims;
3		∴ △ ABC : △ ADE = ▱ FC : ▱ GD.
4	Case 1.	But ∵ BC : DE = △ ABC : △ ADE;
5	D. 3.	& ∵ △ ABC : △ ADE = ▱ FC : ▱ GD;
6	11, V.	∴ BC : DE = ▱ FC : ▱ GD.
7	Rec.	∴ *Triangles and parallelograms, &c.* Q. E. D.



Cor. 1. From this it is evident that, *triangles and parallelograms that have equal altitudes are one to another as their bases; and having equal bases are as their altitudes.*

CASE I.—*they are their bases.*—See fig. 1.

C. 1	*Pon.* 1.	Place the given figures on the same st. line, as KL, or BE;
2	„ 2.	& let the triangles be on the same side of the line.
D. 1	28, I.	Then ∵ the perps. are equal and parallel;
2	33, I.	∴ the line joining the vertices will be ‖ the base.
3	*Sim.* 1, IV.	∴ as in Case I, 1, VI, the △s having the same alts. are as their bases;
4	„	and as in Case II, 1, VI., the ▱s having the same alts. are also as their bases.

CASE II.—*they are as their altitudes.*

D. 1	38, I	∵ the △s = rt ∠d △s on eq. bases & alts.;
2	36, I.	& ∵ the ▱s = rectangles on eq. bases & alts.;
3		∴ the alt. as the base, and *vice versâ.*
4	*Sim.*	So, the dem. becomes an instance of Case I, Cor. 1.

COR. 2. *Any two triangles, or parallelograms,* T, T,′ *are to one another in the ratio compounded of the ratios of their alts.,* a, a' *and of their bases,* b, b'.

C.	Sup.	Take M a △ or ▱ with alt. a and base b'
D. 1	Cor. 1, 1, VI	∵ $T : M = b : b'$, and $M : T' = a : a'$;
2	Conc.	∴ $T : T' = \left\{ \begin{matrix} b : b', \\ a : a' \end{matrix} \right\}$, a R. compounded of the Rs of the bases and altitudes.

COR. 3. *The rectangle under two lines, A & B, is a mean proportional between their squares.*

D. 1	H.	∵ the square on A & the rect. A·B have the same alt. A;
2	1, VI.	∴ $A^2 : A \cdot B = A : B$.
3	H.	And ∵ the sq. and the rect. have the same base B;
4	1, VI.	∴ $A \cdot B : B^2 = A : B$.

Cor. 4. Generally, "*if two triangles or two parallelograms be as their bases, they have equal altitudes; and if they be as their altitudes they have equal bases.*"

Sch. 1. From the principle that rectangles of the same altitude are to one another as their bases, the first Proposition might be directly inferred; for $\square$s are equal to rectangles on the same base and with the same alt.; and $\triangle$s are *one-half* of the Area of the respective $\square$s.

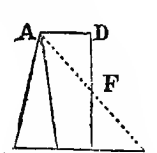
2. When, as in Case II, a first figure, FC, is to a second GD, as the base BC, or alt. AH, &c., of the first to the base DE, or alt. AH, &c., of the second.—they *vary* exactly as the base, or altitude, &c., varies. Propositions of this kind constitute a very numerous class, and are distinguished by the general name of *Variants*. Under the commercial law of supply and demand, the price of a commodity varies as these vary,—the proportion being *inverse; i. e., as* the supply increases *so* the price or estimated value diminishes. But under the mechanical law of power and work done, *as* the power increases *so* the work done increases also, the proportion being *direct*. We use the words *as* and *so* to avoid the longer and fuller enunciation; thus, the work done in a given time by a machine of one degree of power, *is* to the work done in the same time by another machine of a different degree of power, *as* the power of the first machine to the power of the second.

3. We shall do well to remember that "one quantity does not vary *as* another, because it varies *with* that other. A square and its side or root vary together, but the square does not vary *as* the side or root; for instance, if the side or root be doubled, the square is not doubled, but quadrupled;" with a side of 2 the square is 4, but with a side of 2×2, or 4, the square is 16, or four times larger. *Penny Cyclop.*, vol. xxvii, p. 137.

Use & App. This Proposition is very frequently referred to for the demonstration of other propositions. It may also be employed *for dividing a rt. lined surface:* thus,

From a Trapezium A B C D, *with* A D ∥ B C, *to cut off a third part.*

C.	2 & 3, I.	Take CE = AD; and BG = $\frac{1}{3}$ BE, and join A G, A E;
	2 C.	then $\triangle$ A B G = $\frac{1}{3}$ of trap. ABCD.
D.	1 H.	∵ AD ∥ BE, ∴ $\triangle$s ADF and FCE are eq. ang.;
	2 C. 1, & 26, I.	and ∵ A D = C E, ∴ $\triangle$ ADF = $\triangle$ FCE;
	3 Ax. 2, I,	∴ $\triangle$ A B E = trap. A B C D.
	4 C. & 1, VI.	Now $\triangle$ A B G = $\frac{1}{3}$ of $\triangle$ A B E;
	5 Ax. 1, I.	∴ $\triangle$ A B G = $\frac{1}{3}$ of trap. A B C D.



Q. E. D.

PROP. 2.—THEOR.

If a st. line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally; and conversely, *if the sides, or sides produced, be cut proportionally, the st. line which joins the points of section shall be parallel to the remaining side of the triangle.*

DEM. 37, I. △s upon the same base and between the same ‖s are equal to one another.

7, V. Eq. Ms have the same R to the same M; and conversely.

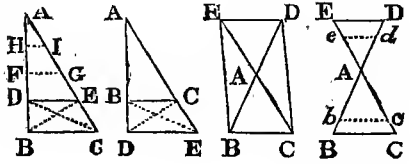
1, VI.—11, V. B, V. If 4 Ms are proportionals, they are proportionals also when taken inversely.

9, V. Ms which have the same R to the same M are eq. to one another; and those to which the same M has the same R are eq. to one another.

39, I. Eq. △s upon the same base and upon the same side of it are between the same parallels.

E. 1	Hyp. 1.	CASE I. Let DE be ‖ BC, one of the sides of △ABC;
2	„	and let DE cut the other sides, AB, AC, in D,E;
3	Conc.	then BD : DA = CE : EA; or AD : DB = AE : EC.
C.	C.	Join BE & CD.

Fig. 1. 2. 3. 4.



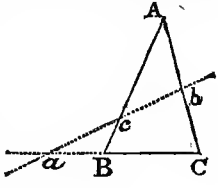
D. 1	H. & C.	∵ On a com. base DE, and between the ∥ s BC, DE,
2	37, I.	∴ △BDE = △CDE.
3	Remk.	But ADE is another △ :
4	7, V.	∴ △BDE : △ADE = △CDE : △ADE.
5	C. & H.	Now, ∵ △s BDE & ADE have the same alt,;
6	1, VI.	∴ △BDE : △ADE = BD : DA.
7	*Sim.*	So, △CDE : △ADE = CE : EA ;
8	11, V. B V.	∴ BD : DA = CE : EA, & *inv.* AD : DB = AE : EC.

E. 1	Hyp. 2.	Case II. Next, in △ABC let AB, AC, or AB, AC produced, be cut in D & E;
2		and so that BD : DA = CE : EA;
3	Conc.	then DE ∥ BC.
C.	Pst. 1, I.	Make the same construction
D. 1	H. 2, 1, VI.	∵ BD : DA = CE : EA; & BD : DA = △BDE : △ADE;
2	1, VI.	& ∵ CE : EA = △CDE : △ADE;
3	11, V.	∴ △BDE : △ADE = △CDE : △ADE;
4	9, V.	∴ △BDE = △CDE;
5	C.	And these △s are on the same base:
6	39, I.	∴ DE ∥ BC.
7	Rec.	∴ *If a st. line be drawn parallel, &c.* Q.E.D.

Cor. In the same manner it may be shown that, *if the sides AB, AC, of an angle be cut by any number of parallels BC, DE, FG, HI, any two parts of the one will have the same ratio to one another, as the corresponding parts of the other*, *i. e.*, the sides will be similarly divided; and every pair of corresponding segments in each side will be proportional; BH : HF = CI : IG; FD : DB = GE : EC, &c.

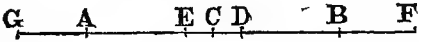
Sch. In the first part of the Proposition the Enunciation is not sufficiently exact, and in the other part it is "strictly speaking *false*, inasmuch as a line may cut two sides *proportionally*, and yet not be parallel to the third side." The Enunciation should be, "1°. If a line be drawn parallel to any side of a triangle, it divides the other sides, or those sides produced; so that their segments between the parallel and the third side shall have the same ratio to their segments between the parallel and the vertex of the opp. angle; and 2°. if a line cut the two sides in this manner, it will be parallel to the third side." Lardner's *Euclid*, p. 178.

2. The Theory of *Transversal* Lines, that is, of lines drawn across several others so as to cut them all either internally or externally, is intimately connected with this second Proposition: where, however, in the last figure, the transversal D E, cutting two of the sides A B, A C, proportionally, does not cut the third side B C, except on the idea that the point of external section is at an infinite distance, the segments being equal. In triangle ABC in the margin, the line the $a\,c\,b$ is the transversal, cutting A C, A B internally in the points b, c, and C B externally in the point a. Here and in all similar cases, $Ac \times Ba \times Cb = Bc \times Ca \times Ab$.



In the Projection of figures, and in Surveying, especially, when inaccessible points are required by the aid only of signal poles and a measuring line, the theory of Transversals is very useful; for its leading principles, however, we refer to Appendix III. pp. 324—332, of LARDNER's Euclid.

3. The ways in which a st. line A B, may be cut in a given ratio, A D : D B, belong to the full consideration of Prop. 2, Bk. VI. Here one point of section will be D. Take C, the point in which A B is bisected, and make C E=CD. Thus, BE = AD and AE = BD; ∴ B E : E A is the given ratio, and E another required point of section. In a given st. line there are thus two points of internal section.



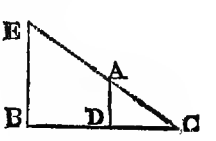
In the same way, if A F : B F & A G : B G be each the given ratio, F & G are two points of external section, cutting the line A B produced in the given ratio. There are, therefore, four points, two internal and two external, at which a line may be cut in a given ratio.

4. This Proposition shows, moreover, that parallel lines, as E D, $e\,d$, $b\,c$, BC, fig. 4, cut diverging lines A B, A C, A D, A E, proportionally.

USE & APP. 1. The second proposition, bk. VI., is of absolute necessity in establishing many of the following propositions.

2. It is used also *for the measurement of the height of an inaccessible object* B E, *which casts an accessible shadow* B D.

Take A D, a staff of known length, and setting it upright at D, the extremity of the shadow B D, measure the shadow D C; also measure the shadow B D.



Now, $\because$ AD ‖ EB,
$\therefore$ CD : DB = AD : EB;
whence $BE = \frac{DB \cdot AD}{CD}$

Or the staff AD may be set along BC the shadow of BE at a point D, where the extremities of the shadows from AD and BE would coincide, as at C;
then $CD : CB = AD : BE$;—whence $BE = \frac{CB \cdot AD}{CD}$.

3. *A given line* AC, as in fig. 1, *may be divided into parts proportional to those in another line* AB.

C. 1	3, I.	From A draw a line, on which the parts in AB are set, *i. e.*, AH, HF, FD, DB; and join BC.
2	31, I.	Through the · s D, F, H, draw parallels to BC;
3	Conc.	then AC is divided into parts proportional to those in AB; *i. e.* CE : EG = BD : DF, &c.
D.	Cor. 2, VI.	The proof as in 2, VI. and Cor. 2, VI.

Prop. 3.—Theor.

If the angle of a triangle be divided into two equal angles by a st. line which also cuts the base; the segments of the base shall have the same ratio which the other sides of the triangle have to one another; and conversely, *if the segments of the base have the same ratio which the other sides of the triangle have to one another, the st. line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.*

Or, "If an angle of a triangle be cut into two equal parts, and the st. line cutting the angle cuts also the base, the segments of the base shall have the same ratio to the other sides of the triangle; and if the segments of the base have the same ratio to the other sides of the triangle, the st. line drawn from the vertex to the section-point cuts into two equal parts that angle of the triangle." Euclid.

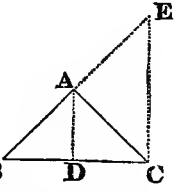
Con. 31, I. Through a given · to draw a st. line ‖ to a given st. line.

DEM. 29, I. If a st. line fall upon two st. lines, it makes the altr, $\angle$s equal, and the ext. $\angle$ = the int. and opp, $\angle$ upon the same side; and likewise the two int. $\angle$s upon the same side = two rt. $\angle$s.

AX. 1, I.—6, I. If two $\angle$s of a $\triangle$ be equal, the sides which subtend the eq. $\angle$s shall be equal.

2, VI.—7, V.—11, V.—9, V.—5, I. The $\angle$s at the base of an isosc. $\triangle$ are equal; and if the eq. sides be produced, the $\angle$s on the other side of the base shall be equal.

E. 1	Hyp.	CASE 1. Let the $\triangle$ be ABC, and let AD make $\angle BAD = \angle CAD$;
2	Conc.	then BD : DC = BA : AC.
C.	31, I.	Through C draw CE $\parallel$ DA, to meet BA prod. in E.
D. 1	C. 29, I.	$\therefore$ AC cuts EC $\parallel$ AD, $\therefore \angle ACE = \angle CAD$;
2	H. Ax. 1, I.	but $\angle CAD = \angle BAD$, $\therefore \angle BAD = \angle ACE$.
3	C. 29, I.	Again, $\because$ BE cuts the $\parallel$s AD, EC, $\therefore \angle BAD = \angle AEC$;
4	D. 1. 6, I.	but $\angle ACE = \angle BAD$, $\therefore \angle ACE = \angle AEC$, & AE = AC.
5	C. 2, VI.	And $\because$ AD $\parallel$ EC in $\triangle$BCE, $\therefore$ BD : DC = BA : AE :
6	D. 4, 7, V.	but AE = AC, $\therefore$ BD : DC = BA : AC.



E. 1	Hyp.	CASE II. Let BD : DC = BA : AC; and join AD;
2	Conc.	then in $\triangle$BAC, $\angle BAD = \angle CAD$.
C.	31, I.	Make the same construction as in Case I.
D. 1	H. 2, VI.	$\because$ BD : DC = BA : AC; & BD : DC = BA : AE;
2	11, V. 9, V. 5, I.	$\therefore$ BA : AC = BA : AE; & $\therefore$ AC = AE, & $\angle AEC = \angle ACE$;
3	29, I.	but $\angle AEC = \angle BAD$, & $\angle ACE = \angle CAD$;
4	Ax. 1, I.	$\therefore \angle BAD = \angle CAD$, *i. e.*, $\angle$BAC is bis. by AD.
5	Rec.	$\because$ *If the angle of a triangle &c.* Q.E.D.

Prop. A.—Theor.

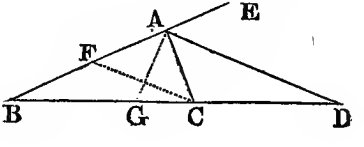
If the outward angle of a triangle, made by producing one of its sides, be divided into two equal angles, by a st. line, which also cuts the base produced; the segments between the dividing line and the extremities of the base, have the same ratio which the other sides of the triangle have to one another; and conversely, *if the segments of the base produced have the same ratio which the other sides of the triangle have; the st. line drawn from the vertex to the point of section, divides the outward angle of the triangle into two equal angles.*

Combining Prop. III and A into one, the General Enunciation might be;

"*If the vertical angle of a triangle and its adjacent exterior angle be bisected by lines cutting the base and the base produced, the base will be cut internally and externally in the ratio of the two sides.*

Con. 31, I. Dem. 29, I. Ax. 1, I.—6, I.—2, VI.—7, V.—11, V. 9, V.—5, I.

E. 1	Hyp. 1	Case I.—Let B A a side of △ A B C be produced to E;
2	„ 2	and let A D, meeting B D the base produced, in D, bisect ext. ∠ C A E;
3	Conc.	then B D : D C = B A : A C.
C.	31, I.	Through C draw C F ∥ A D.
D. 1	C. & 29, I.	∵ A C meets the ∥s A D, F C,
		∴ altr. ∠ A C F = altr. ∠ C A D;
2	H. 2, Ax. 1, I.	But ∠ C A D = ∠ D A E,
		∴ also ∠ D A E = ∠ A C F.



D. 3	C. & 29, I.	Again, ∵ BE cuts the ‖s AD, FC ; ∴ ext. ∠ DAE = int & opp ∠ CFA ;
4	D. 2. Ax. 1, I. 6, I.	but ∠ ACF = ∠ DAE, ∴ ∠ ACF = ∠ CFA, & AF = AC.
5	C. 2, VI.	And ∵ AD ‖ FC, a side of △ BCF ; ∴ BD : DC = BA : AF ;
6	D. 4, 7, V.	but AF = AC, ∴ BD : DC = BA : AC.
E. 1	Hyp.	Case II. Let BD : DC = BA : AC, and join AD ;
2	Conc.	then ∠ CAD = ∠ DAE.
C.	31, 1.	The same construction is to be made.
D. 1	H. & 2, VI.	∵ BD : DC = BA : AC, & BD : DC = BA : AF ;
2	11, V. 9, V. 5, I.	∴ BA : AC = BA : AF. & ∴ AC=AF, & ∠ AFC= ∠ ACF.
3	29, I.	But ∠ AFC = *ext.* ∠ EAD, & ∠ ACF = altr. ∠ CAD,
4	Ax. 1, I.	∴ also ∠ EAD = ∠ CAD.
5	Rec.	∵ *If the outward angle, &c.* Q.E.D.

Cor. 1. *The segments* BD, DC, *of the base produced by the external bisector* A D, *are proportional to the segments* B G, G C, *of the base* B C, *made by the internal bisector* A G.

D. 1	3, VI., A, VI.	∵ BG : GC = BA : AC ; and BD : DC = BA : AC ;
2	11, V.	∴ BG : GC = BD : DC.

"Hence, the bisectors of the internal and external angles cut the base internally and externally in the same ratio."

Cor. 2. *The two lines* AG *and* AD, *which bisect the vertical angle* BAC, *and its adj. ext. angle* CAE, *cut the base produced harmonically ; i. e.*, so as to make BD : DC = BG : GC ;—or the base BC is a harmonic mean between its greater internal and external segments, BG and CD ; *i. e.*, CD : BG = CD~BC : BC~BG.

Cor. 3. *Also the two sides of a triangle* AB, AC, *and the lines* AG, AD, *which bisect the vertical and ext. vertical angles are harmonicals;* for by Def. A, VI., they are four lines passing through the same point A, and dividing a st. line, BD, harmonically.

Cor. 4. *If* BG, BC *and* BD, *in the same st. line. be in harmonical progression,* DC, DG *and* DB, *will also be in harmonical progression.*

D. 1	Def. B, V.	∵ BG, BC, and BD are in harmonical progression,
2		∴ BG : BD = GC : CD.
3	16, V. B, V.	∴ *alt.* BG : GC = BD : CD, and *inv.* DC : DB = GC : BG.
4	Def. B, V.	∴ DC, DG and DB are st. lines such that the 1st DC : the 3rd DB = DC∼DG : DG∼DB.
5	Conc.	∴ DC, DG and DB are in harmonical progression.

Hence, if a st. line B C be divided in G, and produced so that D B : D C = B G : G C, the whole line made up of B G + C D, *i. e.*, B D, is divided harmonically, in the points G & C;—for D B, D G, and D C, are in harmonical progression, and therefore B D, B C, and B G are also in harmonical progression, and G and C are the points in which B D is harmonically divided.

Sch. When the given triangle is *isosceles,* the line which *bisects* the exterior vert. angle is *parallel* to the base; and the point of supposed external section of the base is then infinitely distant; or, inasmuch as two parallel lines never meet, there is in reality no point of external section. But practically, the infinite segments, of which the difference is the base, may be regarded as equal,—for the bisector is infinitely extended in both directions, and the difference between the infinite segments bears an infinitely small ratio to the segments themselves. On this theory, in the case of an isosceles triangle the proportion is preserved between the external segments and the sides. In all other cases;

E. 1	Hyp.	If the line A G, which bis. ∠ B A C cuts the base B C in G;
2	Conc.	then the st. line B D is divided harmonically in s G, C.
D. 1	3, VI. & A, VI.	∵ B G : G C = B A : A C; and B D : D C = B A : A C;

2	11, V.
3	C.
4	D. 2.
5	Def. A, VI.

$\therefore$ B D : D C = B G : G C.
But B G = B D − D G, and G C = G D − D C,
$\therefore$ B D : D C = B D − D G : G D − D C.
$\therefore$ BD, D G and D C, are in harmonical proportion.

USE & APP. 1. *The harmonic mean*, x, between two numbers, m and n, is $\frac{2\,m\,n}{m+n}$, and *is obtained* from the harmonic proportion $m : n = m-x : x-n$;

whence $m \times (x-n) = n \times (m-x)$; *i. e.*, $m\,x - m\,n = m\,n - n\,x$;
transposing and collecting $m\,x + n\,x = 2\,m\,n$;
dividing by $m + n$, we have $x = \frac{2\,m\,n}{m+n}$ the harmonic mean.

2. By combining the Propositions 3 and A, it is proved;

1°. in *optics*,—that the axis of a pencil of rays, incident on a spherical mirror, is divided harmonically by the radiant point, the geometrical focus of the reflected rays, and the centre and surface of the reflector;

2°. in *acoustics*,—that the lengths of three musical strings, of the same thickness, material and texture, and under the same tension, must be in harmonical progression, in order to produce a note, its fifth and its octave.

PROP. 4—THEOR. (*Important.*)

The sides about the equal angles of equiangular triangles are proportionals; *and those which are opposite to the equal angles are homologous sides*, *i. e.*, are the antecedents or the consequents of the ratios.

OR, "Equiangular triangles have their sides proportional."

CON. 22, I. To make a $\triangle$ of which the sides = three given st. lines, of which any two must be greater than the third.
32, I. The three int. $\angle$s of every $\triangle$ are together = two rt. $\angle$s.
DEM. Ax. 2, I.—17, I. Any two $\angle$s of a $\triangle$, are together < two rt. $\angle$s.

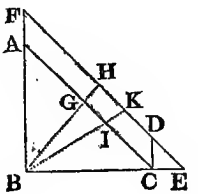
Ax. 12, I. If a st. line meets two st. lines, so as to make the two int. ∠s on the same side of it taken together < two rt. ∠s, these two st. lines being continually produced shall at length meet upon that side on which the ∠s are < two rt. ∠s.

28, I.—34, I. The opp. sides and ∠s of □s are = to one another, and the diam. bisects them.

2, VI.—7. V.—16, V. *alternando.* If four Ms of the same kind be proportionals, they shall also be proportionals when taken alternately.

22, V. *ex æquali.* If there be any number of Ms, and as many others, which taken two and two in order, have the same R; the first shall have to the last of the first Ms the same R which the first has to the last of the others.

E. 1	Hyp.	Let △s ABC, DCE, be eq. ang. *i.e.*, ∠ABC = ∠DCE, ∠ACB = ∠DEC, & ∴ (32, I.) ∠BAC = CDE;
2	Conc. 1.	Then these △s are sim., *i.e.*, by Def, 1, VI. the sides about each pair of eq. ∠s. are proportionals; AB : BC = DC : CE; BC : CA = CE : ED, & BA : AC = CD : DE.
3	Conc. 2.	and those sides are homologous which are opposite to the eq. ∠s.
C.	*Pon.* 22, I.	Place △DCE so that its side CE may be contiguous to BC & in the same st. line with it.
D. 1	H. *Add.*	∵ ∠BCA = ∠CED, to each add ∠ABC;
2	Ax. 2, I.	∴ ∠ABC + ∠BCA = ∠ABC + ∠CED;
3	17, I.	but ∠s ABC + BCA < 2 rt. ∠s;
4	D. 2.	∴ ∠s ABC + CED < 2 rt, ∠s;



D. 5	Ax. 12, I.	∴ BA, ED meet, if produced.
6	Sup.	Let them meet in the · F.
7	H. 28, I.	then ∵ ∠ABC = DCE, ∴ BF ‖ CD;
8	H. 28, I.	& ∴ ∠ACB = ∠DEC, ∴ AC ‖ FE;
9	Def. A, I.	∴ FACD is a ▱,
	34, I.	and ∴ AF = CD and AC = FD;
10	D. 8. 2, VI.	And in △FBE, ∵ AC ‖ FE;
		∴ BA : AF = BC : CE;
11	D. 9. 7, V.	But AF = CD, ∴ BA : CD = BC : CE;
12	16, V.	& *alt.* AB : BC = DC : CE.
13	D. 7. 2, VI.	Again, ∵ CD ‖ BF, & BC : CE = FD : DE;
14	D. 9. 7, V.	but F D = A C, ∴ BC : CE = AC : DE;
15	16, V.	& *alt.* BC : CA = CE : ED.
16	D. 12 & 15.	And ∵ AB : BC = DC : CE,
		& BC : CA = CE : ED;
17	22, V.	∴ *ex æquali* BA : AC = CD : DE.
18	Rec.	∴ *The sides about the eq. angles, &c.* Q. E. D.

Cor. I. *If diverging lines* BF, BK, BE, *cut parallel lines* AC, FE, *the parallel lines will be cut proportionally*, i. e., C I : I A = EK : KF.

D. 1	H.	∵ △ B I C is similar to △ B K E;
2	4, VI.	∴ B I : I K = C I : E K,
		and B I : I K = I A : K F;
3	11, V.	∴ C I : E K = I A : K F;
4	16, V.	and ∴ *alt.* C I : I A = E K : K F.

Cor. 2. *If two par. st. lines*, CA, EF, *be cut by any number of diverging lines*, BF, BH, BK, BE, *the parallels will be similarly cut in the · s of section.*

Cor. 3. *In a triangle* B F E, *a line from the vertex*, B H, *bisecting the base* E F *in* H, *also bisects the parallel to the base*, CA.

Cor. 4. *A parallel to the base of a triangle cuts off a similar triangle.*

Sch. 1. In similar figures the *homologous sides* are those which lie between equal angles as A B and C D, and also B C and C E, the ∠s B F E and F B E, being equal to the ∠s C D E and D C E;—and ∠s F B C, B E F, being also eq. to ∠s D C E and C E D. *Homologous terms* are the antecedents or consequents of a proportion.

2. It can be said only in the case of triangles, that, if their angles are equal their sides about their equal angles are proportionals;—in rectangles and other equiangular figures of more than three sides, the corresponding sides are not necessarily proportionals; neither does it follow, as in triangles, that, if their sides are proportionals their angles are equal.

PROP. 5.—THEOR. *(Important.)*

Conversely. *If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular; and the equal angles shall be those which are opposite to the homologous sides.*

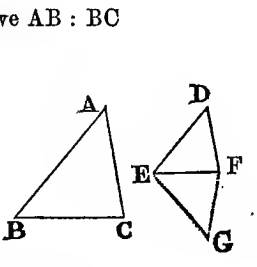
Or, *Triangles whose sides are proportional are equiangular.*

CON. 23, I. At a given · in a given line to make a rectil. ∠ = a rectil. ∠

DEM. 32, I.—4, VI.; 11, V.; 9, V.; 8, I.—Triangles having the three sides of the one equal to the three sides of the other, have the ∠s equal which are contained by eq. sides.

4, I. If two △s have each two sides and their included ∠ equal, the △s are eq. in every respect.

E. 1	Hyp. 1	Let △s ABC, DEF, have AB : BC = DE : EF.
2	,,	& BC : CA = EF : FD; and *ex. æq.* BA : AC = ED : DF;
3	Conc.	then △ A B C is eq. ang. with △ DEF, the eq. ∠s being opp. the homologous sides; *i. e.* ∠ ABC = ∠ DEF; ∠ BCA = ∠ EFD; and ∠ BAC = ∠ EDF.
C. 1	23, I.	At .s E. F, in EF make ∠ FEG = ∠ ABC, and ∠ EFG = ∠ BCA;



2	32, I.	∴ rem. ∠ EGF = rem. ∠ BAC, and △ GEF, is eq. ang. with △ ABC.
D. 1	32, I.	∴ ∠ EGF = ∠ BAC, and △ GEF eq. ang. with △ ABC;
2	4, VI.	∴ AB : BC = GE : EF;
3	H. 11, V.	but ∵ AB : BC = DE : EF; ∴ DE : EF = GE : EF;
4	9, V. and *Sim.*	∴ DE = GE; and so, DF = FG;
5	D. 4 & C.	∴ in △s DEF, GEF, the 3 sides = the 3 sides, *i. e.* DE = GE, EF com. and DF = FG.
6	8, I. & 4, I.	∴ ∠ DEF = ∠ GEF, ∠ DFE = ∠ GFE; and ∠ EDF = ∠ EGF.
7	D, 6. C.	And ∵ ∠ DEF = ∠ GEF, and ∠ GEF = ∠ ABC;
	Ax. 1, I.	∴ ∠ ABC = ∠ DEF;
8	*Sim.*	So ∠ ACB = ∠ DFE, and ∠ at A = ∠ at D.
9	Conc.	∴ △ ABC is eq. ang. to △ DEF.
10	Rec.	∴ *If the sides of two Triangles, &c.* Q. E. D.

Sch. Propositions 47 and 48 of Bk. I, and 4 and 5 of Bk. VI. establish the most important properties in the Elements of Geometry, and may be regarded as universal principles in every kind of rectilineal Measurement. They are the foundation of the application of Algebra to Geometry; and, inasmuch as every rectilineal figure may be resolved into triangles, and each triangle by a perpendicular from the vertex into two rt. angled triangles, they really include the solution of all problems relating to right-lined figures.

Use & App. I. The Theory of Representative Value, by which a small picture, executed with care to keep a due proportion between all its parts, becomes an accurate index to the human features, or of the wide spread earth itself,—*that* Theory finds its sure support on the truth, that Equiangular Triangles have their sides Proportional; and if their sides are proportional they are equiangular. It is on this Principle that Trigonometry is founded, for the sides of a Triangle have the same ratio as the sines of their opposite angles. The *Triangulation* practised in a Survey, ascertains actual distances, but the Map, founded on the measurements and calculations, is throughout constructed on the principle, that the small spaces are proportional to the real spaces; and though the lines are not drawn on the map which represent the triangle formed, for instance, by three hills not in the same st. line—yet the points themselves—

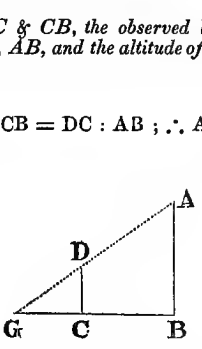
the vertices of the several angles are noted down, and their accuracy depends on the condition, that the triangle on paper is in exact ratio to the triangle in the field.

To give a general idea of the Use of Prop. 4, and of its Converse, we may state, that it is the ordinary practice in ascertaining the Lengths of inaccessible lines to construct or draw small triangles like to the large ones existing, or conceived of as existing on the ground; and as we have just stated, the principle on which this is done is derived from the last two propositions. On their truth also depends the accuracy of such Instruments as the Staff, the Quadrant, and the Geometrical Square, on which are formed triangles similar or equiangular with those which we intend to measure. Besides to take the Plane of a Place, or the outlines of a Country, from two stations, is but a continued application of the principle, that the sides of equiangular triangles are proportional.

II.—The DEDUCTIONS, for Practical Purposes, made from this truth may be arranged under eight Problems and Formulas.

PROB. 1.—*Given GC & CB, the observed length of the shadows of two perpendicular objects, DC, AB, and the altitude of one, DC ; to find the altitude of the other* AB.

$$\because GC : CB = DC : AB ; \therefore AB = \frac{CB \times DC}{GC}.$$

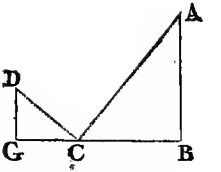


EX.—A tower AB, casts on the level ground a shadow of 140 feet ; a pole DC, standing perpendicularly, a shadow of 6 feet ; the length of DC is 10 feet ; required the height of the tower.

$$\text{Here, } 6 : 140 = 10 : AB ; \therefore AB = \frac{140 \times 10}{6} = \frac{1400}{6} = 233\tfrac{1}{3} \text{ feet.}$$

N.B.—It is said that THALES, B. C. 600, taught the Egyptians to measure the heights of the pyramids by this method.

PROB.—*By means of a mirror placed horizontally at C, the angle of incidence, ACB, being always equal to the angle of reflection, DCG :—to ascertain the height AB of a perpendicular object.*



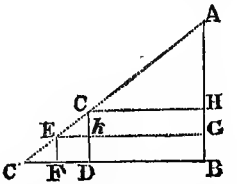
The Observer DG, standing upright, notes the height of his eye above the horizontal line GB; the distance GC of his feet from the mirror; and also the distance CB, of the mirror from the foot of the perpendicular; two equiangular triangles are thus formed, DGC & ABC; again GC : CB = DG . AB; and

$$\therefore AB = \frac{DG \times CB}{GC}.$$

Ex. The height of the eye, DG = 5·5 ft; the distance GC = 6 feet; and CB = 96 ft; required AB.

$$\text{Here } 6 : 96 = 5 \cdot 5 : AB; \therefore AB = \frac{96 \times 5{\cdot}5}{6} = 88 \text{ feet.}$$

PROB III.—*To find the height of a perpendicular object, AB, by means of two unequal rods or poles, CD, EF, placed perpendicularly on the horizontal line CB.*



Set CD and EF, so that A may be seen in the same line with E, C, the tops of the two poles; measure EF & CD, FD and DB, *i. e.*, E*k* & *k* G;

$$\text{then } Ek : kG = kC : GA; \text{ *i. e.* } GA = \frac{kG \times kC}{Ek}; \text{ and } AB = GA + EF.$$

Ex. By measurement EF = 5 ft; CD = 10 ft; FD = E*k* = 6 ft; and DB = *k* G = 24 ft; required the height BA.

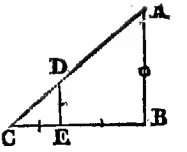
$$\text{Here } 6 : 24 = (10-5) : GA; \therefore GA = \frac{24 \times 5}{6} = 20 \text{ ft};$$

$$\& \ 20 + 5 = 25 \text{ ft} = BA.$$

PROB. IV. *By means of one pole DE, placed perpendicularly, to ascertain the altitude BA of a perpendicular object.*

Let DE be set upright, where A can be seen in a st. line with its top from C; measure CE, DE & EB; then; CE : EB = DE : AB;

whence $AB = \frac{EB \cdot DE}{CE}$

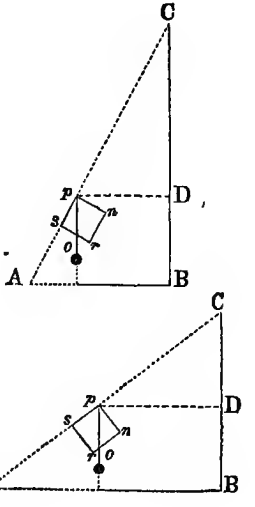


Ex. At a distance CE of 10 ft from DE = 7 ft, A can be seen in a line with point D; and the distance from E to B is 21 feet; what is the altitude BA?

Here 10 : 21 = 7 : BA; *i. e.* $BA = \frac{21 \times 7}{10} = 14 \cdot 7$ feet.

PROP. V. *By means of a Geometrical Square to measure the height of an object, CB, the foot only of which we can reach.*

Let AB represent the horizontal line; *p* D a parallel to it; DB the height of the instrument above AB; CD the height of the object above *p* D; *sp* the edge of the geometrical square along which the top, C, of the object is seen; *sr*, *rn*, and *pn* graduated edges, each of 100 eq. parts, and *p* the point of suspension for the plummet.



From the place of observation measure the distance p D, and the height of the instrument, DB; direct sp towards the object C, so that s, p, and C may be points in one st. line; and note the number of parts in sr, or rn, cut off by the plummet line.

1°. When the plummet line cuts sr. in o, the triangles $p\,s\,o$ and C D p are eq. ang.; for $\because p\,o \parallel$ C B, $\angle\, s\,p\,o = \angle\, p$ C D and $\angle\, s = \angle$ D, both being rt. $\triangle$s;

$$\therefore\ so : sp = p\,\mathrm{D} : \mathrm{CD};\ \text{whence } \mathrm{CD} = \frac{sp \cdot p\,\mathrm{D}}{so};\ \text{and } \mathrm{CB} = \mathrm{CD} + \mathrm{DB}.$$

2°. When the plummet line cuts $r\,n$ in o, the $\triangle$s, $o\,n\,p$ and C Dp, are eq. ang.; for $p\,o \parallel$ C B, $\angle\, o\,p\,n = \angle$ C p D, and $\angle\, p\,o\,n = \angle\, p$ C D;
$\therefore p\,n : n\,o = p\,\mathrm{D} : \mathrm{D\,C}$;

$$\text{whence } \mathrm{DC} = \frac{n\,o \cdot p\,\mathrm{D}}{p\,n},\ \text{and } \mathrm{CB} = \mathrm{CD} + \mathrm{DB}.$$

Ex. 1. The distance p D = 80 ft.; $so = 60$ eq. pts., and D B = 6 ft.; required C B.

$$\text{Here } 60 : 100 = 80 : \mathrm{CD};\ \therefore \mathrm{CD} = \frac{80 \times 100}{60} = \frac{8000}{60} = 133\tfrac{1}{3}\ \text{ft.}$$

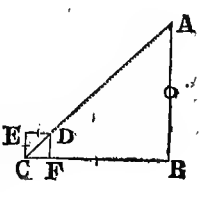
And $133\frac{1}{3}$ ft. + 6 = $139\frac{1}{3}$ = CB.

Ex. 2. The plummet line $p\,o$ cuts off $n\,o = 20$ eq. pts.; DB = 5 ft.; and p D = 90 ft.; required CB.

$$\text{Here } 100 : 20 = 90 : \mathrm{DC};\ \text{whence } \mathrm{DC} = \frac{20 \times 90}{100} = 18\ \text{ft.};$$

and 18 + 5 = 23 ft. = CB.

PROB. VI. *By means of a geometrical square, with an index to point out the extremities, whether altitude or otherwise, of two objects, one of which is in the vertex of a right angle; to measure the distance.*



Place the sqnare, each side of which is divided into 100 eq. parts, on a line, CB, which is at rt. $\angle$s to AB; and along the moveable index CD, observe the object A, and note the number of eq. parts cut off by the index on the edge of the square, DE or CE; then. $\because$ EC $\parallel$ AB, and ED $\parallel$ CB, $\therefore \angle$ E = $\angle$ B, $\angle$ ECD = $\angle$ CAB, and $\angle$ EDC = $\angle$ ACB; thus the $\triangle$s, ABC and CED are eq. ang., and $\therefore$ DE : EC = CB : BA; $\therefore \mathrm{BA} = \frac{\mathrm{EC} \cdot \mathrm{CB}}{\mathrm{DE}}$.

T

Ex. 1. From the foot, B, of the perp. AB, to C, I measure BC,—it is 40 feet; the index CD, cuts off 100 eq. pts., as ED;—EC also being 100 eq. pts.; required AB.

$$\text{Here } 100 : 100 = 40 : BA;\ \therefore BA = \frac{100 \times 40}{100} = 40 \text{ feet.}$$

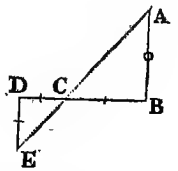
Ex. 2. The index cuts ED at 60; as before CB = 40 feet, EC or DF = 100 eq. pts.; required AB.

$$\text{Here } 60 : 100 = 40 : AB;\ \therefore AB = \frac{100 \times 40}{60} = 66\tfrac{2}{3} \text{ feet.}$$

PROB VII. *To find, by aid of the cross staff, or theodolite, the distance of A from B without approaching A.*

At rt. ∠s to AB, lay down the line BD; and in BD take a · C, at which place a staff; from D, bnt on the other side of BD, lay down DE perp. to BD, and measure along DE, until E, C & A are in one st. line; then, ∵ the ∠s at C are equal, ∠D = ∠B, and ∠E = ∠A; ∴ the △s ABC, CDE are equiangular;

$$\therefore CD : CB = DE : AB;\ i.\ e.\ AB = \frac{CB \cdot DE}{CD}.$$



Ex. The measurement of BC = 200 links; that of CD = 60 links, and of DE = 50; how many links are there between the · s B and A ?

$$\text{Here } 60 : 200 = 50 : BA;\ i..\ e.,\ BA = \frac{200 \times 50}{60} = \frac{10000}{60} = 166\tfrac{2}{3} \text{ links}$$

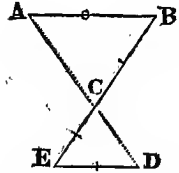
PROB. VIII.—*By means of a line, DE, of which the length is known, to find the length of its parallel, AB, one end of which, B, only can be approached.*

Set out the line BE, and on it ascertain the point C, where a line joining A and D would cross BE; measure BC, CE;

∵ the △s ACB and DCE are eq. ang.,

∴ CE . CB = DE : AB;

$$i.\ e.,\ AB = \frac{CB \,.\, DE}{CE}.$$



Ex. The parallel DE measnres 800 links; the side EC, 900 links, and CB, 1800 links; required the links in AB.

Here $900 : 1800 = 800 : AB$; $\therefore AB = \dfrac{1800 \times 800}{900} = 1600$ links.

Obs. An instrument in common use,—*the Proportional Compasses*, is an example of the last Problem; in this instrument, the common centre C, about which the legs turn, is changed at pleasure, yet so as to preserve any given proportion between EC and CB, and consequently between ED and AB. These compasses give great facility in enlarging or diminishing a plan or map; and with sufficient accuracy for many purposes,—the principle being perfect, but the application liable to fail for want of the requisite care.

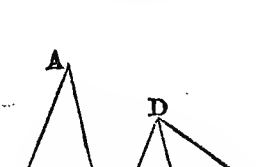
The Pentagraph and Eidograph are also instruments, of which the principle is, that the arms move parallel to each other, and consequently that the triangles formed are always similar, being exemplifications of Euclid's Prop. 4, Bk. VI. For accuracy and precision the Eidograph is far superior to the Pentagraph. See Bradley's *Pract. Geom.* p. 59.

Prop. 6.—Theor.

If two triangles have one angle of the one equal to the one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Con. 23, I.—32, I, Dem. 4, VI. 11, V. 9, V. 4, I. Ax. 1, I. 32, I.

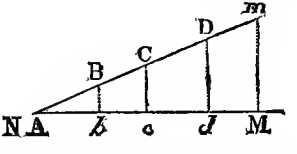
E.		
1	Hyp. 1	Let $\triangle$s ABC, DEF have $\angle BAC = \angle EDF$,
2	„ 2	and $BA : AC = ED : DF$;
3	Conc.	then $\triangle$s ABC, DEF are eq. ang.; *i. e.*, $\angle B = \angle E$, and $\angle ACB \angle DFE$.



C. 1	23, I.	At · s D, F in DF make $\angle FDG = \angle BAC$, or $\angle EDF$; & $\angle DFG$; $= \angle ACB$;
2	32, I.	∴ rem. $\angle B$ = rem. $\angle G$.
D. 1	C, 4, VI.	∵ $\triangle DGF$ is eq. ang. with $\triangle ABC$; ∴ BA : AC = GD : DF;
2	H. 2.	but BA : AC = ED : DF;
3	11, V. 9, V.	∴ ED : DF = GD : DF; & ∴ ED = DG.
4	C.	And ∵ DF com., ED = GD, & $\angle EDF = \angle GDF$;
5	4, I.	∴ EF = FG, $\triangle EDF = \triangle GDF$, $\angle DFG = \angle DFE$, & $\angle G = \angle E$;
6	C. 2. Ax. 1, I.	but $\angle DFG = \angle ACB$, ∴ $\angle ACB = DFE$;
7	H. 1. 32, I.	& ∵ $\angle BAC = \angle EDF$, ∴ rem. $\angle B$ = rem. $\angle E$.
8	Conc.	∴ $\triangle ABC$ is equiangular to $\triangle DEF$;
9	Rcc.	∵ *If two triangles have one angle &c.* Q. E. D.

Cor. 1.—It may be added, *that the sides also about each pair of equal angles shall be proportional*; i. e., by 4, VI, AB : BC = DE : EF; & BC : CA = EF : FD.

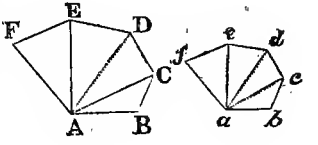
Cor. 2. *If through any points, b, c, d,* &c., *of a straight line* NM, *parallels b* B, *c* C, *d* D &c., *be drawn, which are proportional to the distances* A *b*, *b c*, *c d* &c., *from any point A on the 1st line, then their extremities* B, C, D &c., *will be on the rt. line* DA *passing through* A.



∵ A*b* : *b* B = A *c* : *c* C; & $\angle$ A *b* B = A *c* C;
∴ $\triangle$ A*b* B is similar to $\triangle$ A *c* C; & ∴ $\angle$ BA *b* = $\angle$ CA *c*;
and ∵ A *b* coincides with A *c*; ∴ AB & AC, being on the same side of NM, also coincide.

N. B.—The equation of a rt. line in Analytical Geometry depends on this principle. See LARDNER'S *Euclid*, p. 184.

USE & APP. 1. Since all rectilineal figures may be divided into triangles, if any two rectilineal figures, A B C D E F *a b c d e f*, thus divided have all their angles but two equal in order, $\angle B = \angle b$, $\angle C = \angle c$, $\angle D = \angle d$, $\angle E = \angle e$, and the corresponding sides about the equal angles proportionals; then their remaining angles shall be equal each to each, $\angle A = \angle a$, $\angle F = \angle f$, and their remaining sides AF, *a f*, in the same ratio with any other two corresponding sides A B, *a b*. Thus the one figure shall be similar to the other by Def. 1, VI.



E. 1	Pst. 1, I.	Join AC, AD, AE; *ac, ad, ae.*
D. 1	H.	∵ △s ABC, *a b c*, have $\angle B = \angle b$, & AB : BC = *ab* : *bc*;
2	6, VI.	∴ $\angle$ ACB = $\angle$ *a c b*, and AC : CB = *a c* : *c b*;
3	H. & D. 2.	and ∵ $\angle$ BCD = $\angle$ *b c d*, and $\angle$ACB = $\angle$ *a c b*;
4	Ax. 3, I.	∴ $\angle$ ACD = $\angle$ *a c d*.
5	D. 2. & H.	Also ∵ AC : CB = *ac* : *c b*; and CB : CD = *c b* : *c d*;
6	22, V.	∴ *ex. æquo.* AC : CD = *a c* : *c d*;
7	6, VI.	∴ △s ACD and *a c d* are similar.
8	Sim.	And thus △s ADE, AEF, are sim. to △s *a d e*, *a e f*;
9	Conc.	∴ $\angle$ F = $\angle f$; and $\angle$s at A = $\angle$ s at *a*.
10	H.	Also ∵ AF : AE = *a f* : *a e*; and AE : AD = *a e* : *a d*; and AD : AC = *a d* : *a c*, and AC : AB = *ac* : *a b*:
11	22, V. 16, V.	∴ *ex. æq.* AF : AB = *a f* : *a b*; and *alt.* AF : *a f* = A B : *a b*.

2. Similar rectilineal figures may also be divided into the same number of similar triangles; and their homologous or corresponding sides, are to one another in the same ratio, each to each.

Prop. 7.—Theor.

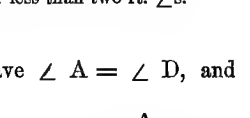
If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a rt. angle, or if one of them be a rt. angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

Or, "If two triangles have one angle equal to one angle, and the sides about the other angle proportional, and the remaining angles either both less, or both not less than a right angle, the triangles will be equiangular, and will have those angles equal about which the sides are proportional."—Euclid.

Dem. 23, I. 32, I. 4, VI. 11, V. 9, V. 5, I. 13 I. The ∠s which one st. line makes with another upon one side of it, are either rt. ∠s, or together, equal to two rt. ∠s

17, I. Any two ∠s of a △ are together less than two rt. ∠s.

E. 1	Hyp. 1	Let △s ABC, DEF have ∠ A = ∠ D, and AB : BC = DE : EF;
2	,, 2	and let the 3rd ∠s ACB, DFE, be < or ⩽ a rt. ∠, or one be a rt. ∠;
3	Conc. 1	then ∠s ABC, ACB, of the one = respectively ∠s DEF and EFD, of the other,
4	,, 2	and AB : BC = DE : EF, and BC : CA = EF : FD.

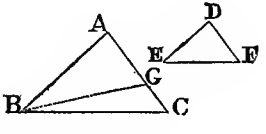


Case I. Let each of the rem. ∠s C and F be < a rt. ∠;

	Conc.	then $\triangle$s ABC and DEF, are eq. ang. $\angle ABC = \angle DEF$, and $\angle C = \angle F$.
D. 1	Sup. 23, I.	If $\angle ABC \neq \angle DEF$, let $\angle ABC > \angle DEF$; and make $\angle ABG = \angle DEF$.
2	H. C. D. 1, 32, I	And $\because \angle A = \angle D$, $\angle ABG = \angle DEF$, and $\angle AGB = \angle DFE$;
3	4, VI.	$\therefore$ $\triangle$ ABG is eq. ang. to $\triangle$ DEF; and $\therefore$ AB : BG = DE : EF;
4	H. 11, V.	but as DE : EF so AB : BC, $\therefore$ AB : BC = AB : BG;
5	D. 4, 9, V.	and $\because$ AB : BC = AB : BG, $\therefore$ BC = BG;
6	5, I.	and $\therefore$ $\angle BGC = \angle BCG$.
7	H. 13. I.	But $\angle BCG <$ rt. $\angle$, $\therefore \angle BGC <$ rt. $\angle$, and $\therefore$ adj. $\angle AGB >$ rt. $\angle$.
8	D. 2.	Now $\angle AGB = \angle F$, $\therefore \angle F >$ rt. $\angle$;
9	H.	but $\angle F <$ rt. $\angle$;—which is absurd.
10	Conc.	$\therefore \angle ABC$ not $\neq$ DEF, *i.e.* $\angle ABC = \angle DEF$, and $\angle A = \angle D$;
11	32, I.	$\therefore$ rem. $\angle C$ = rem. $\angle F$.
12	Conc.	$\therefore$ $\triangle$ ABC is eq. ang. to $\triangle$ DEF.

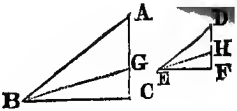
CASE II. Let $\angle$s C & F be each $\not<$ a rt. $\angle$;

	Conc.	then $\triangle$ ABC, is eq. ang. to $\triangle$ DEF.
D. 1	D. 1—5 Case I.	As before BC = BG, $\therefore \angle C = \angle BGC$;
2	H.	but $\angle C \not<$ rt. $\angle$, $\therefore \angle BGC \not<$ rt. $\angle$;
3	17, I.	$\therefore$ two $\angle$s of $\triangle$ BGC, are $\not<$ 2rt. $\angle$s; which is impossible;
4	Conc.	and $\therefore$ $\triangle$ ABC is eq. ang. to $\triangle$ DEF, as in Case I.



CASE III. Let one of the ∠s, C, F, namely ∠C, be a rt. [illegible]

	Conc.	Then also △ ABC is eq. ang. to △ DEF.
D. 1	Sup. 23, I.	If not, at B in AB, make ∠ ABG, = ∠ DEF;
2	Case 1. 5, I.	then BG = BC, and ∴ ∠ BCG = ∠ BGC;
3	H. Ax. 1, I.	but ∠ BCG is a rt. ∠, ∴ ∠ BGC is a rt. ∠;
4	17, I.	∴ two ∠s of △ BGC are ≮ 2 rt. ∠s; which is an impossibility;
5	Conc.	∴ △ ABC is eq. to ang. △ DEF.
6	Rec.	∴ *If two triangles have one angle, &c.* Q. E. D.



SCH. When two angles are both greater, or both less than right angles they are both said to be *of the same affection*; and in enunciating this Prop, instead of "both greater or both less than right angles," it is not unusual to say, "both of the same affection."

USE AND APP.—In Book I. Propositions 4, 8 and 26 contain the *criteria* of the *equality* of two triangles; and in Book VI. Propositions 4, 5, 6 and 7 may be classed together as giving the conditions on which we declare the *similarity* of two triangles. Equality in triangles is absolute, not in area only;—but the similarity is likeness of *shape*, not identity of *size*.

The *criteria* of similarity are;

1°,—The equality of the three angles, 4, VI.;

2°.—The identity of the ratios of the respective sides, 5, VI.;

3°.—The equality of two angles, one in each triangle,—and the identity of the ratios of the containing sides, 6, VI.;

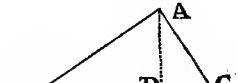
4°.—The identity of the ratios of two sides in each triangle,—the equality of an angle in each opposite one pair of homologous sides;—and each of the remaining angles opposite the other pair of homologous sides less than a right angle, or one of them a right angle.

Universally, if triangles fulfil any one of these four conditions of similarity, the sides about the equal angles are proportional.

PROP. 8.—THEOR. *(Important.)*

In a rt. angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle and to one another.

DEM. Ax. 11, I. All rt. ∠s are equal to one another. 32, I. 4, VI. Def. 1, VI.



E. 1	Hyp. I.	Let △ABC have BAC a rt. ∠;
2	,,	& from A let AD be ⊥ BC the hypotenuse;
3	Conc.	then △s ABD, ADC, are sim. to △ABC, and to each other.
D. 1	Ax. 11, 1. 32, I.	∵ ∠BAC = ∠ADB, & ∠B com.; ∴ rem, ∠ACB = rem. ∠BAD
2	4, VI.	∴ △ABC is eq. ang. to △ABD, and the sides propls.;
3	Def. I, VI.	∴ △ABC is similar to △ABD.
4	*Sim.*	So, △ADC is eq. ang. and sim. to △ABC;
5	D. 2 & 4.	and ∴ △ABD is eq. ang. & sim. to △ACD, which is eq. ang. and sim. to △ABC.
6	Rec.	Therefore *in a rt. angled triangle, &c.* Q. E. D.

Cor. 1. *The perpendicular*, A D, *from the vertex of the rt.* ∠ BAC *to the opposite side* BC, *is* a mean proportional *between the segments* BD, DC *of this side; and also each of the sides*, BA, AC, *including the rt.* ∠ is a mean proportional *between the opposite side* BC, *and the segment of it*, BD, *or* DC *adjacent to that side*, BA *or* AC.

D. 1	H.	∵ △ ADB is sim. to △ ADC,
	Def. 1, VI.	∴ BD : DA = DA : DC;
	Obs. Def. 10, V.	*i. e.* DA is a mean proportional between BD and DC.
2	H. 4, VI.	Also, ∵ △ ABC is sim. to △ DBA,
		∴ BC : BA = BA : BD;
		i. e. A B is a mean proportional between BC and BD.
3	H. 4, VI.	And ∵ △ ABC is sim. to △ ACD,
		∴ BC : CA = CA : CD;
		i. e. AC is a mean proportional between BC and CD.

Cor. 2. *The segments* BD, CD, *of the hypotenuse, made by the perp.* AD, *are to one another as the squares on the sides of the rt.* ∠, $BA^2 : AC^2$.

∵ by D. 2 & 3 of Cor. 1. 8, VI. BD . BC : CD . BC = BA^2 : AC^2;
if we divide the 1st & 2nd terms of the analogy by BC,
then BD : CD = BA^2 : CA^2

Cor. 3. *The squares on the sides about the rt.* ∠ *and on the hypotenuse are to each other as the segments of the hypotenuse, made by the perp.* AD, *and the hypotenuse itself.*

∵ BD.BC : CD.BC : BC.BC = AB^2 : CA^2 : BC^2;
∴ on dividing the three terms by BC,
BD : CD : BC = AB^2 : CA^2 : BC^2.

N. B.—Several other *subsidiary* Corollaries might be added,—but the most important deductions have been given, and we subjoin only ;

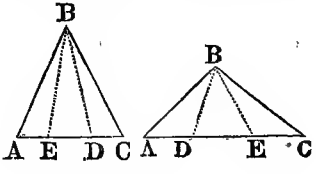
COR. 4. *If the base of a triangle*, BC, *the two sides*, AB, AC, *and the perpendicular* AD, *be four proportionals, the triangle must be right angled.*

D. 1	Hyp.	∵ in △ ABC, and in one of the component △s ABD, two sides are proportionals ;
2	,,	and the ∠s opp. one pair of homologous sides are equal ;
3	,,	and also ∵ of the ∠s opp. the other pair of homol. sides one is a rt. ∠ ;
4	7, VI.	∴ the whole △ ABC is sim. to the component △ ABD ;
	Conc.	and ∴ also △ ABC is rt. angled.

SCH. 1. The 8th Proposition, and the deductions that may be made from it are particular cases of a more general principle; namely,

If from the vertex, B, *of a* △ ABC, *two lines* BD, BE. *be drawn to the base*, AC, *making the angles at the base*, BDA. *and* BEC, *or their supplements each equal to the vertical angle* ABC; *then the* △*s* BDA, BEC, *formed by those lines, and by the segment* DA *or* EC *which each cuts off, shall be similar to the whole triangle and to one another.*

E. 1	Hyp.	Let ABC be a △, and from vertex B, let there be drawn BD, BE, making ∠ BDA = ∠ BEC = ∠ ABC;
2	Conc.	then △ BED is isosc.; and △BDA is eq. ang. to △BEC which is eq. ang. to △ ABC.
D. 1	Remk.	When ∠ ABC > rt. ∠, ∠s BDA and BEC are ext. ∠s. at the base of the isosc. △ BDE;



2	Remk. 2.	but when $\angle$ ABC $<$ rt. $\angle$, then those $\angle$s are internal.
3	„ 3.	As the obtuse $\angle$ ABC decreases, the base DE diminishes and the sides BD, BE approach;
4	„ 4.	when $\angle$ ABC becomes a rt. $\angle$, BD and BE coincide;
5	„ 5.	and after $\angle$ ABC becomes $<$ a rt. $\angle$, BD and BE change sides.
6	„ 6.	In the general proposition, this isosc. $\triangle$ DBE, is what the perpendicular is, when the given $\triangle$ is rt. angled.
7	Conc.	$\therefore$ the sides of this $\triangle$ DBE, and the triangles under them, and the sides of the given $\triangle$ ABC, possess many of the properties already proved in the case of a rt. $\angle$d $\triangle$; for instance, as in Cor. 1, Pr. 8, VI.;

1. AC : AB = AB : AD, where AB is a mean proportional.

2. AC : CB = CB : CE, where CB is a mean proportional.

3. AC : AB = BC : BE,

4. AD : BD = BD : CE, where BD is a mean proportional.

   or since BD = BE; AD . BE = BE : CE, where BE is a mean propl.

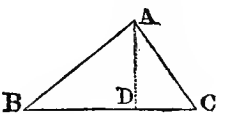
5. $\because$ AD : BD = AB : BC, $\therefore$ the segments AD and EC are in the duplicate ratio of the sides AB and BC.

2. "Hence in a right-angled triangle the segments of the hypotenuse by the perpendicular, are in the duplicate ratio of the sides." LARDNER'S *Euclid*, pp. 187, 188.

USE & APP. 1. The first Corollary of Pr. 8, bk. VI. supplies the principle on which a very clear and brief demonstration may be given of Prop. 47, bk. I.

$\because$ BC $\times$ BD = BA^2,
and BC $\times$ CD = CA^2;

$\therefore$ *Add.* BC . BD + BC . CD
= $BA^2 + CA^2$.

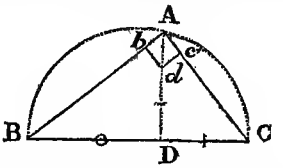


Or, BC $\times$ (BD + CD) = $BA^2 + CA^2$;

i. e. BC $\times$ BC, or $BC^2 = BA^2 + CA^2$.

2. Also according to this Proposition, and by aid of a square, *inaccessible distances*, as DB, *may be measured.*

At D raise the perp. DA, and measure it;
And at A place a square, so that by looking along one of its sides A*b*, the point B may be seen in the same st. line with A*b*;



And along the other side A*c*, the point C may be seen; and measure DC;

Then, $\because$ CD : DA = DA : DB; $\therefore$ DB $= \dfrac{DA^2}{CD}$.

Ex. Suppose AD = 3, and CD = 2·25; then DB $= \dfrac{9}{2.25} = 4$.

3. *In a circle any chord, as* BA, *is a mean proportional between the diameter* BC, *and that segment of the diameter* BD, *which is drawn from one extremity of the chord,* B, *and cut off by a perpendicular,* AD, *let fall from* A *the other extremity of the chord.*

D. 1	31, III. & C.	$\because$ $\triangle$s BAC and ADB have each a rt. $\angle$, and $\angle$ B common;
2	4, VI.	$\therefore$ the $\triangle$ BAC is eq. ang. to $\triangle$ ADB, and $\therefore$ BC : BA = BA : BD; *i. e.* the chord BA is a mean propl. to BC & BD.
3	8, VI.	And $\because$ $\triangle$s BAC, BDA and ADC are similar, the segments of the hypotenuse are in the duplicate ratio of the sides.

PROP. 9.—PROB.

From a given st. line to cut off any part required, i. e., any measure or submultiple.

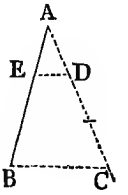
CON.—3, I. 31, I.

DEM.—2, VI. 18, V. *Componendo.* If Ms, taken separately, be propls, they shall also be propls when taken jointly ; *i. e*, if first : 2nd = 3rd : 4th ; then 1st + 2nd : 2nd = 3rd + 4th : 4th.

D, V—If 1st : 2nd = 3rd : 4th, if 1st a *m* or *pt* of the 2nd, the 3rd is the same *m* or *pt* of the 4th.

Def. 1, V.—A less M is a *pt* of a greater when the less measures the greater.

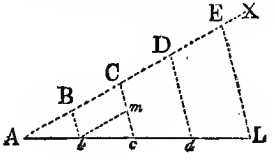
E. 1	Dat.	Let AB be a given st. line ;	
2	Quæs.	to cut off from it any *pt.* required.	
C. 1	Pst. 1, I.	From. A draw AC making any ∠ with AB ;	
2	3, I.	in AC take any . D, and make AC the same *m* of AD that AB is of AE the *pt* to be cut off.	
3	31, I.	join BC, and draw ED ∥ BC ;	
4	Sol.	then AE is the submultiple required ;	
D. 1	C. 3.	∵ ED ∥ BC one of the sides of △ ABC ;	
2	2, VI.	∴ CD : DA = BE : EA ;	
3	18, V.	and *compon.*, CA : AD = BA : AE ;	
4	C. D, V.	but CA is a *m* of AD ; ∴ BA the same *m* of AE ;	
5	Def. 1, V.	∴ AE the same *pt* of AB, that AD is of AC.	
6	Recap.	∵ *From a given st. line, &c.*	Q. E. F.



SCH. Prop. 10, Bk. I. by which a rt. line may be bisected, and its bisections also bisected, is a particular case of this Problem.

USE & APP. A simple extension of the Problem enables us; 1st, *to divide a given line,* AL, *into any number of equal parts.*

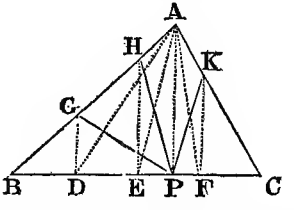
E. 1	Dat. & Quæs.	Let it be required to divide AL, into four eq. pts.	
C. 1	Pst. 1, I.	Draw AX, making any ∠ with AL ;	
2	3, I.	from A on AX set off four eq. spaces, *i. e.* AB = BC = CD = DE ;	



3	31, I.	Join EL, and through D, C, B draw ‖s to EL and cutting AL;
4	Sol.	then AL is divided into four eq. pts. in *b*, *c*, *d*, L.
D. 1	9, VI.	∵ A*b* is the same *pt.* of AL that AB is of A E; *i. e.* the 4th part;
2	*Sim.*	and ∵ A*b*, = *bc* = *cd* = *d* L;
3	Conc.	∴ AL is divided into four eq. parts.

2. *To divide a triangle*, ABC, *into any number of eq. parts*, say four, *by lines from a given point*, P, *in one of the sides, as* BC.

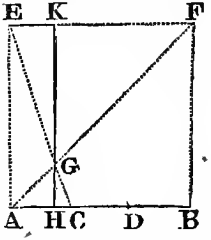
C. 1	Sch. 1. 9, VI.	Divide BC into four eq. pts. in D, E, F, and join AP;
2	31, I.	and through D, E, F draw DG, EH, FK each ‖ AP;
3	Pst. 1, I.	then join AD, AE, AF, and PG, PH, PK;
4	Sol.	and the △s BPG, GPH, HPK, KPC, each = ¼ △ ABC; *i. e.* by lines from P, △ ABC is divided into 4 eq. pts



D. 1	37, I.	∵ △ DPG = △ GAD, both being on GD and between the same ‖s GD, AP;
2	*Add.* Ax. 2, I.	to each add △ BGD, ∴ △ BPG = △ ADB.
3	C 1.	Next, ∵ BD = DE = EF = FC and the altitude com.
4	38, I.	∴ △s ABD, ADE, AEF, AFC, are equal; and each is ¼ △ ABC;
5	D. 3	but △ BPG = △ ADB, ∴ BPG = ¼ △ ABC.
6	*Sim.*	So △ GPA = △ DPA, and △ HPA = △ APE,
7	*Sub.*	take △ HPA from △ GPA, and △ APE from △ DPA,
8	Ax. 3, I. D. 4.	∴ rem. △ GPH = rem. △ AED = ¼ △ ABC.
9	C.	Now fig. PHAK = APH + APK; △ APH = △ APE and △ APK = △ APF;
10	Ax. 1, I.	∴ fig. PHAK = △ APE + △ APF = ½ △ ABC.
11	D. 5, 8, 10,	Now fig. ABPK = ¾ of △ ABC;
12	Ax. 3, I.	∴ rem. fig KPC also = ¼ of △ ABC;
13	Conc.	∴ △ ABC is divided into 4 eq. pts. by lines from P.

3. *Given the nth part of a line AB, to find the* $(n + 1)$ *th part.*

E. 1	Dat.	Given $AC = \frac{AB}{n}$, or $AB = n \,.\, AC$;
2	Quæs.	to find $\frac{AB}{n+1}$.
C. 1	46, I. Pst. 1, I.	On AB desc. a sq. ABEF, and join AF, EC cutting in G;



2	31, I.	through G draw HGK ∥ AE, cutting AB, EF, in H & K;
3	Sol.	then AH, or $EK = \frac{AB}{n+1}$; or $(n+1)\,AH = AB$.
D. 1	C.	∵ △s AHG, FKG, are eq. ang.; and △s AGC & EGF;
2	4, VI.	∴ AH : FK, or BH, = AG : GF = AC : EF, or AB.
3	Conc.	and ∴ BH = n. AH; and AB = $(n+1)$ AH;
		$\therefore AH = \frac{AB}{n+1}$. Q. E. F.

Prop. 10.—Prob.

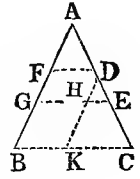
To divide a given st. line similarly, i. e., proportionally, *to a given divided st. line*; or *to divide a given st. line into parts that shall have the same ratios to one another which the parts of the divided given st. line have.*

Or, "To divide a given undivided line similarly to a given divided line." Euclid.

Con. Pst. 1, I. 31, I. 1, I.

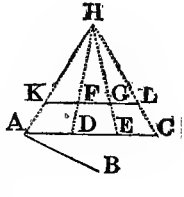
DEM. 34, 1. The opp. sides and ∠s of ▱s are eq. to one another, and the diam. bis. them. 2, VI. 7, V. 6, VI. 28, I.

E. 1	Dat. 1.	Given AB to be divided;
2	„ 2.	& AC a line divided at the · s D, E;
3	Quæs.	to divide AB similarly to AC.
C. 1	*Pon.* Pst. 1, I.	Let AB and AC form an ∠ at A; & join BC;
2	31, I.	through D,E, draw DF,EG, each ∥ BC; and also through D, DHK ∥ AB;
3	Sol.	then AB is div. in F,G, as AC is in D,E.
D. 1	C. 2. 34, I.	∵ FH, HB are ▱s; ∴ DH = FG, and HK = GB;
2	C. 2.	and ∵ HE ∥ KC, a side of △DKC;
3	2, VI.	∴ CE : ED = KH : HD;
4	C. 7, V.	but KH = BG, & HD = GF; ∴ CE : ED = BG : GF.
5	D. 2. 2, VI.	Again, ∵ FD ∥ GE in △AGE,; ∴ ED : DA = GF : FA;
6	D. 4 & 5.	and ∵ CE : ED = BG : GF, and ED : DA = GF : FA;
7	D. 4 & 4.	∴ AB is divided in the same proportion as AC. Q. E. F.



Or,

C. 1	1, I.	On AC desc. an eq. lat. △AHC;
2	3, I.	and from HA, HC, cut HK, HL, each = AB; & join KL;
3	Pst. 1, I.	from H draw lines to D,E, the divisions of AC;
4	Sol.	then KL = AB is div. in F,G, similarly to AC.
D. 1	C. 2. 6, IV.	∵ HK = HL, & HA = AC, ∴ △HKL is eq. an. to △HAC;



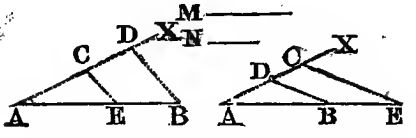
2	C. 1.	but AC = HA, ∴ KL = KH = AB.
3	*Sim.* 28, I.	So ∠HKL = ∠HAC; ∴ KL ∥ AC;
4	Conc.	and ∵ KL = AB ∴ AB is div. similarly to AC.

Q. E. F.

USE AND APP.—By aid of this proposition several nseful Problems may be solved.

PROB. 1. *To divide a given st. line,* AB, *internally, or externally, in a given ratio, as of* M : N.

C. 1	Pst. 1, I.	From A draw AX at any ∠ with AB;
2	3, I.	take AC = M, and CD = N; and join DB;
3	31, I.	Through C draw CE ∥ DB, and meeting AB, or AB produced,
4	Sol.	Then AB is divided in the ratio M : N.

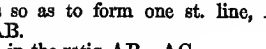


D. 1	C. 3.	∵ in △s AEC and ABD, CE ∥ DB;
2	2, VI.	∴ AE : EB = AC : CD = M : N;
3	Conc.	∴ AB is divided in E in the given ratio.

N. B. A line cannot he cut externally in a ratio of equality.

PROB. 2. *To find a harmonical mean between two given st. lines,* AB *and* AC.

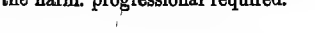
C. 1	*Pon.*	Place the lines so as to form one st. line, AC being set off on AB.
2	Prob. 1.	divide BC in D in the ratio AB : AC;
3	Sol.	then AD is the harmonical mean.



D. 1	2, VI. Def. A. VI.	∵ AB : AC = BD : DC; ∴ AB, AD and AC are in harm. progression;
2	Concl.	∴ AD is a harmonical mean between AB and AC.

PROB. 3. *To find a third harmonical progressional to two given st. lines,* AB *and* AC.

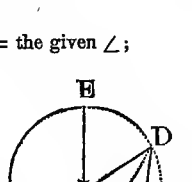
C. 1	*Pon.*	Set off A B on AC, and produce AC;
2	Proh. 1.	divide AC produced, in D, in the ratio AB : BC;
3	Sol.	then AD is the harm. progressional required.



D. 1	2, VI. & 16, V.	∵ AD : CD = AB : BC; and *alt.* AD : AB = CD : BC,
2	Def. A, VI.	∴ AB, AC and AD, are in harm. progression,
3	Concl.	and AD is the third harm. progress. to AB, AC.

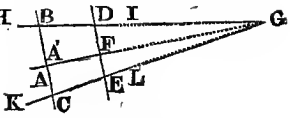
PROB. 4.—*To construct a triangle of which one side AB, the angle ADB opposite to it, and the ratio of the other sides are given.*

C. 1	33, III.	On AB desc. a ⊙ in which the ∠ = the given ∠;
2	1, III.	draw a diam. EC, at rt. ∠s to AB;
3	10, VI.	divide AB so that AF : FB = ratio of the sides;
4	Pst. 1. 2, I.	join CF, and produce CF to D in the ⊙ce;
5	Sol.	join AD, DB; and ADB is the △ required.
D. 1	C. 2. 3, III. 4, I.	∵ ∠s at G rt ∠s; ∴ AG = GB, and chord AC = CB.
2	28, III., 27, III.	Now arc AC = arc CB; ∴ ∠ ADC = ∠CDB, and CD bisects ∠ADB;
3	3, VI,	∴ AD : DB = AF : FB, which is the given ratio;
4	Conc.	∴ △ ADB is on AB, its vert. ∠ = a given ∠, and its sides, DA : DB = a given ratio.



PROB. 5. *Through a given · A, to draw a line, which, on being produced, would pass through the points of intersection of two given lines,* HI, KL, *without their being produced to meet.*

C. 1	Pst. 1, I.	Through A draw any line BC, cutting HI in B, & KL in C;
2	31, I. 10, V.	draw DE ∥ BC, and divide DE so that EF . FD = CA : AB;
3	Sol.	then AF is the line required.



D. 1	Sup.	For should HI, KL, have met in G, and GF cut BC in A′;
2	C.	then ∵ DF : FE = BA′ : A′C; ∴ BA′ : A′C = BA : AC;
3	18, V.	and *comp.* BC : A′C = BC : AC; ∴ A′ coincides with A;
4	Conc.	and AF is the line required.

N.B. This operation is frequently called for; and the *Centrolinead* is an instrument, invented by Mr. P. Nicholson, senior, for the purpose of drawing lines tending to the inaccessible point where two given lines, if produced, would meet. Another instrument for the same purpose is the invention of Mr. John Fary. BRADLEY'S *Pract. Geom.* p. 42. There is, however, a simpler instrument used by draughtsmen, consisting of three rulers stiffly moveable about a common joint; but it is less convenient in form, and less accurate in its results, though depending on the same principle.

PROP. 11.—PROB.

To find a third proportional to two given st. lines.

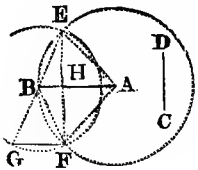
CON. Pst. 2, I. 31, I. DEM. 2, VI, 7, V.

E. 1	Data.	Given the two lines AB and AC;
2	Quæs.	to find a third proportional to them.
C. 1	*Pon.* Pst. 2, I.	Place AB and AC to form an ∠ BAC, and produce AB and AC;
2	3, I. 31, I.	take BD = AC, join BC, and draw DE ∥ BC;
3	Sol.	then CE is a third proportional; *i. e.*, AB : AC : CE.

D. 1	C. 2, 2, VI.	∵ BC ‖ DE, a side of △ ADE; ∴ AB : BD = AC : CE;
2	C. 2, 7, V.	but BD = AC; ∴ AB : AC = AC : CE.
3	Rec.	∴ to the given lines AB, AC, a 3rd prop. CE, has been found. Q. E. F.

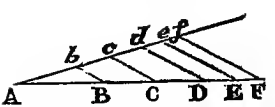
Sch. 1. There are various constructions by means of which this Problem may be solved: we select one which in practice requires the use of the compasses alone;—the given lines being AB and CD.

C. 1	Pst. 3, I.	From A with AB desc. ⊙ BEF; and from B with DC, ⊙EFG;
2		from E on arc EFG, set off CD three times to G;
3	Pst. 1, I.	join EF, AF, BF, BG, and GF;
4	Sol.	the chord FG is the third proportional.
D. 1	Con.	∵ AE, EB = AF, FB, and AB com.;
2	8, I. 5, I. 4, I.	∴ ∠ EBA = ∠ FBA; also ∠ BEF = ∠ BFE, and ∠s at H rt. ∠s;
3	D. 2. 31, III.	and ∵ ∠s at H are rt. ∠s, and ∠ EFG is a rt. ∠;
4	28, I.	∴ AB ‖ FG.
5	C. 1. 29, I.	and ∵ △s ABE, BFG, are isosc.; and ∠BFG = ∠ ABF; ∴ △ ABF is eq. ang. to △ BFG;
6	4, VI.	∴ AB : BF = BF : FG;
7	C. 7, V.	but BF = DC, ∴ AB : CD = CD : FG;
8	Conc.	∴ FG is the third proportional.



Use and App.—1. By repeating the same construction we solve the Problem, *to continue a series of ratios in progression,* AB : BC *being the given antecedent and consequent.*

C. 1	3, I. 31, I.	Take A *b* = BC, and draw C *c* ‖ B *b*;
2	„ „	„ CD = *bc* „ D *d* ‖ C *c*;
3	„ „	„ DE = *cd* „ E *e* ‖ D *d*, &c.
D.	2, VI. 7, V.	It is evident AB : BC : CD : DE, EF, &c.



Arithmetically. Since $CD = \frac{BC \cdot BC}{AB}$, the third proportional to two given numbers is found *by dividing the square of the consequent by the antecedent*; thus, the 3rd proportional of $9 : 6 = \frac{6 \times 6}{9} = 4$; *i.e.*, $9 : 6 = 6 : 4$.

2. From LARDNER'S Notes on Prop. 11, Bk. VI. we take several most USEFUL THEOREMS allied to the last Problem.

THEOR. I. "*If a series of magnitudes* A, B, C, D, *be in continued proportion, their successive differences, a, b, c, d, are also in continued proportion, and in the same ratio.*

D. 1	H.	$\therefore$ A : B : C;	*Arith.*- 2 : 6 : 18;
2	E. V.	$\therefore$ *conv.* A : a = B : b.	2 : 4 = 6 : 12.
3	16, V.	and *alt.* A : B = a : b.	2 : 6 = 4 : 12.
4	*Sim.*	So, $\therefore$ B : C = b : c;	So, 6 : 18 = 12 : 36;
5	D. 3, 4.	$\therefore$ a : b : c.	4 : 12 : 36.
6	*Proced.*	And a : b : c : d, &c.	and 4 : 12 : 36 : 108 &c.

THEOR. II.—*If a series in continued proportion, A : B : C : D &c., be an increasing one, there is no limit to the increase of its terms.*

E.	Hyp.	Let a = B − A ; b = C − B ; c = D − C &c. ; and L be the last term.
D. 1	†	$\therefore$ no magn. so great that we cannot obtain a greater,
2		$\therefore$ let M. be a magn. however great we please.
3		Find what multiple of a, M is;
4		and continue the series to a gr. number of terms than $\frac{M}{a}$;
5		$\therefore$ L > A by $a + b + c + d$ &c. + (the no. of terms = $\frac{M}{a}$)
6	Hyp.	But $\therefore$ a : b : c : d &c. is an increasing series ;
7		$\therefore$ each successive term > a ;
8		$\therefore$ their sum + A > M.

N.B.—The different steps † of this demonstration rest rather on truths that may be implied from Euclid's Elements, than on those expressly taught and proved ; they are, however, so plain that, perhaps, a formal proof is not required. The same remark should be extended to the Theorems 3 and 4 which follow.

THEOR. III. *If a series in continued proportion,* A : B : C : D, *&c, be a decreasing one, there is no limit to the diminution of its terms.*

C. 1		Continue the series until a term be found *less than m,* any assigned magnitude however small;
2		let $m : l = B : A$;
3	Use 11, VI.	and let the ratio, $m : l$, be continued in a series;
4	H.	$\because B < A$, $\therefore m < l$, and the series $m : l$ increases;
5		and $\therefore$ a term may be found, $a > A$,
6		Let the series be continued until $a > A$,
7		and let the series A : B : C &c., be continued the same no. of terms;
8		then its last term M will be $< m$, the assigned magnitude.
D. 1	H.	For, $\because$ A : B : C : D..... : L : M,
		& $a : b : c : d$: $l : m$;
2	22, V. 16, V.	$\therefore$ *ex. æquo.* $A : M = a : m$; and *alt.* $A : a = M : m$;
3	C. 6. 14, V.	but $A < a$, $\therefore M < m$.
4	*Proced.*	And so on by continuing a like process.

THEOR. IV. "*If a series of magnitudes,* decreasing *in continued proportion, be continued, or imagined to be continued, to an infinite number of terms; the sum of all the terms, or the sum of the series, will be a finite and determinate magnitude.*

C. 1	Use 11, VI.	On the line AZ let the decreasing series be set, AB : BC : CD : DE, &c.,
2	Pst. 1, I.	and draw a line ZM, making an $\angle$ with AZ;
3	31, I.	at A, B, C, D, E &c. raise parallels cutting MZ in the points M, N, O, P, Q;
4	31, I.	draw through N, NL $\parallel$ AZ, and cutting AM;
5	Conc.	the Sum of the series S = AZ.

Snp. If $AZ \neq S$, S is either gr. or less than AZ.

CASE. I. S *is not gr. than* AZ.

D. 1	H.	For each parallel DP, EQ is less than DZ, EZ, from which it is to be taken to determine the points for the next parallel;
2	4, VI.	and AM : AZ = DP : DZ = EQ : EZ;
3	14, V.	and $\because$ AM < AZ ; $\therefore$ EQ < EZ.

CASE II. Though the series is unlimited, S *is not less than* AZ.

Sup. If S were $<$ AZ, let S $=$ AY.

D. 1	H.	Now ∵ the parallels AM, BN, CO, &c., continually decrease proportionally;
2	4, VI.	and ∵ AZ, BZ, CZ &c. are in proportion to them;
		∴ AZ, BZ, CZ &c., are in decreasing continued proportion.
3	Use 11, VI.	Continue this series through a determinate number of terms; at last a term will be found $<$ YZ.
4		Thus the sum of the corresponding parallels must be greater than AY; *i. e.*, AM + BN + CO + &c. are $>$ AY.
5		Hence the Sum of a *limited* no. of terms is $>$ AY, the sum of an *unlimited* number;
6	Ax.	∴ a part is $>$ the whole, which is absurd;
7	Conc.	∴ AZ $\not>$ S the sum of an infinite number of terms:
8	Case 1.	nor is AZ $\not<$ S, ∴ AZ $=$ S.

3. CERTAIN PROBLEMS also may be deduced from the foregoing principles.

PROB. I. On the last Theorem, Theor. IV. GREGORY'S Problem, as it is named, is to be solved;— *from the two first terms in a series* AB : BC, *to obtain* S, *the sum of the series.*

C. 1	31, I.	Draw NL ∥ AZ, and meeting AM the parallel to BN &c.
2	Sol.	then AZ $=$ S, the sum of the series.
D. 1	2, VI.	For ML : LN $=$ MA : AZ;
2	C. Use 11, VI	but LN $=$ MA $=$ AB; BC $=$ BN; & ML $=$ MA$-$LA, or AB$-$NB, or AB$-$BC;
3	Conc.	∴ AB$-$BC : AB : AZ;
4	Remk.	*i. e.*, S the sum of the series is a third proportional to the difference of the 1st & 2nd terms, and the 1st term.

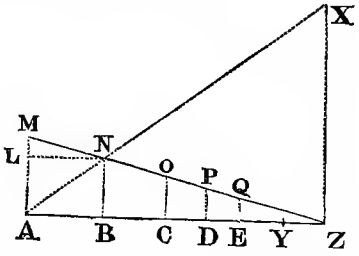
PROB. II. By anticipating Propositions 13 & 16, bk. VI., it follows, that *of the three quantities, the first and second terms,* AB, BC, *&* S *the sum of the series, if any two be given the remaining one may be found.*

1°. Given AB, BC:—then $AZ = \frac{AB^2}{AB-BC}$;

as 8 and 4; then $\frac{64}{8-4} = 16$

2°. Given AZ & AB; then $AB - BC = \frac{AB^2}{AZ} = BC$.

as 16 and 8; then $8-4 = \frac{64}{16} = 4$.



3°. Given AZ & AB—BC; then $AB = \sqrt{AZ \cdot (AB-BC)}$;
as 16 & 8—4; then $\sqrt{16 \times 4} = \sqrt{64} = 8 = AB$;
and $BC = 8-4 = 4$.

4°. Given AZ and BC; then AB : AZ = BC : BZ;
∴ 16, VI. AB . BZ = AZ . BC;

Thus, AZ divided at B, so that rect. AB . BZ = rect. AZ . BC will give the solution.

Let AZ, Cor 14, II., be so divided; and the first term of the series will be either segment, AB, or ZB, the Problem having two solutions; for

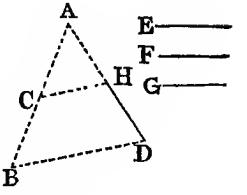
C. 1	Pst 1, 2, I.	Join AN, and produce it to meet the perp. from Z, in the . X.
2		Then BZ = ZX.
D. 1	2, VI.	∵ AB : BN = AZ : ZX; and AB : AZ = BN : BZ, *i. e.*, BC : BZ;
2	16, V.	and ∵ BZ = ZX; ∴ *alt.* AB : BN = AZ : BZ.
3	Conc.	∴ ZA = S, the sum of the series, of which the first term is BZ, and the second term, BN, or its equal BC.

Prop. 12.—Prob.

To find a fourth proportional to three given st. lines.

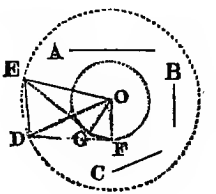
Con. 3, I. 31, I. Dem. 2, VI. 7. V.

E. 1	Data	Let E, F, G, be the three given st. lines;
2	Quæs.	to find a fourth proportional.
C. 1	Pst. 1, I.	From a com. · A draw AB, AD, forming ∠ BAD;
2	3, I.	on AB make AC = E, CB = F; and on AD, AH = G; join CH:
3	31, I.	and draw BD ∥ CH;
4	Sol.	then HD is the 4th proportional to E, F, G.
D. 1	C. 3, 2, VI.	∵ CH ∥ BD, a side of △ABD; ∴ AC : CB = AH : HD;
2	C. 2.	but AC = E, CB = F and AH = G, ∴ E : F = G : HD;
3	Conc.	∴ to E, F, G, a 4th proportional HD, is found. Q. E. F.



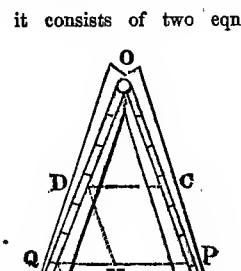
Sch.—Among several other constructions for the Solution of this Problem there is one, for which *the compasses alone* are sufficient,

C. 1	Pst. 3, I.	From any cen. O, with A & B, desc. two ⊙s;
2	„	& from D, a . in the outer ⊙, set off DE = C;
3		from D, E, with DO cut the inner ⊙ in G, F;
4	Sol.	then FG is the 4th proportional required.
D. 1	C. 3, 1.	∵ DF = EG; OE = OD, & OG = OF;
2	D. 1.	∴ △ODF has its sides = the sides of △OEG;



3	8, I.	$\therefore \triangle ODF = \triangle OEG$, and $\angle DOF = \angle EOG$;
4	*Add*, or *Sub.*	In $\angle$s DOF EOG add or take away com. $\angle DOG$;
5	Ax. 3, I.	in this case, $\angle EOD = \angle GOF$.
6	C.	And $\because$ $\triangle$s GOF, DOE are isosc. & $\angle DEO = \angle GFO$;
7	Def. 1, VI.	$\therefore \triangle GOF$ is eq. ang. to $\triangle DOE$;
8	4, VI.	$\therefore$ EO : OG = ED : FG;
9	Conc.	$\therefore$ FG is a 4th prop. to A, B, C. Q. E. F.

USE and APP. 1—The *Sector* is an instrument invented by GUNTER who lived between A.D. 1581 and 1626; it consists of two eqnal legs, or rather rulers, turning like a Carpenter's rule, on a pivot at the centre. The Scales on it OB, OA, converging to the pivot are those which properly belong to the Sector, thongh in the vacant spaces, it is usual to insert various other scales.



This instrument has not inaptly been termed, "a large number of pairs of compasses packed up into one." For explanation we will take one pair of these compasses or scales, OA, and that which corresponds to it, OB; on each is laid down a scale of chords, OA = OB = 90°; OP = OQ being radii, or chords of 60°; and if OP be 4 inches, or 8 inches, and the proper divisions inserted, we have a scale of chords, with a radius of 4, or 8 inches, laid down.

2.—BY AID OF A PAIR OF COMPASSES AND THE SECTOR, the following Problems, among others, may be solved.

PROB. I.—*To find a fourth proportional to three given lines K* 3, *L* 4, *M* 6

Take OD, OC each = K 3, and open the instrument till DC = L 4 : then take OQ, OP, each = M 6, and the distance QP = the 4th proportional = 8.

For $\because$ DC $\parallel$ QP, a side of $\triangle OQP$; $\therefore$ OD 3 : DC 4 = OQ 6 : QP 8.

PROB. II.—*To find a chord*, say of 40°, *to a radius*, as of 5 inches, *from a Sector the rad. of the chord of which OP* = 4 *inches.*

From a scale of inches take 5 in the compasses, and open the Sector so that QP = 5 inches; fix now one leg of the compasses at C, the extremity of OC = 40°, and the other leg at D, of OD = 40°; the distance DC will give the chord of 40°. when the radius is 5 instead of 4 inches.

For, by similar △s, OP 4 : PQ 5 = OC 40° : DC 50°, when the rad. is 5 inches.

PROB. III.—*To divide a given line L,* = 120, *into two parts, x* and *y, which shall be to each other, as two lines, or numbers OD* = 3, *and OQ* = 5.

Take OE, OF each = (OD 3 + OQ 5) = 8; open the sector so that EF = L = 120; then DC = *x*, and QP = *y*.

For, on drawing QG ∥ OF,

∵ OD = QE, ∠EQG = ∠DOC, & ∠QEG = ∠ODC,

∴ △ODC = △QEG, DC = EG, & QP = GF.

Now DC : QP = 3 : 5; ∴ EG : GF = 3 : 5.

Hence the parts will be $x = 3\left(\frac{120}{3+5}\right) = 45$; & $y = 5\left(\frac{120}{3+5}\right) = 75$.

3.—VARIOUS OTHER LINES are marked ON THE SECTOR, and for their use depend on the same principles as those which affect the Line of Chords and Line of Numbers.

1.—*A Line of Sines,* marked S, of which the rad. = 90°.

2.—*Two Lines of Tangents,* one marked from 0° to 45°, the radius being the sine of 90°;—the other, on a smaller scale, marked from 45° to 75°, the rad. being the tangent of 45°, or distance 0 to the beginning of the scale.

3.—*A Line of Secants,* the rad. being the secant of 0° extending to about 75°

4.—*A Line of Polygons,* marked P O L. showing the length of the sides of regular polygons inscribed in circles.

5.—*A Line of equal parts,* by which a fourth proportional to three given numbers may be found; the line being divided into 100 parts,—that number is the limit of the accurate use of this line.

6.—A *line of chords,* marked C, the radius being the chord of 60°. The extent of this line is from 0 to 60°;—and if a chord for a greater arc than 60° be required,—a circle, with a rad. equal to the chord of 60°, must be drawn, in which the chord of 60° must be set off; and then from the point where the chord of 60° cuts the circle, must also be set off the amount of the given arc above the 60°;—the distance in the circle from the extremities of the sum of the two arcs will be the chord required.

Similar Lines to these, with some additional, are marked on GUNTER'S Scale ;—but as this Scale is sometimes of two feet in length, and sometimes of one, it will be found that the Magnitudes, though proportional to the corresponding lines on the six inch Sector are not identical. Besides the Lines of Numbers, Sine Rhumbs. Tangent Rhumbs, &c., on Gunter's Scale, are Logarithmic Lines, the uses of which depends on the principle *that the logarithms of the terms of equal ratios are equidifferent.* . For an explanation, of the Logarithmic Lines on Gunter's Scale, their Construction and Use, the Student may consult KEITH'S *Plane and Spherical Trigonometry,* pp. 18—24. The *General Rule* given is this ; " the extent of the compasses from the first term to the second, will reach, *in the same direction,* from the third to the fourth term. *Or,* the extent of the compasses from the first term to the third, will reach, *in the same direction,* from the second to the fourth."

4.—An Example or two will show USE OF THE SECTOR.

Ex. 1. *In a rt.* $\angle d.$ $\triangle ADC$, *as in fig. to* 13, *VI, to find the side* DC *opposite* $\angle A$, *when* $\angle A = 40°$ *and* $AC = 4$ *inches.*

Take 4 inches in the compasses, and open the Sector, until 90° and 90° on the two identical lines of Sines are 4 inches apart ; then the distance from 40° on one of those lines of Sines to 40° on the other will equal the side DC.

Ex. 2. *Given in a* $\triangle$ *two* $\angle s$, *of* 59° *and* 38° *respectively, and the side opposite to* $\angle$ 59° *equal to* 76 *equal parts ;—required the length of the side opposite to* $\angle$ 38°.

The analogy is,—Sine 59° : Sine 38° : : 76 : x ; *i. e.,* $x = \dfrac{\text{Sine } 38° \times 76}{\text{Sine } 59°}$.

Open the Sector until the counterpart lines of sines, 59° and 59°, are as far apart as the distance from 0 to Sine 38° ; then the distance from 76 to 76, on the corresponding lines of equal parts, will show the value of x.

Ex. 3. *To inscribe a regular polygon of* 10 *sides in a circle of which the radius is* 3 *inches.*

Open the Sector until 6 and 6, on the counterpart lines of Polygons, are 3 inches apart, then the distance from 10 to 10, on the same counterpart lines, will equal the length of one side.

5. The Sector may also be applied, to extract the Square and Cube Roots, and to double the Cube ; to measure Triangles,—to find the Areas of Figures, and the Contents of Solids,—and to increase or diminish any Figure according to any given Proportion.

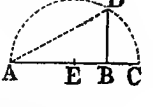
NB. It must however be remembered, that no great accuracy is to be obtained from the small Sectors in common use ; and they are the less to be trusted in proportion to the greater opening of the sides or legs. It is only by considerable practice that a person can become expert and exact in the use of this instrument.

Prop. 13.—Prob.

To find a mean proportional between two given st. lines.

Con. Pst. 3, I, 11, I. Dem. 31, III. Cor. 8, VI.

E. 1	Data	Let the given st. lines be AB and AC;
2	Quæs.	to find a mean proportional between them.
C. 1	Pst. 3, I.	Place AB, BC, in a st. line AC; & on AC descr. a semicircle ADC;
2	11, I.	from B draw BD $\perp$ AB and join AD, DC;
3	Sol.	then BD is a mean proportional to AB & AC.
D. 1	31, III. C. 2.	$\therefore$ $\angle$ADC is a rt. $\angle$, & in the rt. $\angle$'d $\triangle$ABC, BD is $\perp$ the base;
2	Cor. 8, VI.	$\therefore$ DB a mean propl. to the segments of AC, namely, AB, BC;
3	Conc.	$\therefore$ between AB, BC, a mean proportional DB, is found. Q, E. F.



Sch. There are other constructions by the aid of which a mean proportional between two given lines can be obtained,—but none simpler or easier than the above. For *numerical* calculations however the following formula may be used: Let AC $= 2r$; AB $= x$; BC $= 2r - x$; and BD $= y$

then $x : y : 2r - x$; or, $y^2 = 2rx - x^2$, & $y = \sqrt{2rx - x^2}$,

or $y = \sqrt{AB \cdot BC}$.

Ex. Find a mean proportional to 9 and 4.

$\sqrt{4 \times 9} = \sqrt{36} = 6$ the mean propl; *i. e.*, $9 : 6 = 6 : 4$.

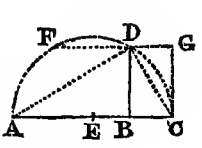
Use and Appl. I.—*Any rectangular parallelogram may be reduced to an equivalent square by this proposition*;—for if AB & BC represent the two sides, then the square on DB $=$ AB $\cdot$ BC;—the demonstration of which follows from Prop 17, VI.

II. Combining Propositions 11 and 13, we are able, *when, of three lines in continued proportion, any two are given, to find the unknown line*; See Use and Appl. 11, VI. Problem II.

III. *Given one of three terms and the sum of the other two; to find the two unknown terms.*

1°. *Given the mean* BD, *and the sum* AC = AB + BC.

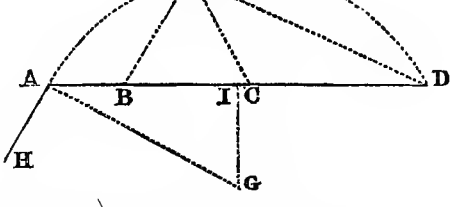
C.	1	Pst. 1, I. 11, I.	On AC desc. a semi. ⊙; and at C draw CG, = BD ⊥ AC;
	2	31, I.	Through G draw GF ∥ AC, cutting the arc in ·s F, D;
	3	12, I.	and from D drop DB ⊥ AC;
	4	Sol.	then the other two terms are AB and BC.
D.	1	C. 4, VI.	∵ △s ABD and DBC are similar,
	2	Conc.	∴ AB : BD = BD : BC; Thus BD being the mean, AB & BC are the extremes.



Q. E. F.

2°. *Given* AB, *one extreme, and the sum of the two terms,* BD = BC + CD; *required the mean, and the other extreme.*

C.	1	2, I.	Set the sum of the three terms, AB + BD, in one st. line AD;
	2	23, I.	at the extremity A draw an ∠, DAH = ext. ∠ of an eq. lat. △;
	3	33, III.	and on AD desc. a seg. of a ⊙ with ∠ = ∠ DAH;
	4	3, I.	then set on AD the given extreme AB;
	5	23, I,	at B make ∠ ABE = ∠ DAH;
	6	23, I,	prod. BE, and at E make ∠ FEC = ∠ DAH;
	7	Sol.	BC is the mean, and CD the other extreme.



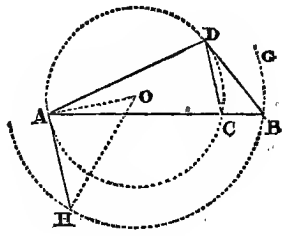
D.	1	C.	∵ EBC is an eq. lat. △, and BC = BE;
	2	Sch. 1. 8, VI.	∴ AB : BC = BC : CD.
	3	Conc.	∴ BC is the mean, and CD the other extreme.

Q. E. F.

IV. When *three lines* are in continued proportion, *if one be given and the difference of the other two, those other two may be found.*

1°. Given the mean BD, and AC the difference of the extremes.

C. 1	33, III.	On AC descr. any seg. of a ⊙ ADC;
2	17, III.	To the ⊙ at A draw a tang. AH = given mean;
3	Psts 1 & 3, I.	join H and cen. of the ⊙, and with OH desc. concentric ⊙ HBG;
4	Psts. 2 & 3, I.	produce AC to B; and from B with rad. = AH cut the inner circle in D;
5	23, I.	join DA and DC;
6	Sol.	then AB and BC are the two terms required;
D. 1	C.	∵ BD touches the ⊙ ACD in D, and DC cuts it in C;
2	32, III.	∴ ∠ BDC = ∠ DAC.
3	C. & D. 2	In △s ADB, DCB we have ∠BDC = ∠DAC, and ∠B com.
4	Def. 1 VI	∴ △s ADB, DCB, are eq. ang., and their sides proportional;
5	4, VI.	∴ AB : DB = DB : BC.
6	Conc	∴ AB and BC are the other two terms.

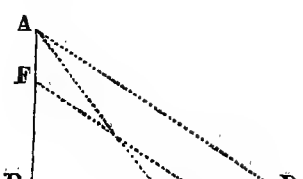


2°. *Given one extreme,* BC, *and the difference between the mean and the other extreme,* AB—BD; *to find both the mean and that other extreme.*

D. 1	Theo. I. Use 11, III.	∵ the differences of successive terms are as the terms themselves,
2	16, VI.	∴ the rect. under one extr. and the dif. of the mean and the other extr. = the rect. under the mean and the difference between it and the given extreme.
3	Remk.	*i. e.*, BC. (AB—DB) = DB. (DB—BC).
4	D3. & H.	Since the Area of this rect. and the difference of the sides are given;
5	Use 3, 10, II.	∴ the sides themselves may be found.
6	Remk.	For $\sqrt{\left(\frac{\text{difference}}{2}\right)^2 + \text{the area}} = \frac{\text{Sum}}{2}$. LARDNER'S Euclid, p. 79.
7	Use 1, 10, II.	and $\frac{\text{Sum}}{2} + \frac{\text{Diff.}}{2}$ = the greater, or the less magnitudes.

V. *To find two st. lines to contain a rectangle equal to a given rectangle* AB . BC, *and to have a given ratio one to the other.*

C. 1	*Pon.*	Place AB and BC so as to be conterm. at $\angle$ B;
2	12, VI.	prod. BC to D so that AB is to BD in the given ratio;
3	13, VI.	join AD, and take BE a mean propl., *i. e.*, BC : BE = BE : BD,
4	31, I.	and through E draw EF $\parallel$ DA, meeting AB in F;
5	Sol.	the lines BE, BF, are the st. lines required.



D. 1	C. 2, 3.	Now the lines are evidently in the given ratio;
2	2, VI.	and $\because$ EF $\parallel$ DA, $\therefore$ AB : BD = FB : BE;
3		and $\therefore$, also BF : BA = BE : BD;
4	20, VI. C. 3.	$\therefore$ BF . BE : BA . BD = BE^2 : BD^2, *i. e.* BC : BD;
5	1, VI.	but AB . BC : AB . BD = BC : BD,
6	9, V.	$\therefore$ EB . BF = AB . BC.
7	Sol.	$\therefore$ BE and BF are the two required lines. Q. E. F.

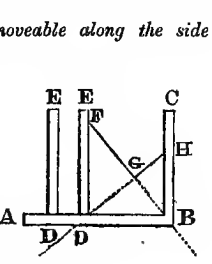
VI. We are also able by the 13th Prop. to find any number of means represented by a power of 2, *minus* 1; *i. e.* 4—1 or 3, 8—1 or 7, 16—1 or 15 &c. For having obtained *one* mean between two magnitudes, we may by the same process determine a mean between it and each of its extremes, and thus we shall have *three* means. And again between each successive pair of the series thus found inserting means, and pursuing the same method, we obtain *seven* means, *fifteen*, *thirty one*, &c.

Addenda to Prop. 13, VI.

I.—The Problem, *to obtain* two mean proportionals *between two given st. lines*, on which depends "the duplication of the cube" in Solid Geometry, cannot be determined by the rule and compasses, the only instruments allowed by Euclid; and consequently it has never received a strictly Geometrical Solution. Various mechanical contrivances have been invented to attain the solution; as Plato's Method of a ruler inserted perpendicularly on the side of a square and moveable along that side; Philo's of Byzantium, of a graduated ruler revolving on a point; the Trammel of Apollonius, or of Nicomedes who lived

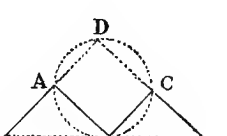
about two centuries before the birth of Christ; and the Method of DES CARTES, consisting of a collection of rulers, two of which move round a pivot, each of the two having a groove on the inner edge, in which other rulers are free to move perpendicularly to the groove. For a description of these Methods reference may be made to LARDNER'S *Euclid* pp 196-199 COOLEY'S *Supplement to Euclid* pp 92, 93; or *Galbraith's Manual* pp. 111, 112. There are several solutions in the *third* Bk. of the Mathematical Collections of PAPPUS, who flourished A.D. 379—395; and ten different solutions occur in a Commentary by EUTOCIUS of Ascalon, who lived AD., 560, *on the Sphere and Cylinder of* ARCHIMEDES.

1°. PLATO'S Method *of a perpendicular moveable along the side of a Square.*



C. 1		Take two st. rulers AB, BC, fitted at a rt. ∠ ABC;
	2	and a third ruler moving along AB at rt. ∠s to AB;
	3	place the given extremes FG and GH at a rt ∠ FGH;
	4	and so that FG produced cuts the rt. ∠ ABC in the vertex B;
	5	while HG produced cuts BA in D,
	6	and the perp. DE meets the . s D & F;
	7	then GB and GD are the two mean proportionals to FG & GH.
D. 1	C.	∵ HBD & FDB are rt. ∠d △s of which the altitudes are GB, GD;
	2 8, VI.	∴ HG : GB : GD, and BG : GD : GF.
	3 Conc.	∵ GB & GD are mean proportionals to the extremes HG & GF.

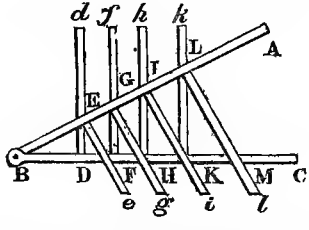
2°. PHILO'S Method *of a graduated ruler FG revolving round, B, the vertex of a rt. angle.*



C. 1	11, I.	Place the given extremes, AB, BC, at a rt ∠ at B;
	2 31, I. 4, IV.	complete the rect. ABCD, and about it descr. a ⊙ ABCD;
	3 Pst. 2, I.	produce DA & DC; and move FG about the . B, until QF = BG;
	4 Conc.	then AF & CG are the two means to AB & BC.
D. 1	C.	∵ BG = QF, & QG = BF; ∴ rect. QG . GB = rect. QF . BF;
	2 36, III.	but QG . GB = DG . GC; & QF . BF = DF . FA;
	3 Ax. 1.	∴ DG . GC = DF . FA;

4	16, VI.	∴ DG : DF = AF : GC ;
5	C.	but ∵ △s GDF, BAF & GCB are equiangular,
6	2, VI.	∴ DG : DF = AB : AF, & DG : DF = GC : CB ;
7		and ∴ BA : AF = AF : GC ; and AF : GC = GC : CB.
8	Conc.	∴ AF & CG are the two means to AB & CB.

3°. Method of DES CARTES, *with a collection of rulers.*



C. 1		Take two st. rulers AB, BC united by a pivot at B ;
2		in each there is a groove, in which rulers move perpendicularly to AB & BC ;
3		so that perp. D*d*, on opening the rulers, pushes forward E *e*, E *e* pushes F *f* &c.
4	*Sup.*	*Let two means be required.*
5		Move D*d* from B, until BD = the less extreme ;
6		close AB upon CB that the perpendicular may move up to D,
7		and fix D *d* in the position where BD = the less extreme ;
8		Now open AB & CB, until BG = the greater extreme, D*d*, during the process, pushing E *e* from B, & E e pushing F *f*, &c. ;
9	Sol.	then the two means are BF & BE.
D. 1	C.	∵ △s BDE & BFG are equiangular ;
2	4, VI.	∴ BD : BF = BF : FE ; & BF : FE = FE : BG ;
3	Conc.	∵ BF & BE are the two means to BD & BG

In a similar way *three* means BE, BF, BG will be found to BD & BI, by opening the rulers AB, CB, until BH = the greater extreme.

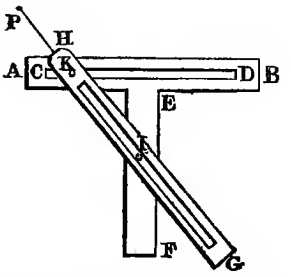
For *four* mean proportionals open the rulers until BI = the greater extreme; and so on for a greater number of means.

Generally, if an *even* number of means be required, the extremes will be on different rulers ; if an *odd* number, on the same ruler.

N.B.—The advantage of the Method of DES CARTES is more apparent than real. "It would appear that any number of mean proportionals may be found ; but the necessary adjustments are almost as deficient in practical facility as in geometrical legitimacy."—COOLEY.

II. The *Trisection of a rectilineal angle*, ABC, is another puzzle for Plane Geometry. The Solution of this Problem, though of no more trouble by aid of the Higher Analysis than the finding of the cube root, is impossible by the circle and right line. NICOMEDES, known as the discoverer of the conchoid curve, effected the Solution by the *Trammel* which he invented, and by which the Duplication of the Cube and the Trisection of an angle are both accomplished.

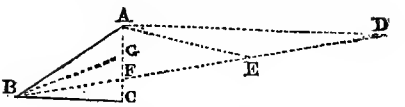
1°. The TRAMMEL of NICOMEDES, *a* T *square with a moveable ruler* HG.



C. 1 Take a T square with a groove CD, in the cross ruler AB;
and a fixed pin, I, in the stem, FE;
2 Let this fixed pin be inserted in the groove of the moveable ruler HG;
3 and HG also have a fixed pin K, inserted in CD.
4 From H, in HG, there issues a sliding stem HP;
5 Its length HP = that part of the line to be intercepted by the sides of the angle.
6 Let the fixed pin I be now placed on the given point;
7 and the groove CD on one side of the given angle;
8 let HG be moved so that the pin K go along one side of the angle;
9 and let the motion be continued until the · P shall come on the other side of the angle;
10 then the line joining P & I, *i. e.*, PI, is the required line.

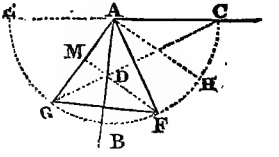
2°.—By this instrument *to Trisect a given angle ABC.*

C. 1 12, I. 31, I. From A draw AC ⊥ BC, and AD ∥ BC;
2 Trammel. inflect BD so that FD = 2 BA;
3 10, I. bisect FD in E, and join AE;
4 Sol. 1. then ∠DBC = ⅓ ∠ABC;
5 Sol. 2. bisect ∠ABD by BG, & ∠ABC is trisected.



D. 1	C. I. 39, I.	∵ ∠FAD is a rt. ∠ ; and AE = ED = EF ;
2	5, I. 32, I.	∴ ∠EAD = ∠ADE ; & ext. ∠AEB = 2 ∠ADB ;
3	29, I.	but ∠ADB = ∠DBC ; ∴ ∠AEB = 2 ∠DBC ;
4	C. 2. 5, 1.	and AE = AB ; ∴ ∠AEB = ∠ABE,
5	D. 2 & 3.	∴ ∠ ABE = 2 ∠DBC ; & ∴ ∠FBC = ⅓ ∠ABC.
6	D. 5. C. 5.	And ∵ ∠ ABD = ⅔ ∠ABC, and is bisected ;
7		∴ ∠CBF = ∠FBG = ∠GBA ;
0	Conc.	∴ ∠ABC is trisected by the lines BF, BG. Q. E. F.

3°.—*Or*, by Cooley's Tentative Method, *to trisect a given ∠BAC.*



C. 1	Pst. 2, I.	Produce the side, as AC beyond A, the vertex ;
2	Pst. 3, I.	from A descr. a semi ⊙ EBC ;
3	*Superp.*	on the . A place the vertex of an eq. lat. △, made of card or board ;
4		and let the △ move round the . A on a pin.
5		At M, the middle · of AG, a side of the △, or of the part between A & the circumference, fix a fine thread ;
6		and attach the thread to F ;
7	Sim.	So, fix a thread at G, and keep it extended through C.
8		Now, if the eq. lat. △ move round A till the intersection in D of the two threads falls on the line AB,
9		then AF will cut off ⅓ of ∠CAB.
D. 1	C. Sch. 32I.	∵ fig. AGF, is an eq. lat. △ and ∠GAF = ⅓ of 2 rt. ∠s ;
2	13, I.	∴ ∠GAF = ⅓ of the ∠s EAG, GAF, FAC ;
3	D. 2.	∴ ∠GAF = ½ of ∠s EAG + FAC.
4	20, III.	But ∠EAG at centre = 2 ∠ACG at ⊙ce.
5		& ∵ ∠ACG = ∠FAB, ∴ ∠EAG = 2 ∠FAB, *i. e.*, ∠FAB = ½ ∠EAG ;
6	26, III.	Now ∵ ∠FAB = ½ ∠FAC : ∴ arc BF = ½ arc, FC ;
7		∴ arc BF = ⅓ arc BC, & ∠BAF = ⅓ ∠BAC ;
8	Conc.	∴ ∠BAC is trisected by AF and AH. Q. E. F.

N.B.—Either of these methods solves the Problem, and may be adopted in practice when the Trisection of an angle is required.

Prop. 14.—Theor.

Equal Parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and conversely, *parallelograms that have one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional, are equal to one another.*

"When ▱s are eq. ang. the sides which are about the eq. ∠s, are reciprocally proportional; and when the ▱s are eq. ang., and the sides, which are about the eq. ∠s, are reciprocally proportional, those ▱s are equal."—Euclid.

Con. 14, I If at a · in a st. line, two other st. lines, upon the opp. sides of it, make the adj. ∠s together eq. to two rt. ∠s, these two st. lines shall be in one and the same st. line.

31, I. To draw a st. line through a given · ‖ to a given st. line.

Dem. Def. 2, VI. Reciprocal △s and ▱s have their sides about two of their ∠s proportionals, so that a side of the 1st fig : a side of the other : : the rem. side of the other : the rem. side of the 1st.

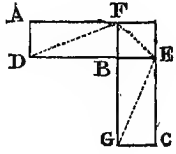
7, V. Eq. Ms have the same ratio to the same M; and the same has the same ratio to eq. Ms.

1, VI. △s and ▱s of the same alt. are to one another as their bases.

11, V. Ratios that are the same to the same ratio, are the same to one another.

9, V. Ms which have the same ratio to the same M are eq. to one another: and those to which the same M has the same ratio are eq. to one another.

E. 1	H. Case I.	Let ▱ AB = ▱ BC, & ∠DBF = ∠EBG
2	Conc.	then DB, BF and GB, BE are reciprocally proportionals.
	Def. 2. VI.	*i. e.*, DB : BE = GB : BF.
C. 1	*App.*	Apply ▱ AB to ▱ BC so that DB, EB form one st. line;
	14, I.	and FB, BG also one st. line:
2	31, I.	and complete the ▱ FE.



D. 1	H. & C.	∵ $\square$ AB = $\square$ BC, & EF is another $\square$;
2	7. V.	∴ $\square$ AB : $\square$ FE = $\square$ BC : $\square$ FE;
3	1, V.	but $\square$s AB : FE = bases DB : BE, and $\square$s BC : FE = bases GB : BF;
4	11, V.	∴ sides DB : BE = sides GB : BF;
5	Conc.	∴ the sides of $\square$s AB, BC, about the eq. ∠s are reciprocally proportionals.
E. 1	H. CASE II.	Let sides DB : BE = GB : BF;
2	Conc.	then $\square$ AB = $\square$ BC.
D. 1	H. & 1, VI.	∵ DB : BE = GB : BF; and DB : BE = $\square$ AB : $\square$ EF;
2	1, VI.	and ∵ GB : BF = $\square$ BC : $\square$ EF;
3	11, V.	∴ $\square$ AB : $\square$ FE = $\square$ BC : $\square$ FE;
4	9, V.	∴ $\square$ AB = $\square$ BC.
5	Rec.	Therefore, *Equal parallelograms, &c.* Q.E.D.

SCH. 1. There is a *third* case, not introduced in the proposition, and depending for its proof on Prop. 16, bk. VI;—it is, "*When two $\square$s have eq. areas and their sides are reciprocally propl., they will be eq. ang.*" It will be sufficient to prove that one ∠ in the one is eq. to the corresponding ∠ in the other $\square$; for from the proof it follows, that each separate ∠ in the one is eq. to each corresponding ∠ in the other.

2. Inasmuch as △s = the halves of their equiangular $\square$s,—what is proved respecting eq. $\square$s might be proved respecting eq; △s. (Ax. 5, I); and the following Proposition, the 15th, form a corollary of the 14th; for if △DBF = △ EBG, and ∠ DBF, = ∠EBG, then also DB : BE = GB : BF. The parallelogram and triangle are united into one Theorem, Prop. 1, VI, and might also be united here.

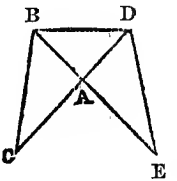
PROP. 15.—THEOR.

Equal angles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally pro-

portional: and conversely, *triangles which have one angle in the one equal to one angle in the other, and the sides about the equal angles reciprocally proportional, are equal to one another.*

CON. 14, Pst. 1, I. DEM. 7, V. 1, VI. 11, V. Def. 2, VI. 9, V.

E. 1	Hyp.	CASE I.—Let $\triangle ABC = \triangle ADE$, and $\angle BAC = \angle DAE$;
2	Concl.	then CA, AB, EA, AD are reciprocally proportional.
	Def. 2, VI.	*i. e.*, CA : AD = EA : AB.
C. 1	*Appl.*	Place CA & AD in one st. line CD;
2	14, 1. Pst. 1, I.	∴ EA, AB also in one st. line EB; join DB.

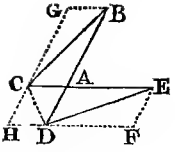


D. 1	H. & C.	∵ $\triangle ABC = \triangle ADE$, & fig. ABD is another $\triangle$;
2	7, V.	∴ $\triangle ABC : \triangle ABD = \triangle AED : \triangle ABD$.
3	1, VI.	but $\triangle ABC : \triangle ABD$ = bases CA : AD, and $\triangle AED : \triangle ABD$ = bases EA : AB;
4	11, V.	∴ CA : AD = EA : AB.
5	Conc.	∴ the sides about the eq. $\angle$s are reciprocally proportional.

E. 1	Hyp.	CASE II.—Let CA : AD = EA : AB;
2	Conc.	then $\triangle ABC = \triangle ADE$.
C.	14, I.	Make the same construction as in Case I.
D. 1	H.	∵ CA : AD = EA : AB;
2	1, VI.	and ∵ CA : AD = $\triangle ABC : \triangle ABD$, and EA : AB = $\triangle ADE : \triangle ABD$;
3	11, I.	∴ $\triangle ABC : \triangle ABD = \triangle ADE : \triangle ABD$;
4	D. 3. 9, V.	and ∵ $\triangle ABD$ is com., ∴ $\triangle ABC = \triangle ADE$.
5	Rec.	Therefore, *equal triangles which have one angle, &c.* Q. E. D.

Or, for Case II.

C. 1	*Appl.*	Place as before, CA, AE in one st. line, and DA, AB in another.
2	31, I.	and complete the ▱s AF, AG, AH.
D. 1	H.	∵ BA : AD = EA : AC,
2	1, VI.	∴ BA : AD = ▱ AG : ▱ AH; and EA : AC = ▱ AF : ▱ AH;
3	9, V. 41, I.	∴ ▱ AG = ▱ AF, and △ ABC = △ ADE.



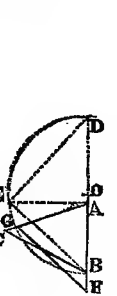
N. B, Case I, may be proved on the same construction.

Schol. 1. As a *criterion* for the equality of triangles this Proposition may be classed with Props. 4, 8, & 26, bk. I.

2. When in each triangle the equal angle, included by the reciprocal sides has the same supplement, the sides are also reciprocally proportional.

Use & App. *To construct an isosceles triangle equal to a given scalene triangle* A B C, *and with the same vertical angle* B A C.

C. 1	3, I.	Produce BA, and make AD = AC;
2	Pst. 3, I. 11, I.	on BD desc. a ⊙ DEB; and draw AE ⊥ BD;
3	3, I.	in AB prod., take AF = AE, and in AC, AG = AE;
4	Sol.	join GF; then △ AGF is the isosc. △ required.
D. 1	Cor. 8, VI.	∵ BA : AE = AE : AD, ∴ BA : AE = AE : AC;
2	C. 3, Def. 25, I.	but AF, AG, each = AE; ∴ AGF is an isosc. △
3	D. 1, 2.	and ∴ BA : AG = AF : AC. *i.e.*, the sides about the vert. ∠ are recip. props.
4	15, VI.	∴ the isosc. △ AGF = the scalene △ ABC.



Q. E. F.

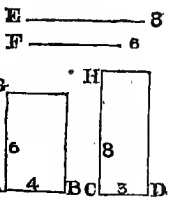
Prop. 16.—Theor. (*Very Important.*)

If four st. lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and conversely, *if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four st. lines are proportionals.*

Con. 11, I. 3, I. 31, I.

Dem. 7, V. 14, VI. Def. 1, II. A rect. is contained by any two of its conterminous sides. Ax. 1, I.

E. 1	Hyp.	Case I.— Let AB : CD = E : F;
2	Conc.	then □ AB . F = □ CD . E.
C. 1	11, I.	From A, C, draw AG ⊥ AB, and CH ⊥ CD,
2	3, I. 31, I.	make AG = F, CH = E, and complete □s BG, DH.
D. 1	C. 1 & 2.	∵ AB : CD = E : F, and ∵ E = CH and F = AG.
2	7, V.	∴ AB : CD = CH : AG; *i. e.*, sides about the eq. ∠s are recip. propl.;
3	14, VI.	∴ □ BG = □ DH;
4	C. 2, Def. 1, II.	but ∵ AG = F, ∴ □ BG is contained by AB and F;
5		and ∵ CH = E, ∴ □ DH is contained by CD and E;
6	Conc.	∴ □ AB . F = □ CD . E.



E. 1	Hyp.	Case II.—Let □ AB . F = □ CD . E
2	Conc.	then AB : CD = E : F.
C.	*Sim.*	Using the same construction;
D. 1	H. & C.	∵ AB . F = CD . E, and ∵ AG = F and CH = E;
2	Def. 1, II.	∴ □ BG is contained by AB . F, and □ DH by CD . E;

3	Ax. 1.	$\therefore \square BG = \square DH$.
4	C. 14, VI.	and $\because \angle A = \angle C$, $\therefore$ AB : CD = CH : AG, *i. e.*, they are reciprocally proportional;
5	C. 7, V.	but CH = E, and AG = F; $\therefore$ AB : CD = E : F.
6	Rec.	$\therefore$ *If four st. lines be proportionals &c.* Q. E. D.

Alg. & Arith. Hyp. Let $a\ 4 : b\ 3 = c\ 8 : d\ 6$.

Alg. $\because \frac{a}{b} = \frac{c}{d}$,

$(\times b \times d)$ $\frac{a\,b\,d}{b} = \frac{b\,c\,d}{d}$, i. e., $ad = bc$;

conversely, $\because ad = bc$.

$(\div b \times d)$ $\frac{ad}{bd} = \frac{bc}{bd}$, i. e., $\frac{a}{b} = \frac{c}{d}$.

Arith. $\because \frac{4}{3} = \frac{8}{6}$,

$(\times 3 \times 6)$ $\frac{4 \times 3 \times 6}{3} = \frac{8 \times 3 \times 6}{6}$, *i. e.*, $4 \times 6 = 3 \times 8$;

conversely $\because 4 \times 6 = 3 \times 8$.

$(\div 3 \times 6)$ $\frac{4 \times 6}{3 \times 6} = \frac{3 \times 8}{3 \times 6}$, *i. e.*, $\frac{4}{3} = \frac{8}{6}$.

COR. *Rectangles which have their sides about the right angles reciprocally proportional are equal; and if the rectangles are equal, the sides about the right angles are reciprocally proportional.*

SCH. I. DEFINITION. Two ratios A : B and $a : b$ are said to be reciprocally proportionals, when the antecedent is to the consequent in the one as the consequent to the antecedent in the other; as A : B = $b : a$.

II. As a necessary consequence, the 16th Prop., Bk. VI. may be deduced from the principles resting on this definition;—1°. That a ratio compounded of reciprocal ratios is a ratio of equality; and 2°. That if a ratio of equality be compounded of two ratios they must be reciprocals.

1°. *A ratio compounded of reciprocal ratios,* as, A : B and $a : b$, *is a ratio of equality.*

D. 1	H.	$\because A : B = b : a$; $\therefore \left.\begin{matrix} A : B \\ a : b \end{matrix}\right\} = \begin{matrix} A : B \\ B : A \end{matrix}$,
2	Conc.	$\therefore A : b = A : A$; and $A : A$ is a ratio of equality.

N. B. The form $\left.\begin{matrix} A : B \\ a : b \end{matrix}\right\}$ denotes a ratio componnded of $A : B$ and $a : b$.

2°. *If a ratio of equality,* A : A, *be compounded of two ratios,* $a : b$, *and* $c : d$, *they must be reciprocals.*

D. 1	C.	Let $c : d = b : x$;
2		then $\left.\begin{matrix} a : b \\ c : d \end{matrix}\right\} = \left\{\begin{matrix} a : b \\ b : x \end{matrix}\right\}$, or $a : d = a : x$;
3	H.	bnt $a : x$ is a ratio of equality; *i. e.* $x = a$.
4	Conc.	$\therefore b : x$ is the the recip· of $a : b$; and $c : d$ is the recip. of $a : b$.

3°. *The sides of equal rectangles are four proportionals.*

D. 1	Cor. 2, 1, VI.	∵ △s and □s are in a ratio compounded of the bases and altitudes.
2	Theor. 2 Sch. II.	∴ if the △s and □s are equal, those bases and alts. are recip. propls.,
3	Theor. 2 Sch. II.	and also those △s or □s are equal,
4	Remk.	Now, ∵ the means are sides of one, and the extremes the sides of the other;
5		∴ those sides of the eq. □s are propls.;
6		and ∵ the sides are propls.
7	Conc.	∴ the □ under the means = □ under the extremes;
8	16, VI.	and ∴ the sides of those eq. □s are proportionals.

"Thus the sixteenth Proposition," says LARDNER, p. 203, "follows immediately from the first."

III. By the principle, that if four st. lines are proportional, the rectangle of the extremes is equal to the rectangle of the means, *we convert the equality of the two ratios,* or the proportion of the four lines, *into the equality of two rectangles.* In numerical proportion the expression, *the product of the extremes equals the product of the means,* points out a very similar, if not an identical process; and the phrase, "product of two lines," is sometimes used instead of "rectangle under two lines." The word *product* however, supposes the four magnitudes which are proportionals to be commensurable, and really denotes the result by multiplication of the numbers which stand for the magnitudes: bnt in general, it may be demonstrated both of *commensurable* and of *incommensurable* magnitudes, that, *if four st. lines,* A, B, C, D *be proportionals,* A : B :: C : D, *the rectangles under the extremes* A . D = *the rectangle under the means,* B . C.

IV. *The doctrine of* LIMITS, however, is needful for the *full* demonstration of this Proposition, when incommensurable magnitudes are introduced. Briefly stated, by a *limit* we mean a *fixed* magnitude to which another and a *variable* magnitude may approach as nearly as we please; and yet the variable magnitude is never able to attain an exact equality with the fixed magnitude. Thus, the circle is the limit of an inscribed polygon; the polygon may approach in area, as near as we choose, to the area of the circle, but never actually attains that area. Take any fixed quantity, A, and a variable quantity P;—in order that A may be named the limit of P, the two conditions have to be fulfilled—1st, that P never attains equality to A;—and 2nd, that P shall be capable of being made as nearly as we please on an equality with A. Again,

"Two fixed magnitudes, A and B, are the limits of two others, P and Q, when P and Q by increasing together, or by diminishing together, may be made to approach more nearly to A and B respectively, than by any the same given difference, but can never become equal to, much less pass A and B."—GEOM. *Plane, Sol. & Spher.* p. 248.

V. For the proper elucidation of this subject two leading Theorems are needed, given as M, V. p. 235; and P. V. p. 241.

THEOR. I. *If there be two fixed magnitudes* A and B *which are the limits of two others,* P and Q, *and if P be to Q always in the same given ratio of C to D; then A shall be to B in the same ratio.*

E. 1	Hyp. 1.	CASE 1. *Let P and Q, approach A and* B *by a continual increase.*
2	,, 2.	Let P and Q never = A and B, much less exceed A and B,
3	Hyp. 3.	but P & Q approach A and B more nearly than by any given difference.
4		Let now a M, B′ be taken, so that A : B′ = C : D;
5		if B′ ≠ B, B′ > or < B.

SUP. 1°. *Let B′ be < B, by any difference, as b;*

D. 1	H. 11, V.	∴ P : Q = C : D; & A : B′ = C : D; ∴ A : B′ = P : Q;
2	H.	but A always > P, ∴ B′ always > Q.
3	D. 2, & Sup. 1°.	∵, ∵ Q always < B′, and B′ < B by *b*;
4		∴ Q cannot approach B within the difference *b*;
5		but this is against the Hyp,; ∴ B′ ⋠ B.

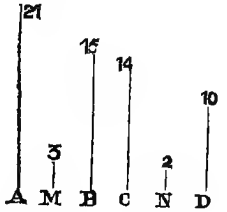
SUP. 2°. *Let B′ > B; and take A′ so that A′ : B = A : B′.*

D .1	H. 14, V.	∵ B < B′ ∴ A′ < A, as by some difference *a*.
2	Sup. 2°. H. 11, V.	And ∵ A′ : B = A : B′ & P : Q = A : B′; ∴ A′ : B = P : Q;
3	H.	but B always > Q ; ∴ A′ always > P ;
4		∴, ∵ P always < A′, & A′ < A by *a* ;
5		∴ P cannot approach A within the difference *a*;
6		but this is contrary to the Hyp. ; ∴ B′ ⩾ B ;
7	Conc.	∴ B′ cannot but = B, *i. e.* A : B = C : D
E.	H.	CASE II. Let P & Q approach A & B respectively by a continual decrease.
D. 1	*Sim.*	In this case the same demonstration may be given, taking care to substitute the word "greater" for "less" and "less" for "greater."
2	Rec.	Therefore; *if there be two fixed magnitdues, &c.* Q. E. D.

THEOR. II. *If four st. lines,* A B C D, *be proportionals.* (whether commensurable or otherwise) *the rectangle under the extremes will be equal to the rectangle under the means.*

CASE I. Let A and B be *commensurable,* & ∴ also C & D.

E. 1	Hyp.	Let their com. R. be 7 : 5, and their com. meas. M and N;
2	Pst. 2, V.	*i. e.* let M be contained 7 times in A, and 5 times in B,
3		and N be contained 7 times in C, and 5 times in D;
D. 1	E. 2, 3.	∵ A = 7 M, and D = 5 N; ∴ A . D = 7 × 5 times M.N;
2	E. 2, 3.	and ∵ B = 5 M, and C = 7 N; ∴ B . C = 5 × 7 times N . M;
3	Ax. 1. I.	∴ A . D = B . C.



CASE II. Let A & B be *incommensurable,* and ∴ also C and D.

D. 1	H.	∵ A : B = C : D;
2	Pr M, V.	and P, Q may approach nearer A, C than any assigned difference,
3		P and Q also containing like parts of B and D;
4		∴ P . D = Q . B.
5	Pr. M, V.	And ∵ there may be taken like parts of B, D continually less and less,
6		and ∵ P and Q increase towards A and D within any assigned difference;

7		and also ∵ P . D and Q . B, by increasing together, approach nearer to A . D and C . D than any assigned difference;
8	Theor. I.	∴ A . D = C . B, or B . C.
9	Rec.	Therefore, *if four st. lines be proportionals, &c.* Q. E. D.

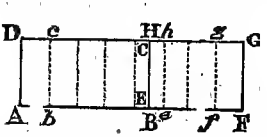
USE & APP. I. The Theory of Limits may be applied to establish Prop. 1, VI., both for bases that are commensurable, and for those which are incommensurable; thus,

The Rectangles AC *and* EG, *having the same alt.* AD = EH, *are to each other as their bases,* AB *and* EF.

CASE I. Let AB, EF be *commensurable, i. e.*, exact multiples of A*b* = E*f*.

C. 1	*Sup.*	Let AB = 5 A*b*, and EF = 3 E*f*;
2	11, I.	and from each . of the divisions of the bases erect perpendiculars, making 5 eq. □s on AB, and 3 eq. □s on EF.
D. 1	C. 2.	∵ □ AC = 5 D*b*, & □ EG = 3 H*f*;
2		∴ □ AC : □ EG = 5 : 3;
3	C. 1.	and ∵ AB = 5 A*b* and EF = 3 E*f*;
4		∴ □ AC : □ EG = AB : EF.
5	*Sim.*	In like manner, whatever may be the ratio.

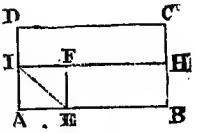
CASE II. Let the bases, AB, EF, be *incommensurable*, i. e., not exact multiples of A*b* and E*f*.



C. 1	Sup.	Let AB, EF form one st. line, AF = 8 A*b*;
2		and let *e* be the · in the division nearest to B or E.
D. 1	C. 1 & 2	∵ AF and A*e* are commensurable, AF = 8 A*b* & A*e* = 5 A*b*;
2		∴ A*eh*D : AFGD = A*e* : AF;
3		or, $\frac{\text{AEHD}}{\text{AFGD}} + \frac{\text{E}eh\text{H}}{\text{AFGD}} = \frac{\text{AE}}{\text{AF}} + \frac{\text{E}e}{\text{AF}}$.
4		But E*e h* H and E*e* may be reduced as small as we please, while AFGD and AF remain unchanged;
5	Sch. 4. 16, VI.	∴ by the theory of limits, AEHD : AFGD = AE : AF.

II. *The Rectangles* AC *and* AF *are to each other as the product of their bases by their altitudes;* i. e., as AB . AD : AE . AI.

C.	Pst. 2, I.	Produce the line IF to cut BC in H.
D. 1	Last Th.	$\frac{ABHI}{AEFI} = \frac{AB}{AE}$ & $\frac{ABCD}{ABHI} = \frac{AD}{AI}$;
2	× the equat.	and $\frac{ABHI \times ABCD}{AEFI \times ABHI} = \frac{AB \times AD}{AE \times AI}$;
		or, $\frac{ABCD}{AEFI} = \frac{AB \cdot AD}{AE \cdot AI}$.
3	*Sup.*	Let AI = AE = *unity;* then □ AEFI is a *unit of surface.*
4	Bk. II. p. 145.	∴ Units of surface in ▭ AC = AB . AD; *i. e.* = the linear units in AB × the linear units in AD.



N. B. In this sense the *Area of a* ▭ = product of base and perpendicular.

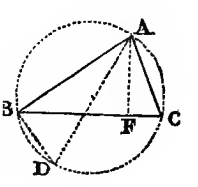
And ∵ *a* △ = ½ ▭ on the same base and of the same altitude;

∴ *Area of a* △ = ½ the product of the base and perpendicular.

III. Among other Theorems the following is easily deducible;

The rect. contained by AB, AC, *any two sides of a* △, ABC, *is equal to the rect. contained by* AF, *its alt., or perp. to the third side,* BC, *from the opp.* ∠ BAC, *and by the diam.* AD, *of the circumscribing* ⊙ BACD.

D. 1	31, III, & H.	∵ ∠ ABD is a rt. ∠ and also ∠ AFC;
2	21, III.	and ∠ ADB = ∠ ACF;
3	Cor. 33, 32, I.	∴ △s ADB and ACF are equiangular;
4	4, VI. 16, VI.	and ∴ AB : AD = AF : AC; and AB . AC = AD . AF.
5	Rec.	Therefore, the rect. contained by any two sides, &c. Q. E. D.

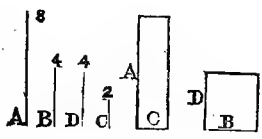


PROP. 17.—THEOR.

COR. to Prop. 16. *If three st. lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean*; and conversely, *if the rectangle contained by the extremes be equal to the square of the mean, the three st. lines are proportionals.*

CON. 3, I. DEM. 7, V. 16, VI.

E. 1	Hyp.	CASE I. Let A : B = B : C;
2	Conc.	then A . C = B . B, or B^2.
C.	3, I.	Take D = B



D. 1	H & 7, V.	∵ A : B = B : C, & B = D; ∴ A : B = D : C;
2	16, VI.	& ∴ □ A . C = □ B . D.
3	C. Conc.	but ∵ B = D ∴ □ B . D = B^2; & ∴ □ A . C = B^2.

E 1.	Hyp.	CASE II. And if □ A . C = B^2
2	Conc.	then A : B = B : C.
D. 1	H.	∴ A . C = B^2, & B^2 = B . D; ∴ A . C = B . D;
2	16, V.	& ∴ A : B = D : C;
3	C. Conc.	but B = D; ∴ A . B = B : C.
4	Rec.	Therefore, *If three st. lines be proportionals &c.*

Q. E. D.

Alg. & Arith. Hyp. Let $a\ 2 : b\ 4 = b\ 4 : c\ 8$.

Alg. If $\frac{a}{b} = \frac{b}{c}$;

$\times (b \times c)$ $\frac{abc}{b} = \frac{bbc}{c}$, i. e. $ac = b^2$;

Conversely, ∵ $ac = b^2$;

$\div (b \times c)$ ∴ $\frac{ac}{bc} = \frac{b^2}{bc}$, i. e. $\frac{a}{b} = \frac{b}{c}$.

Arith. If $\frac{2}{4} = \frac{4}{8}$;

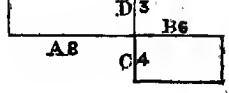
$\times (4 \times 8)$ $\frac{2 \times 4 \times 8}{4} = \frac{4 \times 4 \times 8}{8}$; i. e., $2 \times 8 = 4 \times 4$.

Conv. $\because 2 \times 8, = 4 \times 4$;

$\div (4 \times 8)$ $\therefore \frac{2 \times 8}{4 \times 8} = \frac{4 \times 4}{4 \times 8}$; i. e., $\frac{2}{4} = \frac{4}{8}$.

Use & App. I. *The Demonstration of Proportion in Arithmetic,* commonly named "*The Rule of Three,*" depends on the last four Props. *viz.,* 14, 15, 16 & 17, bk. VI.; and from that demonstration we deduce the Rule for *finding a fourth Proportional,* D, *to three given terms,* A 8, B 6, and C 4.

E.	Hyp.	Suppose the 4th propl. D to be found.
D. 1	16, VI.	$\because A.D = B.C$; $\therefore D = \frac{B.C}{A}$



Deduction, 1. $\frac{B.C}{A} = \frac{6 \times 4}{8} = \frac{24}{8} = 3 = D.$

„ 2. $\frac{B.C}{D.} = \frac{6 \times 4}{3} = \frac{24}{3} = 8 = A.$

„ 3. $\frac{A.D}{B} = \frac{8 \times 3}{6} = \frac{24}{6} = 4 = C.$

„ 4. $\frac{A.D}{C} = \frac{8 \times 3}{4} = \frac{24}{4} = 6 = B.$

II. The Properties of Equiangular triangles established in Prop. 4 and 5, and those of Proportionals in Prop. 16 & 17, lead to a very clear demonstration of 47, I; *that in a rt.* $\angle$ *d* $\triangle$ ABC, *the square of the hypotenuse* AC, equals *the sum of the squares of the other two sides,* AB & BC.

C.	12, I.	From the vert of the rt. $\angle$ B, draw BD $\perp$ AC.
D. 1	8, VI.	$\because$ $\triangle$ ADB is eq. ang. to $\triangle$ ABC which is eq. ang. to $\triangle$ BDC.
2	4, VI.	$\therefore$ AC : AB = AB : AD, and AC : BC = BC : DC;
3	17, VI.	$\therefore$ $AB^2 = AC.AD$; and $BC^2 = AC.DC$.
4	Ax. 2, I.	+ *equals*; $\therefore$ $AB^2 + BC^2 = AC.AD + AC.DC$.

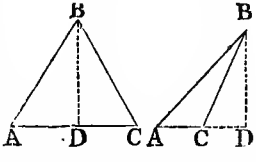
D. 5	C. 1, II.	Now ∵ side AC is com.
		∴ $AB^2 + BC^2 = AC \cdot (AD + DC)$;
6	Conc.	but $AD + DC = AC$;
		∴ $AB^2 + BC^2 = AC \cdot AC = AC^2$.
7	Rec.	∴ *In a rt. ∠d triangle, &c.* Q. E. D.

Thus the MOST IMPORTANT PROPERTIES of Plane Geometry are reduced to the *single property*; that "in similar triangles the sides about the equal angles are proportional.

III. Some of the deductions, made in connection with this demonstration, may be stated *in other words*, yet still in accordance with Prop. 16 & 17; thus, instead of the inference, "*if a perp. be drawn from the rt. ∠ to the hypotenuse the square of either side is equal to the rect. under the hypotenuse and seg. adj. to that side*, we say, "either side, as AB, is a mean proportional between the whole hypotenuse AC, and the seg. AD adj. to that side."

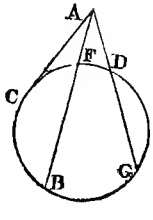
Also, instead of "*the square of the perp. BD, on the hyp. AC, will equal the rect, under the segs. of the hyp. AD . DC*," we declare,—"the perp. BD, is a mean propl. between the segs. of the hyp. AD & DC."

And in place of the deduction, "*in every △ if a perp. BD be drawn from the vertex B, to the base AC*, fig. 1, *or base produced AD*, fig. 2, *the difference of the squares of the sides BA, BC, shall be equal to the difference of the squares of the segs of base, AD, CD, or of the base produced*," we substitute, "the base is to the sum of the sides as the difference of the sides is to the difference of the segments of the base, or sum of the segments of the base produced.



IV. As instances in which the use of Props. 16 & 17 very considerably shortens the demonstrations of other propositions, let us take;

Ex. 1. *If from a point A, there be drawn two st. lines one AC, a tangent to a circle, the other, AB, a secant, then the tang. will be a mean propl. between the whole secant AB, and its ext. segment, AF.*



D. 1	H. & 36, III.	∵ AC touches and AB cuts the ⊙,
		∴ $AC^2 = AB \cdot AF$;
2	17, VI.	∴ BA : AC = AC : AF; *i. e.*, the tang. is a mean propl. to AB & AF.

Ex. 2. *If from a . A there be drawn several st. lines AB, AG cutting the circle, then the whole secants, AB, AG, will be one to another* inversely *as their external segments., AF, AD.*

D. 1	36, III.	∵ $AC^2 = BA . AF$ & also $= GA . AD$;
2	Ax. 1, I.	∴ $BA . AF = GA . AD$;
3	16, VI.	& ∴ $BA : GA = AD : AF$.

N.B. For additional Examples of the Use and Application of Propositions 16 & 17, the long series may be consulted given in Lardner's *Euclid* pp. 204—208.

Prop. 18.—Prob. *(Important.)*

Upon a given st. line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

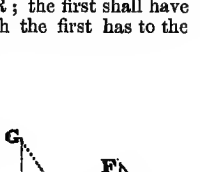
Con, 23, I. At a given . in a given st. line to make a rectl. ∠ equal to a given rectl. ∠.

32, I. If a side of a △ be prod. the ext. △ = two int. & opp. ∠s ; and the three int. ∠s of every △ together = two rt. ∠s.

Cor. 3, 32, I. In △s if two ∠s in each be eq. the third ∠s also eq.

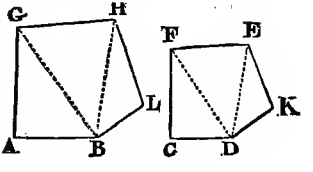
Dem. 4, VI. 22, V. If there be any no. of Ms, and as many others, which taken two and two in order have the same R ; the first shall have to the last of the first Ms, the same R, which the first has to the last of the others. Def. 1, VI. Ax. 2, 3, I.
Def. 12, V, Homologous sides.

E. 1	Dat.	Case I. Given AB and △ CDF ;
2	Quæs.	on AB to desc. a △ sim. to △ CDF, & AB homogous to CD.
C. 1	23, 1.	At A make ∠A = ∠C, & at B, ∠B, = ∠D ;
2	Cor. 3, 32, I.	then rem. ∠G = rem. ∠F, & △ ABG is eq. ang. to △ DF.
D. 1	C. 2, 4, VI.	∵ △ ABG is eq. ang. to △ CDF,
2	Def. 12, V.	∴ those △s are similar ; ∴ BA : AG = CD : CF ; & AB, CD are homolog. sides.
3	Hence	Hence on the given line AB &c.



E. 1	Dat.	CASE II. Given AB & rectl fig. CDEF of four sides;
2	Quæs.	on AB to desc. a rectl fig. similar, and similarly situated to fig. CDEF.
C. 1	Pst. 1. 23, I.	Join DF; at A make $\angle A = \angle C$, and at B, $\angle ABG = \angle CDF$;
2	23, I. Ax. 3.	then rem. $\angle AGB$ = rem $\angle CFD$; and $\triangle FCD$ is eq. ang. to $\triangle GAB$.
3	32, I.	Again at G, make $\angle BGH = \angle DFE$; & at B, $\angle GBH = \angle FDE$;
4	32, I. Ax. 3.	then rem. $\angle H$ = rem. $\angle E$, & $\triangle GBH$ is eq. ang. to $\triangle FDE$.
D. 1	C. 2 & 3.	$\because \angle AGB = \angle CFD$, & $\angle BGH = \angle DFE$;
2	Ax. 2.	$\therefore$ the whole $\angle AGH$ = the whole $\angle CFE$.
3	*Sim.*	Thus $\angle ABH = \angle CDE$, $\angle A = \angle C$, & $\angle H = \angle E$:
4	Cor. 3, 32, I.	$\therefore$ rectl. fig. ABHG is eq. ang. to rectl. fig. CDEF.
5	D. 3. 4, VI.	Also $\therefore \triangle GAB$ is eq. ang. to $\triangle FCD$, $\therefore$ BA : AG = DC : CF;
6	C. 4.	& $\because \triangle BGH$ is eq. ang. to $\triangle DFE$;
7	4, VI.	$\therefore$ AG : GB = CF : FD; & GB : GH = FD : FE;
8	22, V.	$\therefore$ *ex æq.* AG : GH = CF : FE.
9	*Sim.* 4, VI.	So AB : BH = CD : DE, & GH : HB = FE : ED.
10	Rec.	Now, $\because$ rectl. fig. on AB. is eq. ang. with the rectl. fig. on CD, and their sides proportional;
11	Def. 1, VI.	$\therefore$ figures, ABHG & CDEF are similar.

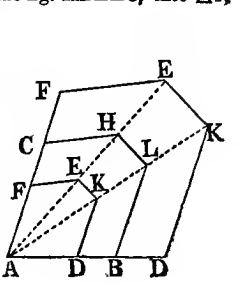
E. 1	Dat.	CASE III.—Given AB and rectl. fig. CDKE F of five sides;
2	Quæs.	on AB to desc. a rectl. fig. similar, and similarly situated to fig. CDKEF.



C. 1	Pst. 1, I, & Case II.	Join DE, and on AB desc. rectl. fig. ABHG sim- and similarly situated with fig. CDEF;
2	23, I.	at B made ∠ HBL = EDK, and at H, ∠BHL = ∠ DEK;
3	32, I. Ax. 3	then rem. ∠ L = rem. ∠ K.
D. 1	C. Def. 1. VI	∵ fig. AH is sim. to fig. CE, ∴ ∠ GHB = ∠ FED;
2	C. 2.	and ∵ ∠ BHL = ∠ DEK, ∴ whole ∠ GHL = whole ∠ FEK.
3	*Sim.*	So ∠ ABL = ∠ CDK; ∴ pent. GL is eq. ang. with pent. FK.
4	C. Def. 1, VI	And ∵ fig. AH is sim. to fig. CE; ∴ GH : HB = FE : ED;
5	4, VI. 22, V.	but HB : HL = ED : EK; ∴ *ex. æq.* GH : HL = FE : EK
6	*Sim.*	So AB : BL = CD : DK.
7	C. 2, 4, VI.	And ∵ △ BLH is eq. ang. with △ DKE; ∴ BL : LH = DK : KE.
8	D. 3.	∵ pent. GL is eq. ang. with pent. FK, and their sides proportional.
9	Conc.	∴ pent. GL is similar to pent. FK.
10	*Sim.*	By a like process a *hexagon* &c., may be described on AB.
11	Rec.	Therefore, *Upon a given st. line, &c.* Q. E. F.

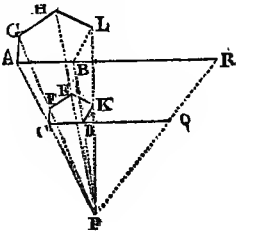
Sch. I. A more simple way *for making a rectil. fig., as* ADKEF, *similar to a rectil. fig.*, ABLHC, *would be;*

C. 1	3, I.	*Method* 1. On AB or AB produced set AD:
2	Pst. 1, III.	From the centre of similarity A, divide fig. ABLHC, into △s, by indefinite lines AK, AE;
3	31, I.	make DK ∥ BL, KE ∥ LH, and EF ∥ HC;
4	Sol.	then fig. ADKEF is similar to fig. ABLHC.
D. 1	C. 3.	∵ the respective lines are parallel;
2	29, I.	∴ ∠s at D = ∠s at B; ∠s at K = ∠s at L, &c. and ∠s at A com. *i. e.*, the respective △s are eq. ang.;
3	4, VI.	∴ the sides about the eq. ∠s are propl.
4		∴ AK : AL = DK : BL, and AK ; AL = KE : LH;



5		∴ DK : BL = KE : LH.
6	*Sim.*	So KE : LH = EF : HC, and so on;
7	Conc.	∴ the sides about the eq. ∠s are proportional and the figures similar. Q. E. F.

Method 2. On AB to place a fig. similar to the rectl. fig. CDKEF.

C. 1	31, I.	At a convenient distance from CD place AB ∥ CD;	
2	Pst. 1, I.	join AC and BD, if AB ; > CD, AC and BD will meet on being produced; let them meet in the . P;	
3	Psts. 1, 2, I.	join PF, PE, PK, and produce them indef;	
4	31, I.	through . A draw AG ∥ CF, through . G, GH ∥ FE, and through . H, HL ∥ EK; and join BL;	
5	Conc.	then fig. AL on AB is sim. to fig. CK on CD.	
D. 1	C. 1, 4.	∵ the respective lines CF, AG, FE and GH, EK and HL, &c. are parallel;	
2	29, I.	∴ the ∠s about the resp. points C & A, F & G, E & H &c. are equal;	
3	4, VI.	& ∴ the resp, △s are equiangular and similar;	
4		∴ CD : DP = AB : BP : & DP : DK = BP : BL;	
5	22, V.	∴ *ex æq.* CD : DK = AB : BL.	
6	*Sim.*	So the sides about the other eq. ∠s are proportional.	
7	Conc.	∴ fig. AL on AB is similar to fig CK on CD. Q.E.D.	

N.B.—Should AB = CD the demonstration must be derived from 32, I.

II. *To construct a rectl. fig.*, as in the last diagram, *similar to a given rectl. fig. AL on AB, and having its perimeter equal to a given st. line CQ.*

C. 1	Pst. 2, I. 3, I.	Prod. AB until AR = the perimeter of the given fig. AL
2	12, VI.	to AR, CQ & AB take a 4th propl. CD;
3	3, I.	set the 4th propl CD on CQ;
4	18, VI.	and on CD desc. a rectl fig. CK sim. to fig. AL;
5	Conc.	then the perimeter of fig. CK = the st. line CQ.
D. 1	C. 2	Now per. AR : per. CQ = AB : CD;
2	C. 1, & .	but per. of fig. AL = AR, and per. of fig. CK = CQ;
3	Conc.	∴ a fig. has been drawn, similar to AL & with perimeter = CQ.

III. "As many figures of the same species with different areas can be constructed on the same right line as the figure of the proposed species has sides of different lengths."

IV. Anticipating the next two propositions, and using the same construction as in the last figure but one, we may also prove, that *Similar triangles and polygons are to one another as the squares of their homologous sides.*

For, $\frac{\text{Area AEF}}{\text{Area AHC}} = \frac{AE^2}{AH^2} = \frac{AK^2}{AL^2} = \frac{AD^2}{AB^2}$, &c.

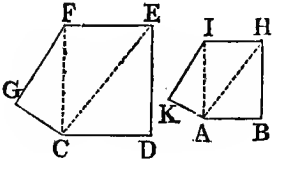
$\therefore$ Area AEF : AD^2 = Area AHC : AB^2.

In like manner, $\frac{\text{Area AKE}}{AD^2} = \frac{\text{area ALH}}{AB^2}$; and $\frac{\text{area ADK}}{AD^2} = \frac{\text{area ABL}}{AB^2}$

Adding equals, Area ADKEF : AD^2 = area ABLHC : AB^2;

$\therefore$ Area ADKEF : area ABLHC = AD^2 : AB^2.

Use & App. Nearly all *the practical methods of taking a Plan* of any building or place, or of drawing a Map of a field, of an estate, or of a whole country, are dependent on this Proposition. The lines made use of in the Plan or Map are *proportionals* to the existing lines in a building, estate, or country, and *Representatives* of the same values. The object of the Surveyor is to lay down a figure, as ABHIK, exactly like in shape to the *Prototype*, or original, as CDEFG, possessing similar angles and proportional sides; and this he accomplishes by actual measurements or calculations of the sides and angles, and then by reducing the lengths of the lines in his drawing or plan in exact proportion to the original Features or Outlines. The very same angles are given in the Map that exist in the field or the country that is surveyed; thus the figure ABHIK is a reduced copy of the fig. CDEFG, preserving the $\angle$s at A, B, &c., equal to those at C, D, &c.,—but giving the lines AB, BH, &c. in proportion only to the lines CD, DE, &c.



Whenever we have to make a Design, or a Model, we use this Proposition either actually or virtually, so that its application extends to all the Problems of Geodœsia, or the Art of Measuring and representing surfaces,—to Plans, Maps, and Geometrical Drawings of every kind.

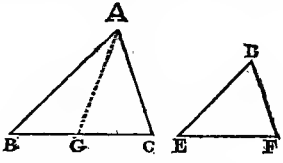
Prop. 19.—Theor. (*Very important.*)

Similar triangles are to one another in the duplicate ratio of their homologous sides; i. e., as the squares of their like sides.

Con. 11, VI. Dem. Def. 12, V.—16, V. If four Ms of *of the same kind* be propls., they shall also be propls. when taken alternately.

11, V. 15, VI. Def. 10, V. When three Ms are propls. the 1st is said to have to the 3rd the duplicate R. of that which it has to the 2nd. 1, VI.

E. 1	Hyp. I.	Let $\triangle ABC$ be sim. to $\triangle DEF$, & $\angle B = \angle E$;
2	,, 2.	& let AB : BC = DE : EF,
	Def. 12, V.	the side BC being homol. to EF;
3	Conc.	then $\triangle ABC : \triangle DEF = BC^2 : EF^2$.



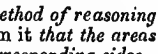
C. 1	11, VI.	Take BG a 3rd propl. to BC & EF, *i.e.*, BC : EF = EF : BG; and join GA.
D. 1	H. 2. 16, V.	∵ AB : BC = DE : EF; ∴ *altern.* AB : DE = BC : EF;
2	C. 11, V.	but BC : EF = EF : BG; ∴ AB : DE = EF : BG;
3	Sc. 1, 16, VI.	∴ the sides about the eq. $\angle$s. are recip. propl.,
4	15, VI.	∴ $\triangle$ ABG = $\triangle$ DEF.
5	C.	And ∵ BC : EF = EF : BG;
6	Def. 10, V.	∴ BC : BG has the duplicate R of BC : EF;
7	1, VI.	but BC : EF or BG = $\triangle$ ABC : $\triangle$ ABG;
8	D. 6, 7.	∴ $\triangle$ ABC : $\triangle$ ABG = $BC^2 : EF^2$;
9	D. 4.	but $\triangle$ ABG = $\triangle$ DEF; ∴ $\triangle$ ABC : $\triangle$ DEF = $BC^2 : EF^2$.
10	Rec.	Therefore, *Similar triangles are &c.* Q. E. D.

Cor. *If three st. lines are proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.*

Sch I. The duplicate ratio of two st. lines is the same with the ratio of their squares; thus similar triangles are to one another as the squares of their homologous sides.

II. This Proposition might be deduced as a particular case of Prop. 1, VI, by the 2nd Cor. "that triangles are to one another in the ratio compounded of the ratios of their bases and altitudes;" for, when triangles are similar, their altitudes are in the proportion of their bases, and by Def. 12, V. this compound ratio is the duplicate ratio of their bases.

USE and APPL. I. The 19th Prop. supplies a *general method of reasoning* concerning all similar geometrical figures; for we declare from it *that the areas of similar triangles are to one another as the squares for the corresponding sides.*



E. 1	Hyp. 1.	Let $\triangle$s ABC, AEF be similar;
2	,,	& AD $\perp$ BC; and AD produced, or AG, $\perp$ EF;
3	Cor. 3,32,I.	& $\triangle$s ADB, AGE equiangular, then Area of $\triangle$ABC : Area of $\triangle$AEF $= BC^2 : EF^2$, or $AB^2 : AE^2$;
D. 1	4, VI.	$\because$ AD : AG = BC : EF; & BC : EF = BC : EF;
2		on multiplying these equals together. AD . BC : AG . EF = $BC^2 : EF^2$.
3	Use 2, 16, VI.	But AD . BC = Area of $\square$ on BC with alt. AD;
4		and AG . EF = Area of $\square$ on EF ,, AG.
5	Use 2, 16, VI.	And Area of $\triangle$s = $\frac{1}{2}$ Area of $\square$s on the same base and altitude,
6	Sch. 1. 19, VI.	$\therefore$ Area of $\triangle$ ABC : Area of $\triangle$ AEF = $BC^2 : EF^2$.
7	*Sim.*	Thus are the Areas as the squares of any other pair of corresponding sides; for Area of $\triangle$ ABC : Area of $\triangle$ AEF = $AB^2 : AE^2$.

Ex. 1. From a given $\triangle$ AEF to cut off a part ABC = $\frac{1}{4}$ of $\triangle$AEF in area, by a line BC $\parallel$ EF.

Since, Area $\triangle$AEF : Area $\triangle$ABC = $EF^2 : BC^2$; *or* $1 : \frac{1}{4} = EF^2 : BC^2$,

$$\therefore BC^2 = \frac{EF^2}{4}, \text{ or } BC = \frac{EF}{2}.$$

Ex. 2. If the Area of a triangle be 36000 sq. feet, of which the base is 900 feet, what will be the area of a similar triangle of which the base is 450 feet.

Take x as representative of the unknown area;

$\because$ $36000 : x = 900^2 : 450^2$; $\therefore$ $810000\,x = 7290000000$;

and $x = 7290000000 \div 810000 = 9000$ sq. feet.

Ex. 3. The base EF, of a triangle is 3240 links, and its perpendicular 840 links; required the distance, from the base, of the parallel line, BC, which shall divide the triangle into two equal parts.

Here, Area of $\triangle$ AEF $= \frac{3240 \times 840}{2} = 2721600$ sq. links;

and Area of $\triangle$ ABC $= \frac{2721600}{2} = 1360800$ „ „

Now AG−AD = DG distance of parallels BC & EF;

To find AD; Area $\triangle$ AEF : Area $\triangle$ ABC $=$ AG^2 : AD^2,

Then AD$^2 = \frac{1360800 \times 840^2}{2721600} = 352800$;

$\therefore$ AD $= \sqrt{352800} = 593.978$ links.

And $840 - 593.978 = 246.022 =$ DG, distance of parallel BC from EF.

II. When in similar triangles, any side of the one, as AE, is double that of the other, the area of the one triangle is four times that of the other; for the Areas are as the squares of the sides; and $2^2 : 1^2 = 4 : 1$.

PROP. 20.—THEOR.

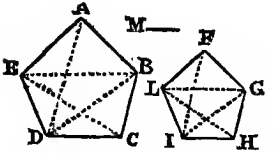
Similar Polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have, and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Similar polygons are divisible into similar triangles equal in number and homologous to the whole polygons; and the polygon has to the polygon the duplicate ratio which the homologous side has to the homologous side."—EUCLID.

CON, Pst. 1. I. Dem. Def. 1, VI. 6, VI. 4, VI. 32, I. Ax. 3, I. 22, V. 11 V. 12, V. If any number of Ms be propls, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents. 19, VI.

E. 1	Hyp. 1.	Let pol. AD be similar to pol. FI, and the $\angle$s at A, B, C, D, E respectively $=$ $\angle$s at F, G, H, I, L;
2	„ 2.	and let the sides AB, BC, CD, DE, EA be respectively homol. to the sides FG, GH, HI, IL, LF;
3	Conc. 1.	then pols. AD and FI, are divisible into $\triangle$s of which each contains the same number of sim. $\triangle$s;

4	,,	2.	of these sim. △s each has to each the same ratio as the pol. AD to the pol. FI.
5	,,	3.	and pol. AD : pol FI has the duplicate ratio of the R, AB : FG.



C.		Pst. 1, I.	Join E with the ∠ . s B, C; and L with the ∠ : s G,H.

Case I. The pols. AD & FI are divisible into ∠s, &c.

D. 1		H. 1, Def. 1, VI.	∵ pol. AD is sim. to pol FI ∴ ∠BAE= ∠GFL, & BA : AE = GF : FL;
	2	H. 1, 2.	And ∵ △s ABE, FGL have ∠A = ∠F, and their sides proportionals;
	3	6, VI. 4, VI.	∴ △ABE is eq. ang. & sim. to △FGL,
	4		∴ ∠ABE = ∠FGL.
	5	Hyp.	And ∵ pol. AD is sim. to pol. FI,
		Def. 1, VI.	∴ whole ∠ABC = whole ∠FGH;
	6	32, I. Ax. 3.	& ∴ rem. ∠EBC = rem. ∠LGH.
	7	D. 3. 4, VI.	And ∵ △ABE is sim to △FGL, ∴ EB : BA = LG : GF;
	8	Def. 1, VI.	Also ∵ of sim. pols. ∴ AB:BC=FG:GH;
	9	22, V.	∴ *ex. æq.* EB : BC = LG ; GH.
	10	D 9.	And ∵ the sides about eq. ∠s are propls,
	11	6, VI. 4, V1.	∴ △EBC is eq. ang. & sim to △LGH.
	12	*Sim.*	So △ECD is eq. ang. and sim to △LHI.
	13	Conc.	∴ Sim. pols. AD, FI are divided into the same number of sim. △s.

Case II. Also $\left.\begin{array}{l} \triangle ABE : \triangle FGL \\ \triangle EBC : \triangle LGH \\ \triangle ECD : \triangle IHL \end{array}\right\}$ = pol. AD : pol. FI.

CASE III. And pol. AD : pol. FI has the duplicate R of AB : FG.

D.	1	Case I. 19, VI	$\therefore$ $\triangle$ ABE is sim. to $\triangle$ FGL,
			$\therefore$ $\triangle$ ABE : $\triangle$ FGL = BE^2 ; GL^2
	2	*Sim.*	So $\triangle$ BEC : $\triangle$ GLH = = BE^2 : GL^2.
	3	11, V.	$\therefore$ $\triangle$ ABE : $\triangle$ FGL = $\triangle$ BEC : $\triangle$ GLH.
	4	Case I. 19, VI	Again $\because$ $\triangle$ EBC is sim. to $\triangle$ LGH,
			$\therefore$ $\triangle$ EBC : $\triangle$ LGH = EC^2 : LH^2;
	5	*Sim.*	So $\triangle$ ECD : $\triangle$ LHI = EC^2 : LH^2,
	6	11, V.	$\therefore$ $\triangle$ EBC : $\triangle$ LGH = $\triangle$ ECD : $\triangle$ LHI;
	7	D. 3.	but $\triangle$ EBC : $\triangle$ LGH = $\triangle$ ABE : $\triangle$ FGL,
	8		$\therefore$ $\triangle$s ABE : FGL = EBC : LGH = ECD : LHI;
	9	12, V.	$\therefore$ antec. : conseq. = all the antecs. : all the conseqs.
			i. e. $\triangle$ ABE : $\triangle$ FGL = pol. AD : pol. FI.
	10	19, VI.	But $\triangle$ ABE : $\triangle$ FGL = AB^2 : FG^2,
	11		$\therefore$ pol. AD : pol. FI = AB^2 : FG^2.
	12	Rec.	Wherefore, *Similar Polygons &c.*
			Q. E. D.

COR. I. *Similar figures of four sides, or of any number of sides,* as already proved of triangles 19, VI.; *are to one another in the duplicate ratio of their homologous sides.*

COR. II. *If three lines be proportionals, then,* as was proved for triangles, Cor. 19, VI., *the first shall be to the third as any polygon on the first is to the similar, and similarly described polygon on the second.*

C.		11, VI.	To AB, FG two homol. sides take M, a third proportional.
D.	1	Def, 10. V.	Then AB : M has the dupl. R of AB : FG;
	2	Cor. 1,20, VI.	but pol. AD : pol. FI = AB^2 : FG^2;
	3	11, V.	$\therefore$ AB : M = fig. on AB : fig. on FG.
	4	Rec.	$\therefore$ *If three lines be proportionals, &c.* Q.E.D.

Cor. III. Because all squares are similar figures, *the ratio of any two squares to one another is the same with the duplicate ratio of the sides;* and hence also, *any two similar rectilineal figures are to one another as the squares of the homologous sides.*

Cor. IV. *In similar figures, their perimeters are to one another as the ratio of the homologous sides.*

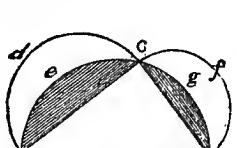
$\because$ AB,BC &c. : FG,GH &c. = AB : FG = BC : GH;
$\therefore$ AB + BC + CD &c. : FG + GH + HI &c. = AB : FG.
i. e., Perim. of fig. AD ; perim. of fig. FI = AB : FG.

Cor. V. The homologous diagonals being sides of similar triangles are also homologous sides ; and therefore, *perimeters of similar figures are as their homologous diagonals.*

Cor. VI. A Circle and its inscribed Polygon of an infinite number of sides do not differ from each other in any degree however small, but that a smaller difference might be assigned; and *practically*, they may be considered of identical values. As therefore it has been predicated of similar polygons, 20, VI, that "they are to each other as the squares of their corresponding sides," whether diagonals or sides;—so it may be predicated of *all circles*, that being similar figures, they *are to each other as the squares of their respective diameters, radii, and circumferences.*

Cor. VII. *If on the three sides*, AB, AC, CB, *of a rt. ∠d triangle* ACB, *similar figures be described* as semicircles, A *e* C *g* B, A *d* C and C *f* B; *the figure on the hypotenuse*, AB, *will be equal to the sum of the similar figures on the two other sides*, AC, CB.

D. 1	Cor. 6, 20, VI.	$\because$ Semicircles, or any sim. fig., on the three sides, are as the squares on those three sides;
2	47, I.	& $\because AB^2 = AC^2 + CB^2$,
3		$\therefore$ Semicircle on AB = Semic. on AC + Semic. on CB.



HIPPOCRATES, of Chios, a Pythagorean philosopher, who lived about 460 B.C., is said to have made this application of the universal principle, which under different forms appears in 47, I., and 31, VI., that "if *three similar figures be described upon the sides of a rt. angled triangle, the contents of that which is described upon the hypotenuse will be equal to the sum of the contents of the figures described upon the sides.*"

Hence, was deduced,

COR. VIII. That *the Lunes* A*e* C*d*, C*g* B*f*, formed by describing semicircles, as A*e* C*g* B, A *d* C & C*f* B, on the sides of a rt. ∠ d triangle, *are equal in area to the right angled triangle* ACB.

D. 1	Cor. 7. 20, VI.	Semic. on AB = semic. on AC + semic. on CB;
2	*Sum.*	from semic. on CB take away the shaded segments A *e* C & C *g* B;
3		The remainder is the rt. ∠ d △ ACB;
4	*Sum.*	From semicircles on AB, CB take away the same segments;
5		The remainders are the two lunes A*e*C*d*, B*g*C*f*;
6	AX. 4, VI.	∴ the Lunes A*e*C*d* + B*g*C*f* = △ ACB.

N. B. This is the first known instance in which a curvilinear space was reduced to an equivalent rectilinear space.

SCH. 1. Further elucidation of Prop. 20, will be found in LARDNER'S Euclid, pp. 210—214, and in pp. 127, 134 of this work.

2. In circles, which are all similar, when the segments are similar, the radii and the chords become homologous sides.

USE & APP. I. By this proposition, *a rt. lined figure may be increased or diminished in any ratio*

Thus, to make a pentagon *five* times the size of the pent. on DC, (in last figure but one.) ;

Take a side AE = 2; and between AE and 5 times AE, *i. e.*, 10, find the mean proportional;—it is $\sqrt{2 \times 10} = \sqrt{20} = 4.472136$;

A sim. fig. on a side of 4.472136 will be five times the pent. on AE.

II. When the homologous sides are known *the Proportion of one figure to another* will be obtained by finding a third proportional to any two corresponding sides.

III. The *increase or diminution of Circles* is effected in the same way.

Thus one circle is on a diam. of 1,—another is to be constructed 4 times larger; required the diam. of the second circle.

The mean propl. to 1 & 4 is $\sqrt{1 \times 4} = \sqrt{4} = 2$.

$\therefore$ the diam. is 2 for the required circle.

Or, Given the diameters 1 & 2, required the magnitude of the second circle when compared with the first.

Here, $\frac{2 \times 2}{2} = \frac{4}{1} = 4$,—the second circle is four times larger than the first.

IV. In two similar figures, if *of the areas and corresponding sides, any three be given, the fourth may readily be found.* The principle employed is, that the areas of similar figures are to one another as the squares of their homologous sides, and *vice versâ*.

For, (fig. 20, VI.) $\triangle$ ABE : $\triangle$ FGL = $AB^2 : FG^2$;
$\triangle$ BCE : $\triangle$ GHL = $BC^2 : GH^2$;
and $\triangle$ CDE : $\triangle$ HIL = $CD^2 : HI^2$.

Hence, $\triangle$s ABE + BCE + CDE : $\triangle$s FGL + GHL + HIL
= $AB^2 : FG^2$.

i.e. fig. ABCDE : fig. FGHIL = $AB^2 : FG^2$;.

Thus $FG^2 = \frac{\text{FGHIL} \times AB^2}{\text{ABCDE}}$; $AB^2 = \frac{\text{ABCDE} \times FG^2}{\text{FGIHL}}$;

Area ABCDE $= \frac{\text{FGHIL} \times AB^2}{FG^2}$;

and Area FGHIL $= \frac{\text{ABCDE} \times FG^2}{AB^2}$.

Ex. 1. Two similar hexagons are respectively of the Areas of 2500 sq. yards and 3000 sq. yards; a side of the first is 5 lineal yards; required the length of the corresponding side of the other hexagon.

Here, the unknown side $x = \sqrt{\frac{3600 \times 5 \times 5}{2500}} = \sqrt{36} = 6$ lineal yards.

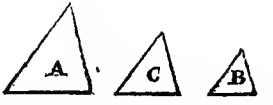
Ex. 2. One of the sides of the base of a pyramid measures 50 yards, and the area of the base 7500 sq. yards;—in a similar pyramid, what is the area of the base when the corresponding side is 60 yards ?

The Area $x = \frac{7500 \times 60 \times 60}{50 \times 50} = \frac{270000}{25} = 10800$ sq. yards.

PROP. 21.—THEOR.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

DEM. Def 1, VI. AX. 1, I. 11, V.

E. 1	Hyp.	Let figures A & B be each sim. to fig. C.
2	Conc.	then fig. A is sim. to fig. B,
D. 1	H.Def.1,VI.	∵ A is sim. to C, ∴ A is eq. ang. with C, & their homol. sides are proportionals. 
2	H. Def. 1, VI	∵ B is sim. to C, ∴ B is eq. ang. with C, and their homol. sides. propls.
3	Ax. 1, I. 11, V.	∴ A & B are each eq. ang. with C, and their homol. sides propls.
4	Def. 1, VI.	∴ rectl. fig. A is sim. to rectl. fig. B.
5	Rec.	Therefore, *Rectilineal figures which are similar, &c.* Q. E. D.

SCH. This proposition follows evidently from Def. 1, VI., of similar rectilineal figures; it is equivalent to an Axiom, and in this respect agrees with Prop. 30, I. St. Lines parallel to the same st. line are parallel to each other.

„ 11, VI. Ratios that are the same to the same ratio are the same to one another;

variations of the General Principle,—Things equal to the same thing are equal to one another.

Prop. 22.—Theor.

If four st. lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and conversely, *if the similar rectilineal figures similarly described upon four st. lines be proportionals, those st. lines shall also be proportionals.*

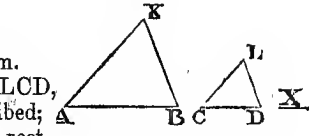
Con. 11, VI. 12, VI. 18, VI.

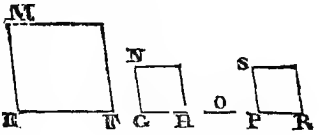
Dem. 11, V. 22, V. Cor. 2, 20, VI. 9, V. Magnitudes which have the same R to the same M are eq. to one another; and those to which the same M have the same R are eq. to one another.

7, V. Eq. Ms have the same R to the same M; and the same has the same R. to eq. Ms.

Case I. *Let four st. lines be proportionals*, &c.

E. 1	Hyp. 1.	Let AB : CD = EF : GH;
2	,, 2.	on AB & CD let sim. rect. figs. KAB & LCD, be similarly described;
3	,, 3.	& on EF & GH sim. rect. figs. MF & NH, be also similarly described;
4	Conc.	then KAB : LCD = MF : NH.
C.	11, VI.	To AB, CD take a 3rd propl. X, & to EF, GH a 3rd propl. O.
D. 1	H. 1.	∵ AB : CD = EF : GH,
	11, V.	∴ CD : X = GH : O;
2	22, V.	∵ *ex æq.* AB : X = EF : O.
3	Cor. 2, 20, VI.	but AB : X = fig. KAB : fig. LCD; and EF : O = fig. MF : fig. NH;
4	11, V.	∴ fig. KAB : fig LCD = fig. MF : fig. NH.





CASE II. *If the sim. rectilineal. figures, &c.*

E. 1	Hyp.	Let fig. KAB : fig. LCD = fig. MF : fig. NH;
2	Conc.	then AB : CD = EF : GH.
C. 1	12, VI.	Make AB : CD = EF : PR,
2	18, VI.	and on PR descr. fig. SR sim. and similarly situated to MF or NH.
D. 1	C. 1.	∵ AB : CD = EF : PR;
2	C. 2.	and on AB, CD are sim. and sim. desc. figs. KAB, LCD;
3	C. 2.	and on EF, PR „ „ figs. MF and SR.
4		∴ figs. KAB : LCD = MF : SR.
5	Hyp.	But KAB : LCD = MF : NH;
6	9, V.	∴ rect. fig. NH = rectl. fig, SR.
7	C. 2.	and ∵ they are sim. and sim. situated, ∴ GH = PR.
8	C. 1. D. 7.	And ∵ AB : CD = EF : PR, and PR = GH;
9	7, V.	∴ AB : CD = EF : GH.
10	Rec.	*If therefore four st. lines be proportionals, &c.* Q .E. D.

COR. As a particular case, *if four st. lines* A,B,C,D, *be proportionals their squares* A^2, B^2, C^2, D^2, *shall also be proportionals*; and conversely; *i. e.*,

if A : B = C : D; then $A^2 : B^2 = C^2 : D^2$;
and if $A^2 : B^2 = C^2 : D^2$; then A : B = C : D.

SCH. Thus, this proposition is equivalent to the theorem, that, "if two ratios be equal, their duplicates and subduplicates will also be equal," or in other words their powers and roots.

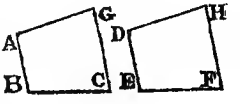
USE & APP. The principle contained in this Theorem is often employed in Arithmetic and Algebra; for if $a\ 3 : b\ 4 = c\ 6 : d\ 8$; then $a^2\ 9 : b^2\ 16 = c^2\ 36 : d^2\ 64$. *If four quantities or numbers are in proportion, their like powers or roots are also proportionals.*

Lemma.—Theor.

If rectilineal figures be equal and similar their homologous sides are equal.

Dem. 16, V. 20, VI.

E. 1	Hyp. 1.	Let fig. AC be =& sim. to fig. DF;
2	,, 2.	& let BC : BA = EF : ED;
3	Conc.	then side EF = side BC.



D. 1	Sup.	If EF ≠ BC, let EF > BC.
2	H. 2.16, V.	∵ EF : ED = BC : BA; ∴ *altern.* EF : BC = ED : BA;
3	D. 1. 20, VI	but EF > BC ∴ ED > BA; ∴ fig. DF > fig. AC;
4	H. 1.	now fig. DF also = fig. AC; —which is impossible;
5	Conc.	∴ EF not ≠ BC; *i. e.*, EF = BC. Q. E. D.

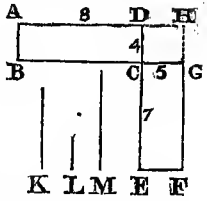
Prop. 23.—Theor.

Equiangular parallelograms have to one another the ratio which is compounded of the ratio of their sides.

Con. 14, I. 31, I. 12, VI.

Dem. Def. A. V. of Compound Ratio. When there are any no. of Ms of the same kind, the 1st is said to have to the last of them the R compounded of the R. which the 1st has to the 2nd, and of the R which the 2nd has to the 3rd, and of the R. which the 3rd has to the 4th, and so on unto the last magnitude. 1, VI. 11, V. 22, V. *ex æquali.*

E. 1	Hyp.	Let $\square$ AC be eq. ang. to $\square$ CF, having $\angle BCD = \angle ECG$;
2	Conc.	then $\square$ AC : $\square$ CF = R compounded of the ratio of the sides, BC : CG & DC : CE.
C. 1	*Pon.* 14, I.	Place BC & CG in a st. line and also DC & CE;
2	31, I.	complete the $\square$ DG; and take a st. line K;
3	12, VI.	make BC : CG = K : L; & DC : CE = L : M.
D. 1	C. 1.	Now K : L & L : M = BC : CG & DC ; CE;
2	D f. A.V.	but K : M is compounded of K : L & L : M;
3		∴ K : M is a R compounded of the R of the sides.
4	1, VI, C. 3.	And ∵ BC : CG = $\square$ AC : $\square$ CH, but BC : GC = K : L;
5	11, V.	∴ K : L = $\square$ AC : $\square$ CH.
6	C. 3	Again ∵ DC : CE = $\square$ CH : $\square$ CF, but DC : CE = L : M;
7	11, V.	∴ L : M = $\square$ CH : $\square$ CF.
8	D. 5 & 7,	And ∵ K : L = $\square$ AC : $\square$ CH;
9	22, V.	∴ *ex. æq.* K : M = $\square$ AC : $\square$ CF.
10	Remk.	But K : M is compounded of the R of the sides;
11		∴ $\square$ AC : $\square$ CF is compounded of the Rs of the sides; *i. e.* of BC : CG and DC : CE.
12	Rec.	∴ *Equiangular parallelograms have &c..* Q. E. D.



N, B. This 23rd proposition would follow as a Corollary from the Theorem, " that any two rectangles are to one another in the ratio which is compounded of the ratios of the sides."

A more brief demonstration would be,

E.	Hyp.	Let AC & CF be two eq. ang. parallelograms;
C.	31, I.	Complete the $\square$ CH.

D. 1	Def. A, V.	∵ $\square$ AC : $\square$ CF has the R compounded of the ratio of AC : CH, and of CH : CF;
2		but $\square$ AC : $\square$ CH = BC : CG, and $\square$ CH : $\square$ CF = DC : CE;
3	Conc.	∴ $\square$ AC : $\square$ CF is a ratio compounded of Rs which are the same with the Rs of the sides.

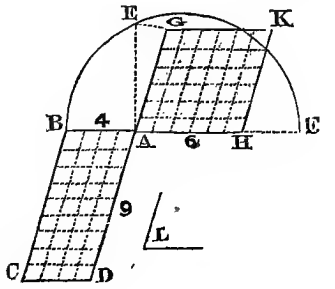
Arith. Hyp. Let BC = 8, CG = 5, DC = 4; and CE = 7.
then 4 : 7 = 5 : 8.75; and of the three numbers 8, 5, & 8.75,
8 : 5 = $\square$ 4 × 8 : $\square$ 4 × 5;
and 5 : 8.75 = $\square$ 4×5 : $\square$ 4 × 8.75;
thus 8 : 8.75 is the ratio compounded of both; or 32 : 35.

Cor. 1. *If the terms of two analogies are lines, the rectangles under their corresponding terms are proportional.*

Cor. 2. *Hence, also, Rectangles whose bases are proportional and also their altitudes, are themselves proportionals.*

Use & App. *To describe a rhombus equal to a given rectilineal figure* R, *and having an angle equal to a given angle* L.

C. 1	45, I. 23, I.	Draw $\square$ AC = given figure R. and with ∠BAD = given ∠L;
2	Pst. 2, I.	Produce BA, DA, indefinitely to G & F;
3	3, I. Pst. 3, I.	Make AF = AD; and on BF desc. a semicircle;
4	11, I.	from A raise a perp. AE cutting the semicircle in E.
5	3, I.	make AG, AH each = AE;
6	Sol.	and AG, AH are sides of the required rhombus.



D. 1	C. 1, 2. &c.	∵ the ▱s AC, AK are equiangular;
2	23, VI.	∴ ▱ AC : ▱ AK has a ratio compounded of the ratio of the sides;
3	Remk.	Thus AB : AH or AE = AG or AE : AD;
4	17, VI.	∴ $AG^2 = AB . AD$.
5	Remk.	Now AG is a side of the rhombus AK;
6	Conc.	∴ the rhombus AK = the rhomboid AC. Q. E. F.

PROP. 24.—THEOR.

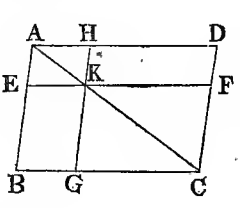
Parallelograms about the diameter of any parallelogram, are similar to the whole and to one another.

DEM. 29, I. If a st. line fall upon two ‖ st. lines, it makes the alternate ∠s = to one another; and the ext. ∠ = the int. and opp. ∠ upon the same side; and likewise the two int. ∠s upon the same side = 2 rt. ∠s. 4. VI.

34, I. The opp. sides and ∠s of ▱s are eq. to one another, and the diam. bisects them. 7. V. Def. 1, VI.

21, VI. Rectl. figures sim. to the same, are similar to each other.

E. 1	Hyp. 1.	Let BD be a ▱ of which AC is the diam;
2	,, 2.	& EH, GF ▱s about the diam.;
3	Conc.	then ▱ EH is sim. to ▱ GF; & each sim. to ▱ BD.



D. 1	H. 29, I.	∵ BC ‖ EF, ∴ ∠ABC = ∠AEF;
2	H. 29, 1.	& ∵ DC ‖ HK, or HG, ∴ ∠ADC = ∠AHK;
3	34, I.	And ∵ ∠DCB = ∠BAD,
		& ∠HKE = ∠BAD;
4		∴ ∠DCB = HKE = ∠BAD;

D. 5	Conc.	∴ ▱s BD & EH are equiangular.
6	D. 2.	And ∵ ∠ADC = ∠AHK, & ∠DAC com.;
7	4, VI.	∴ △DAC is eq. ang. to △HAK;
		& ∴ AD : DC = AH : HK.
8	34, I.	And ∵ opp. sides of ▱s are equal;
9	7, V.	∴ AD : AB = AH : AE;
		BC : CD = EK : KH;
		& BC : BA = KE : EA;
10	Def. 1, VI	∴ sides of ▱s BD, & EH about eq ∠s are proportionals.;
		& ▱ BD sim. to ▱ EH.
11	*Sim.*	So ▱ BD is sim to ▱ GF.
12		∵ ▱s EH & GF are each sim to ▱BD;
13	21, VI.	∴ ▱ EH is sim to ▱ GF.
14	Rec.	Wherefore *parallelograms about, &c.* Q. E. D.

Sch. Proposition 24 should have changed places with Prop. 25.

Use & App. In Perspective this proposition is available to show that a copy is drawn like the original, by the help of a parallelogram.

Prop. 25.—Prob. *(Of extensive use.)*

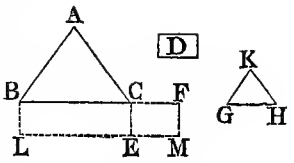
To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure, i. e. in area.

Con. Cor. 45, I. To desc. a ▱ equal to a given rectil. fig. and having an ∠ equal to a given rectil. ∠.
29, I. 14, I. 13, VI. 18, VI.

Dem. Cor. 2, 20, VI. 1, VI. 11, V. 14, V. If the 1st has the same R to the 2nd which the 3rd has to the 4th; then, if the 1st be > = or < the 3rd, the 2nd shall be > = or < the 4th.

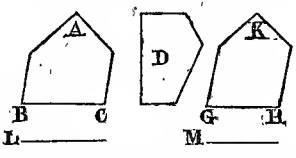
E.	1 Data.	Given rectil. fig. ABC, and rectil. fig. D;
	2 Quæs.	to desc. a rectl. fig. sim. to ABC. and eq. to D.

C. 1	Cor. 45, I.	On BC descr. $\square$ BE = fig. ABC;
2	,,	and on CE, $\square$ CM = fig. D, and having $\angle$ FCE = $\angle$ CBL.
3	29, I. 14, I.	Now BC, CF are in a st. line BF, and LE, EM, in st. line LM;
4	13, VI.	find then GH a *mean* propl. to BC, and CF;
5	18, VI.	and on GH descr. a rectl. fig. KGH, sim. to ABC, and similarly situated,
6	Sol.	then the fig. KGH is the rectl. fig. required.
D. 1	C. 4.	$\because$ BC : GH = GH : CF;
2	Cor. 2. 20, VI	$\therefore$ BC : CF = fig. ABC : fig KGH;
3	1, VI.	but BC : CF = $\square$ BE : $\square$ EF;
4	11, V.	$\therefore$ fig. ABC : fig. KGH, = $\square$ BE : $\square$EF.
5	C. 1. 14, V.	And $\because$ fig. ABC = $\square$ BE, $\therefore$ fig. KGH = $\square$ EF;
6	C.	but fig. EF = fig. D, $\therefore$ KGH = D and is sim. to ABC;
7	Sol.	Therefore fig. KGH is sim. to ABC and equal to D. Q. E. F.



Or, varying the figures.

C. 1	14, II.	Find L,M, the sides of squares eq. to the figures ABC & D;
2	12, VI.	take GH a 4th propl. to L, M, & BC;
3	18, VI.	and on GH desc. the fig. KGH sim. to ABC.



D. 1	20, VI.	$\because$ fig. ABC : fig. KGH = BC^2 : GH^2; *i. e.*, as L^2 : M^2;
2	C. 1.	and $\because$ fig. ABC = L^2, and fig. KGH = M^2 = fig. D;
3	C. 3.	and $\because$ also KGH is sim. to fig. ABC.
4	Conc.	$\therefore$ KGH is the fig. required.

Sch. The chief point of this problem is to find, as in the last fig. but one, a mean proportional to BC, a side of the given figure to which a similar one is to be constructed, and to CF, a side of the rectangle made equal to fig. D;— that mean proportional is GH, which becomes the side of a figure similar to ABC, and equal to D.

Use & App. By this proposition, while we keep always the same area, we may change the form or shape of the figure,—a process of great use in Practical Geometry; and especially convenient, if we do, what has very often to be done; *i. e.*, reduce an irregular rectilineal figure to its equivalent square.

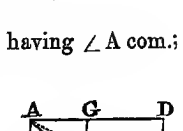
Prop. 26.—Theor.

If two similar parallelograms have a common angle, and be similarly situated; they are about the same diameter;

"If from a parallelogram a parallelogram be taken away similar to the whole, and similarly placed and having a common angle with it,—it is about the same diameter with the whole."—Euclid.

Con. Pst. 2, I. 31, I. Dem. 24, VI. Def. 1, VI. 11, V. 9, V. Ax. 9, I.

E. 1	Hyp. 1.	Let ▱ BD be sim. to ▱ EG, having ∠ A com.;
2	,, 2.	& let AE, AG, sides of ▱ EG fall on AB, AD, homologous sides of ▱ BD;
3	Conc.	then diags. AF, AC are in one line, *i. e.* ▱s BD & EG are about the same diam.:
C. 1	Sup. 1.	If not, and it be possible, let AHC be the diag. of ▱ BD;
2	,, 2.	& AHC be different from AF the diag. of ▱ EG;
3	Pst. 2, I,	let GF meet AHC in H,
4	31, I.	& hrough . H draw HK ‖ AD or BC.



D. 1	C. 1—4.	∵ ▱s BD & KG are about the same diag. AHC;
2	24, VI.	∴ ▱ KG is sim. to ▱ BD;
3	Def. 1, VI.	& DA : AB = GA : AK;
4	H. Def. 1, VI.	but ∵ ▱ BD is sim. to ▱ EG, ∴ DA : AB = GA : AE;
5	11, V.	∴ GA : AE = GA : AK;
6	9, V. Ax. 9	& ∴ AK = AE,—which is impossible:
7	Conc.	∴ diag. of ▱ BD through ∠A cannot be otherwise than on AF, *i. e.*, AF & AC are in the same st· line.
8	Rec.	Therefore, *If two similar parallelograms &c.* Q. E. D

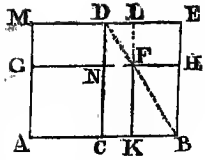
SCH.—This Proposition is the *converse* of Prop. 24, and properly should follow it immediately, or be incorporated with it.

PROP. 27.—THEOR.

Of all parallelograms applied to the same st. line, and deficient by parallelograms similar and similarly situated to that which is described upon the half of the line; that which is applied to the half and is similar to the defect, is the greatest.

N. B. More easily to understand this and the two following propositions, attention must he given to the Subsidiary Definitions D, E, & F. p. 251, 2.

CON. Pst. 1. I. 31, I. DEM. 43, I. The complements of the ▱s which are about the diam. of any ▱, are eq. to one another. 26. VI. Axs. 1, 2, 9. I. 34, I. 36, I. ▱s upon eq. bases, and between the same ‖s are eq. Ax. 4, I.

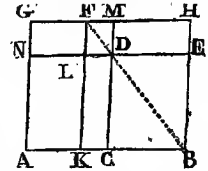


E. 1	Hyp. 1.	Let the st. line AB be bisd. in C;
2	„ 2.	& on the half AC let a $\square$ AD be applied, deficient from $\square$ AE on the whole line AB, by the $\square$ CE on the other half line CB;
3	Conc. 1.	then, of all $\square$s applied to any other pts. of AB, deficient by $\square$s sim. &similarly situated to $\square$ CE, $\square$ AD shall be the greatest.
4	Hyp. 3.	Let any $\square$ AF be applied to AK, a pt. of AB and $\neq$ AC or CB, so as to be deficient from $\square$ AH on AB by $\square$ KH,— $\square$ KH being sim. and similarly situated to $\square$ CE;
5	Conc 2.	then $\square$ AD on AC is $>$ $\square$ AF.

Case I. *Suppose* AK, *base of* $\square$ AF, $>$ AC *the half of* AB; and $\because$ by Hyp. 3, $\square$ CE is sim. to $\square$ HK,
$\therefore$ by 26. VI., they are about the same diam. DB.

C.	31, I.	Draw diag. DB, and complete the scheme.
D. 1	43, I.	$\because$ $\square$ CF = $\square$ FE, to each add $\square$ KH,
2	Ax 2, I.	$\therefore$ the whole $\square$ CH = the whole $\square$ KE;
3	H. 1. Ax. 1.	But $\because$ AC = CB,
		$\therefore$ $\square$ CH = $\square$ CG = KE;
4	*Add.* Ax. 2,	to each add CF,
		$\because$ $\square$ AF = gnomon CHL;
5	Ax. 9.	$\therefore$ $\square$ CE or AD $>$ $\square$ AF.

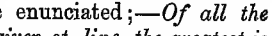
Case II. *Let AK, base of* $\square$ *AF, be* $<$ *AC, the half of AB.*



C.	31, I.	A similar construction is to be made.
D. 1	H. 1. 34, I.	$\because$ BC = CA,
		$\therefore$ HM = MG;
2	36, I. Ax. 9.	and $\because$ $\square$ DH = $\square$ DG.
		$\therefore$ $\square$ DH $>$ $\square$ LG.
3	43, I.	but $\square$ DH = $\square$ DK,
		and $\square$ DK $>$ $\square$ LG;
4	*Add.* Ax. 4.	to each ad $\square$ AL, $\therefore$ $\square$ AD $>$ $\square$ AF.

D. 5	Conc.	Thus in both cases $\square$ AD > $\square$ AF.
6	Rec.	Therefore, *of all* $\square$*s applied to* AB, *and deficient by* $\square$*s each sim. and similarly situated to* AB, *the* $\square$ AD *is the greatest.* Q. E. D.

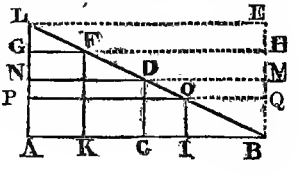
Otherwise the Proposition may be enunciated;—*Of all the rectangles contained by the segments of a given st. line, the greatest is the square which is described on half the line.*

E. 1	Hyp.	Given AB bisected in C and unequally divided in D; 
2	Conc.	then $AC^2 > \square$ AD . DB.
D. 1	H.	∵ AB is div. equally in C and uneq. in D;
2	5, II.	∴ AD . DB + $CD^2 = AC^2$
3	Conc.	∴ AC^2 > AD . DB.

Sch. De Chales, Lardner, and some other geometricians recommend that Propositions 27, 28, and 29 should be omitted as unnecessary; but they were frequently employed by the ancient mathematicians, and are required especially for the solution of several problems.

Use and App.—*In a given* $\triangle ABL$ *to inscribe the greatest parallelogram possible, having an angle, A, in common with the triangle.*

C. 1	31, I.	Complete the $\square$ AE of which LB is the diag.
2	10, I.	bis. AB in C, AC in K; AL in N and LN in G;
3	31, I.	and through . s K, C, I, draw ‖s to AL,
4	„	and through . s G, N, P draw ‖s to AB.
5	Sol.	then parallelog. AD, with ∠ A com. to $\triangle$ABL, is the *greatest* parallelog. possible in $\triangle$ABL.



D. 1	24, VI.	∵ the defects of parallelogs AO, AD &c. are parallelogs. sim. to parallelog. CM ;
2	26, VI.	∴ those parallelogs. are about the same diag. BL ;
3		∴ pargs. AO, AD, AF &c. are inscribed in the given △ABL;
4	C. 2-4.	And ∵ parallelog. AD is described on AC half the base,
5	27, VI.	∴ parallelog. AD is the greatest parallelog. in the △ABL, and the ∠A is com. to the parallelog. & the △. Q. E. F.

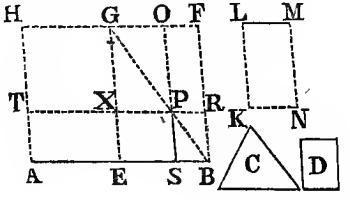
Prop. 28.—Prob.

To a given st. linê to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram ; but the given rectilineal figure to which the parallelogram to be applied is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied ; that is, to the given parallelogram.

"To given rt. line to apply a parallelogram equal to a given rectilineal figure and deficient by a figure similar to the given parallelogram."—Euclid.

Con. 10, I. To bis. a given finite st. line, 18, VI. 31, I. 25, VI. 3, I.

Dem. 21, VI. 26, VI. Ax. 3. 43, I. 36, I. Ax. 1. 24, VI.



E. 1	Dat. 1.	Let AB be the given st. line, and C the given fig. to which the parallelogram is to be equal;
2	,, 2.	let fig. C be $>$ the $\square$ on the half line AE, of which the defect is sim. to the defect of that $\square$ which is to be applied;
3	,, 3.	and let fig. D be the $\square$ to which this defect is to be similar.
4	Quæs.	To apply to AB a $\square$ = C, and deficient from the $\square$ on the line AB, by a $\square$ sim. to fig. D.
C. 1	10, I. 18, VI.	Bis. AB in E; and on EB desc. $\square$ EF sim. and similarly situated to fig. D;
2	31, I.	complete the $\square$ AG, either = C, or $>$ C.

CASE I. *Suppose* $\square$ *AG* = *fig. C*;

D.	Conc.	then on AB is applied $\square$ AG = fig. C, and deficient by $\square$ EF sim to fig. D. Q.E.D.

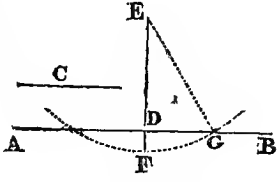
CASE II. *Suppose* $\square AG \neq$ *fig. C, but* $> C$; and by 36, I, $\square$ EF = $\square$ AG, & $\therefore$ $\square$ EF also $>$ fig. C.

C.	25, VI.	Make $\square$ KM = excess of $\square$ EF above fig. C, & sim. and simly situated to fig. D.
D. 1	C.1.21,VI	$\because$ fig D is sim. to $\square$ EF, $\therefore$ $\square$ KM is sim. to $\square$ EF.
2	Sup.	Let KL be homol. to EG, & LM to GF.
3	C. 3.	And $\because$ $\square$ EF = C + KM, $\therefore$ EF $>$ KM,
4		$\therefore$ line EG $>$ KL, & GF $>$ LM.
5	3, I. 31, I.	Make GX = LK, GO = LM; and complete $\square$ XO.
6	C. D. 1.	$\because$ $\square$ XO is = & sim. to $\square$ KM, & KM sim. to $\square$ EF,
7		$\therefore$ $\square$ XO is sim. to $\square$ EF,
8	26, VI.	& $\therefore$ $\square$s XO & EF *both* are about diag. GB.
9	Sup. 31, I.	Let GPB be their diag.; and complete the scheme.
10	D. 3, 6.	then $\because$ $\square$ EF = C + KM, and a part $\square$ XO = $\square$ KM a part of the other;

D.11	Ax. 3.	∴ rem. gnomon ERO= rem. C.
12	43, I.	And ∴ □ OR = parlm. XS, ∴ on adding □ SR to each, □ OB = parlm. XB;
13	C. 1, 36, I. Ax. 1.	but AE = EB; ∴ parlm. XB = parlm. TE & parlm. TE = parlm. OB;
14	*Add.*	to each add parlm. XS; ∴ the whole □ TS = the gnomon ERO;
15	D. 11.	but ERO = fig. C; ∴ also parlm. TS = fig. C.
16	Rec. 24, VI.	∵ *parlm.* TS = fig. *C is applied to st. line AB, deficient by parlm. SR sim. to fig. D, because SR is sim. to EF.* Q. E. F.

Sch. 1. The Proposition may be thus enunciated; *To divide a given st line AB, so that the rectangle contained by the segments may be equal to a given space, as the square on C; but that given space must not be greater than the square of half the given line.*

E. 1	Dat.	Given st. line AB, & C^2 the space to which the seg. of AB must be equal, but $C^2 > \left(\frac{AB}{2}\right)^2$
	2 Quæs.	to divide AB so that the □ contained by the segs = C^2.



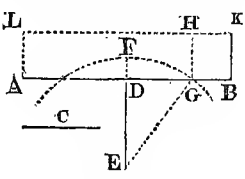
C.	10, I.	Bis. AB in D.
D. 1	Sup. 1.	If $AD^2 = C^2$ the Problem is solved;
2	Sup. 2.	but if $AD^2 \neq C^2$, & AD > C;
3	Sim.	then the Solution though differing in form will be like the foregoing Prop. 28 in substance.
C. 1	3, I. 11, I.	At rt. ∠s to AB draw DE= C;
2	3, I.	& prod. ED so that EF = AD or DB;
3	Pst. 3 & 1, I	from E with rad, EF cut AB in G, and join EG;
4	Sol.	then AB is div. in G so that AG . GB = C^2
D. 1	C.	∵ AB is divided equally in D and unequally in G;
2	5, II.	∴ □ AG . GB + DG^2 = DB^2 = EF^2 = EG^2;
3	47, I.	but $ED^2 + DG^2 = EG^2$;
4	Ax. 1. 1.	∴ □ AG . GB + DG^2 = $ED^2 + DG^2$;
5	*Sub.* Ax. 3	take away DG^2, & □ AG . GB = $ED^2 = C^2$. Q. E. F.

Sch. 2. Prop. 28. bk. VI. is equivalent to the Problem;—"*To inscribe in a given △ a parlm. equal to a given figure not greater than the maximum inscribed parlm. and having an ∠ in common with the △.*" *Manual of Euclid,* Pt. II. *p.* 98.

The Demonstration is similar to that in Use and App. 27, VI.

USE & APPL. There are two cases of this Proposition which are not unfrequently employed by geometers.

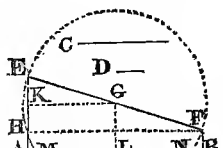
1°. A variation of Sch. 1, 28, VI. *To a given st. line* AB *to apply a rect.* AH, *deficient by a square* GK, *which rect. shall be equal to a given square, that on line* C; *but the given square on* C *must not be greater than the square on the half* AD *of line* AB.



C. 1	10, 1.	Bis. AB in D; and if $AD^2 = C^2$, the prob. is solved.
2	Sup.	But if $AD^2 \neq C^2$ and $AD > C$;
3	11, I. 3, I.	draw DE ⊥ AB, and make DE = C;
4	3, I.	produce ED so that EF = AD or DB;
5	Pst. 3 & 1, I	from cen. E, with EF desc. arc meeting AB in G; and join EG;
6	46, I. 31, I,	on GB desc. the square GK, and complete rect. GL;
7	Sol.	then □ AH = C^2, and deficient by GK, has been applied to AB.
D. 1	C.	∵ AB is div. equally in D, and unequally in G;
2	5, II. 47, I,	∴ AG . GB + DG^2 = DB^2 = EF^2 = EG^2 = ED^2 + DG^2.
3	Sub. Ax. 3.	Take DG^2 from each, ∴ AG . GB = ED^2, *i. e.* = C^2.
4	C. 6.	But AG . GB is □ AH, ∴ GH = GB;
5		∴ □ AH = C^2.
6	Conc.	∴ □ AH = C^2 has been applied to AB, deficient by the sq. GK. Q. E. F.

2°. *To a given st. line to apply a rectangle, which shall be equal to a given rectangle, and be deficient by a square; but the given rectangle must not be greater than the square upon half the given line.*

E.	Dat.	Let AB be the given line, and on lines C & D a given □ not greater than $\left(\frac{AB}{2}\right)^2$;
2	Quæs.	to apply to AB a □ = C . D, deficient by a square.
C. 1	11, I.	On the same side of AB, at A & B, draw AE, BF each ⊥ AB;
2	2, I. 10, I.	make AE = C and BF = D; join EF and bisect it in G;
3	Pst. 3, I.	from G, with rad. GE, desc. a ⊙ meeting AE in H;



4	31, I.	join HF; and draw GK ∥ HF, and GL ∥ AE, meeting AB in L.
D. 1	31, III. Ax. 11, I.	∵ ∠ EHF in a sem. c. is a rt. ∠, and = ∠ EAB;
2	28, I. 34, I.	∴ AB ∥ HF and AH ∥ BF, ∴ AH = BF;
3	Rem. C. 2.	and □ EA . AH = □ EA . BF, *i. e.*, C.D.
4	C. 2, 4.	and ∵ EG = GF, and AE ∥ LG ∥ BF;
5	34, I. 3, III.	∴ AL = LB, & EK = KH; and C . D ⋗ AL² or $\left(\frac{AB}{2}\right)^2$;
6	Conc.	∵ EA . AH ⋗ AL², *i. e.* KG²;
7	*Add.* 47, I.	add KE², ∴ AK² ⋗ EK² + KG². *i. e.* EG²;
8		and ∴ AK or GL ⋗ GE.
9	Def. 2, III.	Now if GE = GL, ⊙ EHF touches AB in L;
10	36, III.	∴ AL² = EA . AH, *i. e.* C . D.
11	Sup.	But if EG ≠ GL, and EG > GL; ∴ ⊙ EHF cuts AB;
12	Sup.	Let the ⊙ cut AB in . s M and N;
13	46, I. 31, I.	on NB desc. NB² and complete the □ AP.
14	3, III. Ax. 3	∵ LM = LN, and AL = LB, ∴ AM = NB,
15	C.	and ∴ AN . NB = NA . AM, *i. e.* EA . AH or C . D.
16	C.	But AN . NB is □ AP; ∴ PN = NB;
17	Rec.	∴ □ AP = C. D, and AP has been applied to AB, deficient by BN². Q. E. F.

This last Problem may be thus enunciated; "*To cut a given line* AB *in the point* N *so as to make the rectangle* AN . NB *equal to a given space.*"

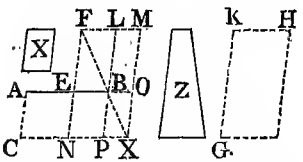
Or, which is the same thing, "*Having* AB *the sum of the sides of a rectangle given, and also its magnitude, or area, to find the sides.*"

Prop. 29.—Prob.

To a given st line to apply a parallelogram equal to a given rectilineal figure, and exceeding by a parallelogram similar to another given parallelogram.

Con. 10, I. 18, VI. 25, VI. 21, VI. 2, I. 31, I.
Dem. 26, VI. 36, I. 43, I. 24; VI.

E. 1	Dat.	Given AB, ▱ X, & rectil. fig. Z,
2	Quæs.	to apply to AB a ▱ = Z, and exceeding by a ▱ sim. to ▱ X.



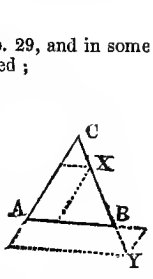
C. 1	10, I. 18, VI.	Bis. AB in E, and on EL descr. ▱ EL sim. & sim. sit. to ▱ X;
2	25, VI.	make ▱ GH = EL + Z, & sim. & sim. to ▱ X.
3	21, VI.	∴ ▱ GH is sim. to ▱ EL.
4		Let KH be homol. to FL, & KG to FE;
5	C. 2.	then ∵ ▱ GH > ▱ EL, ∴ HK > FL & KG > FE;
6	Pst. 2, I. 3, 1.	Produce FL & FE, & make FLM = KH, & FEN = KG;
7	31, I.	& complete the ▱ MN;
8	C, 6, 7.	∴ ▱ MN is eq. & sim. to ▱ GH;
9	C. 3.	but GH sim to EL, ∴ MN is sim to EL;
10	26, VI.	& ∴ EL & MN are ▱s about the same diag.
11	31, I.	Draw their diag. FX, & complete the scheme,
12	Sol.	then to AB is applied ▱ AX = Z, & exceeding by ▱ PO sim. to ▱ X.
D. 1	C. 2, 8. Ax. 1.	∵ GH = EL + Z, & ▱ GH = ▱ MN, ∴ ▱ MN = ▱ EL + ▱ Z;
2	*Sub.* Ax. 3.	take away ▱ EL; ∴ rem. gnomon NOL = Z.
3	C. 1, 31, I. 43, I.	And ∵ AE = EB ∴ ▱ AN = ▱ NB ▱ BM.
4	*Add.* Ax. 2.	Add ▱ NO, ∴ the whole ▱ AX = the whole gnomon. NOL;
5	D. 2.	but NOL = Z, ∴ ▱ AX = fig. Z.
6	Conc.	∴ to AB is applied a ▱ AX = fig. Z, exceeding by ▱ PO, sim. to fig. X; ∵ ▱ PO is sim. to ▱ EL. Q. E. F.

N.B.—In the diagram, X by mistake of the engraver occurs twice.

Sch.—Of the thirteen books of Elements written by Euclid, the tenth bears evident traces of the greatest attention having been bestowed to render it complete. The doctrine of *Incommensurables* there receives its developement, and is treated with great exactness. "The most conspicuous propositions of elementary geometry," says an eminent writer, "which are applied in the tenth book, are the 27th, 28th, & 29th of the sixth book, of which it may be useful to give the algebraical signification. The first of these (the 27th) amounts to shewing that $2x - x^2$ has its greatest value when $x = 1$, and contains a limitation necessary to the conditions of the two which follow. The 28th proposition is a solution of the equation $ax - x^2 = b$, upon a condition derived from the preceding proposition, namely, that $\frac{1}{4}a^2$ shall exceed b. It might appear more correct to say that the solution of this equation is one particular case of the proposition, namely, where the given parallelogram is a square; but nevertheless, the assertion applies equally to all cases. Euclid, however, did not detect the *two* solutions of the question; though if the diagonal of a parallelogram in his construction be produced to meet the production of a line which it does not cut, the second solution may be readily obtained. This is a strong presumption against his having anything like algebra; since it is almost impossible to imagine that the propositions of the tenth book, deduced from any algebra, however imperfect, could have been put together without the discovery of the second root. The remaining proposition (the 29th) is equivalent to a solution of $ax + x^2 = b$; but the case of $x^2 - ax = b$ is wanting, which is another argument against Euclid having known any algebraical reasoning.—Penny Cyc. XII, p. 38.

Use and App.—Several Problems of a like kind to Prop. 29, and in some respects equivalent to it may be here advantageously introduced;

Prob. 1°. *exscribe To to a given triangle ABC, a parallelogram equal to a given rectilineal figure, AX, and having an angle equal to one of the angles of the given triangle.*



Here AX is the *inscribed* ▱ & AY the *exscribed* ▱

N.B. This Prob. is equivalent to to the foregoing Prop. 29. *Manual of Euclid*, p. 100.

Prob. 2°. *To a given st. line AB to apply a rectangle which shall be equal to a given square, that on C and exceeding by a square..*

C. 1	10, I. 11, I.	Bis. AD in D, and draw BE ⊥ to AB so that BE = C ;
2	Pst. 1, 3, I.	join ED, and from D with DE desc. a ⊙ meeting AB produced in G ;
3	36, I, 31, I.	on BG desc. the sq. BH, and complete □ GL
4	Sol	then □ AH = C², & exceeding by GK is applied to AB.
D. 1	C.	∵ AB is divided equally in D and produced to G,
2	6, II	∴ AG . GB + DB² = DG² = DE.² = EB² + BD² ;
3	*Sub.* Ax. 3,I.	from each take DB², ∴ rem. AG . GB = BE² = C²
4	C. 3.	but ∵ GH = GB, ∴ AG, GB is rectangle AH ;
5		∴ □ AH = C²
6	Conc.	∴ AH = C², and exceeding by GK, is applied to AB.

Q. E. F.

N.B.—This Problem is the same as, "*To produce a given st. line, AB, so that the rectangle contained by the external segments of the given line may be equal to a given space, as C².*" The foregoing Con. and Dem. may be used, and then it will appear that the problem is, *to produce AB so that AG . GB = C².*

C. 1	2, I. 11, I.	Draw AE = C, & BF = D, ⊥s AB on contrary sides ;
2	Pst. 1, I. 31,I.	join EF & bisect EF in G ;
3	Pst. 3, I.	from G with GE desc. ⊙ meeting AE in H ;
4	Pst. 1, 31, I.	join HF, & draw GL ∥ AE.
5		Let the ⊙ meet AB produced in M,N ;
6	46, I, 31, I.	on NB desc. sq. NO, & complete □ NQ.
7	Sol.	to AB is applied a □ AP = C . D & exceeding by square BP.
D. 1	C. 1, 28, I.	∵ ∠EHF a rt. ∠ = ∠EAB, ∴ AB ∥ HF;
2	34, I.	∴ AH = BF ; & EA . AH = EA . BF = C . D,
3	3, III. Ax. 1.	And ∵ ML = LN, & AL = LB, ∴ MA = BN ;
4	35, III.	& ∴ AN . NB = MA . AN = EA . AH = C . D ;
5	Ax. 1, I.	∴ AN . NB, *i.e.*, AP = C . D.
6	Conc.	∴ to AB is applied AP = C . D & exceeding by sq. BP.

Q. E. F.

N.B. This Problem is the same as,—"*To find a point N in a given st. line AB produced, so as to make the rectangle AN . NB = a given space.*"

Or, which is the same thing,—"*Having given AB the difference of the sides of a rectangle and the magnitude of it, to find the sides.*"

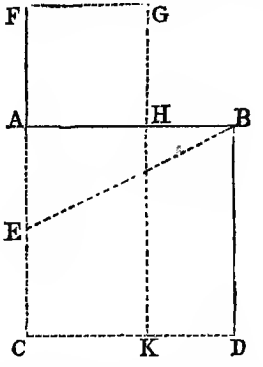
Prop. 30.—Prob.

To cut a given st. line in extreme and mean ratio; i. e., so that the whole line shall be to the greater segment as the greater segment to the less.

Con. 46, I. To describe a sq. on a given st. line.
11, II. To divide a given st. line into two pts, so that the rect. contained by the whole and one of the pts, shall be equal to the sq. of the other part.

Dem. Ax. 3, I. 14, VI. 34, I. Def. 30, I. A sq. has all its sides eq. and all its ∠s rt. ∠s. 14. VI. 17, VI. Def. 3, VI.

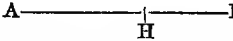
E, 1	Dat.	Given AB a st. line;
2	Quæs.	to cut it in extreme and mean ratio.



C. 1	45, I.	On AB construct a square AD;
2	29, VI.	to AC apply □ CG = □ AD, exceeding by fig. AG, sim. to AD;
3	Sol.	then AB is cut in extreme & mean ratio in . H.
D. 1	C. 1.	∵ AD is a square, ∴ AG is a sq.
2	C. 2. Ax.	∵ sq. AD = □ CG,
	3, I.	∴ rem. BK = rem. AG;
3	C. 14, VI.	& ∵ BK is eq, ang. to AG, ∴ KH : HG = AH : HB;

4	34, I.	but KH = AC = AB;
	Def. 30, I.	& HG = AH;
5		∴ BA : AH = AH : HB;
6	Ax. 9, I.	but AB > AH;
	14, V.	AH > HB.
7	Con. Def. 3, VI.	∴ AB is cut in extreme and mean ratio, *i. e.*, AB : AH = AH : HB. Q. E. F

Otherwise,



C.	11, II.	Divide AB in H so that AB . BH = AH².
D. 1	C. 17, VI.	∵ AB . BH = AH², ∴ AB : AH = AH : HB;
2	Def. 3, VI.	∴ AB is cut in extreme and mean ratio in . H.

Sch. 1. A st. line thus divided is also said to be divided *medially*; an the ratio of its segments is called the *medial ratio.* The same division take place in Prop. 11, bk. II, when the rectangle of the whole and one of its parts is equal to the square on the other part.

2. The dividing a line into extreme and mean ratio belongs to a class of Problems which relate to incommensurable magnitudes. The following is the General Theorem respecting them.

"*Let there be two Magnitudes of the same kind,* P & Q; *and let* P *be contained in* Q *a certain number of times which is to* P *as* P *is to* Q; *then the Magnitudes* P *&* Q *shall be incommensurable.*

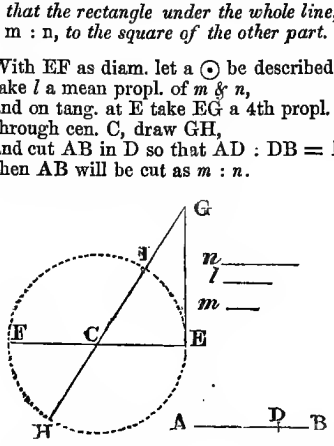
C.	*Sup.*	Let 5 P = Q — R a remainder.
D. 1	Hyp. 15, V.	Then ∵ R : P = P : Q, and 5 R : 5 P = P : Q;
2	16, V. 17, V.	∴ *alt.* 5 R : P = 5 P : Q, and *div.* 5 R ~ P : P = 5P ~ Q : Q.
3	16, V. 23, V.	But P : R = Q : P, ∴ *ex æq.* 5 R ~ P : R = 5 P ~ Q or R : P.
4	A, V.	Now R < P, ∴ 5 R ~ P < R; *i. e.* 5 R = P with a rem. 5 R ~ P or R.
5		And R_2 : R = R : P;
6		∴ 5 R_2 = R, with a rem. R_3.
7	*Sim.*	So 5 times every Rem. = preceding Rem. with a new Rem.;
	Sch. N, V.	∴ No Rem. is contained exactly in the preceding Rem.;
	Conc.	∴ P & Q have no common measure.

Now the segments AH, HB of a line AB, medially divided are magnitudes like P and Q; for the gr. seg. AH is contained in the whole line AB once, with a rem. HB; and HB : AH = AH : AB; ∴ HB also is contained once in AH with a rem.; and R_2 : HB = HB : AH; ∴ AH and HB are incommensurable.

3. Or, the dividing of a line into extreme and mean ratio may be considered as a particular case of the following more general problem;

To divide AB *so that the rectangle under the whole line, and one part shall bear a given ratio, as* m : n, *to the square of the other part.*

C. 1	10, I. Pst. 3, I.	With EF as diam. let a ⊙ be described;
2	13, VI.	take *l* a mean propl. of *m* & *n*,
3	17, III. 12, VI.	and on tang. at E take EG a 4th propl. to *l*, *m* and EF;
4	Pst. 1, I.	through cen. C, draw GH,
5		and cut AB in D so that AD : DB = HI : IG.
6	Sol.	then AB will be cut as *m* : *n*.



D 1	C. 3,	∵ $EG^2 : EF^2 = m : n$.
2	*Note* †	∴ EG : EF = *m* : *l*, i. e. in the subduplicate ratio of *m* : *n*.
3	36, III.	but EG^2 = HG . GI; ∴ HG . GI : $HI^2 = m : n$;
4		∴ HG is cut as required at I.
5	C. 5.	and AD : DB = HI : IG,
6	Conc.	∴ AB also is cut as required in the . D. Q. E. F.

N. B. In the solution of this problem it is assumed that if HI : IG = AD : DB, then HG . GI : HI^2 = AB . BD : AD^2. The proof is simple, and is given in LARDNER's *Euclid*, p. 220.

"Hence, *if two lines be cut in extreme and mean R, they are cut similarly*; and *if a line be cut in extreme and mean R, any line cut similarly will be also cut in extreme and mean R.*"

† "The ratio $\frac{A}{B}$ is *subduplicate* of $\frac{a}{b}$ when it is equal to the ratio of *a* to a mean proportional between *a* and *b*.

4. *Numerically*, as in Sch. 2., *we may approximate to the ratio of incommensurables* by a process similar to that of Continued Fractions; thus,

Take *two* magnitudes P & Q; and let 2 P + Rem. (remainder) = Q;
also let Rem. : P = P : Q:

Then, by making *each new term* = 2 *last*, + *the last but one*,
P : Q lies between any two consecutive ratios of the series,
1 : 2, 2 : 5, 5 : 12, 12 : 29, 29 : 70, 70 : 169, &c.,

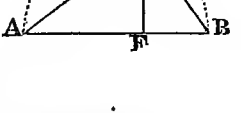
i. e., the rat. P : Q shall lie between 1 : 2, and 2 : 5; again between 2 : 5 and 5 : 12 &c.; the ratios of each successive pair approaching always more nearly to one another, and therefore to the ratio sought.

In the same manner, by making *each new term* = 5 *last* + *the last but one*,
when 5 P + Rem. = Q,
we may approximate to the ratio P : Q by means of the series
1 : 5, 5 : 26, 26 : 135, 135 : 701, &c.

N. B. In the case of *the medial ratio* the series is the simplest possible; namely, 1 : 1, 2 : 3, 3 : 5, 5 : 8, 8 : 13, 13 : 21, 21 : 34, 34 : 45, &c., where each successive term is equal to the sum of the last two.

USE & APP. This Proposition is employed in EUCLID'S 13th Book, treating of the construction of the five regular or Platonic solids,—the Tetrahedron, the Cube, the Octahedron, the Dodecahedron, and the Eicosahedron: it enables us also to solve the following, among other Problems:—

PROB. 1°. *On a given line*, AB, *to construct a rt.* ∠*d triangle, the sides of which shall be in continued or geometrical progression.*



C. 1	Pst. 3, 1.	On AB construct a semicircle.
2	30, VI.	divide AB medially in E, *i. e.* AB : AE = AE : EB;
3	11, I.	from E raise a perp. EC cutting the semic in C;
4	31, III.	on joining CA & CB, ACB is a rt. ∠;
5	Sol.	and the sides of △ ACB are continued proportionals, *i. e.* AB : AC = AC : CB.
D. 1	C.2. Def.10, V.	∵ AB : AE = AE : EB, ∴ AE a mean propl.;
2	8, VI.	and ∵ AB : BC = BC : EB, ∴ CB a mean propl.
3	9, V.	and ∴ AE = CB.
4	8, VI	But AB : AC = AC : AE, and AE = CB.
5	Conc.	∴ AB : AC = AC : CB. *i. e.* the sides of the rt. ∠d △ are in continued proportion. Q. E. F.

N. B. In this Demonstration, steps 1—3, it appears, that if the perp. CE cuts the hypotenuse AB, in extreme and mean ratio, then *the less side* CB = *the alternate segment* AE, and *vice versâ*.

PROB. 2. *"The altitude* EC *of a rt.* $\angle d$ $\triangle$ *being given, of which the sides are in a given ratio, to find the sides.*

By aid of the alt. EB, and from the ratio of the sides the segments AE and EB may he found; for

E. 1	Dat. 1	Let M : N represent the R. of the greater side to the less.
2	„ 2	& let G & L represent the gr. and less segs· of AB, CE being the altitude ;
3	Quæs.	to find the sides of the rt. $\angle$d $\triangle$.
D. 1	8, VI.	Here M : N = CE : L, & N : M = CE : G.
2	E. 2 & 1.	And ∵ CE is given, and M : N known ;
3		∴ G and L the segments may be found ;
4	D. 1.	for $L = \frac{N \cdot AE}{M}$, and $G = \frac{M \cdot CE}{N}$;
5	By last Prob	Now G = the less side, & $AB^2 - G$ = greater side.

Q. E. F.

PROB. 3°, *Given, a, b, two sides of a* $\triangle$, *and the diameter, d, of the circumscribing* circle, to find the other side, *c*.

C. 1		Take $\frac{a \cdot b}{d}$, then $a^2 - \left(\frac{a \cdot b}{d}\right)^2$ & $b^2 - \left(\frac{a \cdot b}{d}\right)^2$;
2		And $\sqrt{a^2 - \left(\frac{a \cdot b}{a}\right)} + \sqrt{b^2 - \left(\frac{a \cdot b}{d}\right)^2}$ c the third side
D. 1	U&A 16 VI	For $a \cdot b = d.\ alt.$; ∴ alt. $= \frac{a \cdot b}{d}$.
2		And a^2 — alt.^{2} = greater seg.^{2} ; b^2 — alt.^{2} = less seg.
3		then $\sqrt{}$ gr. seg.^{2} + $\sqrt{}$ less seg.^{2} = c, the third side.

Ex. Two sides of a $\triangle$ measure respectively 8 feet and 6 feet, the diameter of the circumscribing $\odot$ being 10 feet; required the third side.

Here $\frac{8 \times 6}{10} = 4.8$; then $\sqrt{64 - 23.04} = \sqrt{40.96} = 6.4$;

& $\sqrt{36 - 23.04} = \sqrt{12.96} = 3.6$

∴ $6.4 + 3.6 = 10$ the third side.

PROP. 31.—THEOR. *(Important.)*

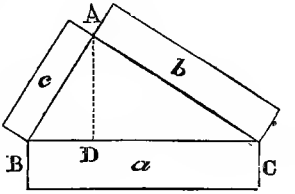
In right angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

CON. 12, I. To draw a st. line ⊥ a given st. line from a given . without it.

DEM. 8, VI. 4, VI. Cor. 2, 20, VI. B, V. If four Ms are propls. they are propls. also when taken inversely.
24, V. If the 1st has to the 2nd the same R. which the 3rd has to the 4th; and the 5th to the 2nd the same R. which the 6th has to the 4th; the 1st and 5th together shall have to the 2nd the same R which th e 3rd and 6th together have to the 4th. AX. 8, I.

A, V. If the 1st of four Ms. has the same R. to the 2nd which the 3rd has to the 4th; then, if the 1st be $>$ $=$ or $<$ the 2nd, the 3rd also is $>$ $=$ or $<$ the 4th.

E. 1	Hyp. 1.	Let ABC be a rt. ∠d △, the rt. ∠ being ∠BAC;
2	,, 2.	and let fig. *a* on BC be sim. and similarly desc. to figures *b* & *c* on AC, AB;
3	Conc.	then fig. *a* = fig. *b* + fig. *c*.
C.	12, 1.	Draw from . A, AD ⊥ BC.
D. 1	C.	∵ from A, the vertex of the rt. ∠ AD ⊥ base BC,
2	8, VI.	∴ △s ABD, ADC, are sim. to △ ABC and to each other.
3	D. 2.	And ∵ △ABC is sim. to △ADB, & to △ADC,
4	4, VI.	∴ CB : BA = BA : BD, & CB : CA = CA : DC;
5	Cor. 2, 20, VI.	and ∵ in 3 propls, as 1st : 3rd so is fig. on 1st : fig. on the 2nd,



6		∴ CB : BD = fig. *a* on CB : sim. and similarly desc. fig. *c* on BA ;
7	B. V.	and *inv.* DB : BC = fig. *c* on BA : fig. *a* on CB ;
8	*Sim.*	So DC : CB = fig. *b* on CA : fig. *a* on CB ;
9	24, V.	∴ BD + DC : BC= fig. *c* on BA + fig. *b* on AC : fig. *a* on BC
10	Ax. 8, I.	but BD + DC = BC.
11	A. V.	∴ fig. *a* on BC = sim. and similarly desc. figs. *c* + *b* on BA and AC.
12	Rec.	Wherefore, *in rt.* ∠*d triangles, the rectilineal fig. &c.* Q. E. D.

Or,

D. 1	23, VI.	∴ sim. figs. : one another in the duplicate R. of homol. sides,
2		∴ fig. *a* on BC : fig. *c* on BA, in the duplicate R of CB to AB ;
3	Cor. 1, 20, VI	but $BC^2 : BA^2 = BC : BA$;
4	11, V.	∴ fig. *a* on CB : fig. *c* on BA = $CB^2 \cdot AB^2$.
5	*Sim.*	So fig. *a* on BC : fig. *b* on CA = $BC^2 : CA^2$;
6	24, V.	∴ fig. *a* on BC : figs. *c* + *b* on BA, AC = $BC^2 : AB^2 + AC^2$;
7	47, I.	but $BC^2 = BA^2 + AC^2$;
8	Conc.	∴ fig. *a* on BC = fig. *c* on BA + fig *b* on AC. Q. E. D.

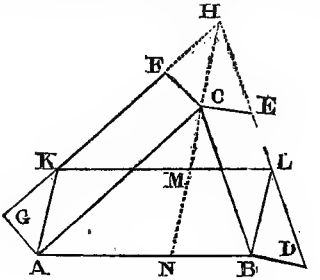
Or,

$$\text{Since } \frac{DB}{DC} : BC = \frac{\square \text{ on BA}}{\square \text{ on CA}} : \square \text{ on BC},$$

Therefore, 24, V. DB + DC : BC = □ on BA + □ on CA : □ on BC.

Sch. 1. Proposition 31, bk. VI, is very comprehensive and renders Prop. 47, bk. I only a particular case. There is however a theorem still more general; it is given in his *Mathematical Collections* by Pappus, one of the later of the Greek Geometricians, who flourished at Alexandria during the reign of Theodosius, AD. 379—395 ;

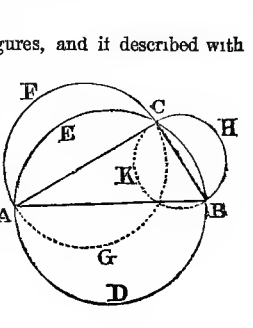
"If any □s AF, BE, be described on two sides, AC, BC, of any △ACB; and if the sides of the □s, DE, GF, be produced to meet, as in H, and if that point of intersection H and the vertex C of the triangle be joined, and the line HC produced to N; then these □s AF, BE are equal in area to a □AL, described on the base AB, and having two of its sides AK, BL parallel to CN the line produced through the point of intersection H and the vertex C. and limited by the sides, DE, GF, of the two □s.



D. 1	C. 33, I.	∵ fig AKHC is a □, ∴ AK = CH ;
2	C. 31, I.	& ∴ fig. ABLK is a □, ∴ AKL = LB ;
3	Ax. 1, I. C.	∴ LB = CH ; & AL is a
4	35, I.	Now □NL = □BH = □CD ;
5	35, 1.	& □NK = □AH = □AF ;
6	Ax. 2, I.	∴ □NL + □NK, *i. e.*, □AL = □CD + □AF
7	Rec.	∴ *If any □s be described on the two sides; &c.* Q. E. D.

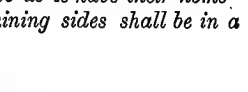
N.B. When ∠ABC is a rt. ∠, the s CD, AF & AL become squares, and the 47, I occurs, that the square on the hypotenuse is equal to the squares on the sides including the rt. angle.

2. Circles, as well as squares, are similar figures, and if described with the sides, AB, AC, BC, of a rt. ∠d △ for their diameters, then the circle with the hypotenuse AB for its diameter is equal in Area to the two circles, that have AC, BC, the sides about the rt. ∠, for their diameters. See Cor. 5 & 6 Pr. 20, VI.



Prop. 32.—Theor.

If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another; the remaining sides shall be in a st. line.



E. 1	Hyp. 1.	Let $\triangle$s ABC, DCE have BA : AC = CD : DE;
2	,, 2.	and let AB be $\parallel$ DC and AC $\parallel$ DE;
3	Conc.	Then BC, CE form one st. line BE.
D. 1	H.	$\because$ AB $\parallel$ DC and AC meets them,
2	29, I.	$\therefore \angle$ BAC = $\angle$ ACD.
3	*Sim.* Ax. 1, I.	So $\angle$ CDE = $\angle$ ACD,
		$\therefore \angle$ BAC = $\angle$ CDE.
4	D. 3. & H.	And $\because$ $\triangle$s ABC, DCE have $\angle$ A = $\angle$ D, and BA : AC = CD : DE;
5	6, VI.	$\therefore$ $\triangle$ ABC is eq. ang. to $\triangle$ DCE, and $\angle$B = $\angle$ DCE;
6	D. 2. Ax. 2, I.	and $\because \angle$A = $\angle$ ACD,
		$\therefore$ whole $\angle$ ACE = $\angle$ ABC + $\angle$BAC;
7	*Add.*	add $\angle$ACB;
		then $\angle$s ACE+ACB = $\angle$s A+B+ACB.
8	32, I.	But $\angle$s A + B + ACB = 2 rt. $\angle$s;
9		$\therefore$ $\angle$s ACE + ACB = 2 rt. $\angle$s;
10	D. 8, 9.	and $\because$ at C in AC, the lines BC and CE on opp. sides of C make adj. $\angle$s = 2 rt. $\angle$s.
11	14, I.	$\therefore$ the lines BC, CE are in one st. line BE.
12	Rec.	Therefore, *if two triangles which have two sides &c.* Q. E. D.

Sch. The position of the given sides AC, DC, which are not homologous should be such as to form an angle, ACD, at the point of junction C; otherwise BC and CE may not be in one and the same st. line BE.

PROP. 33.—THEOR.

In equal circles angles, whether at the centres, or circumferences have the same ratio which the circumferences, on which they stand, have to one another; so also have the sectors

CON. 1, IV. In a given ⊙ to place a st. line equal to a given st. line which is not greater than the diam of the ⊙. Pst. 1, I.

DEM. 28, III. In eq. ⊙s, eq. st. lines cut off eq. arcs.

27, III. In eq. ⊙s the ∠s on eq. arcs are equal, whether they be at the centres or ⊙ces.

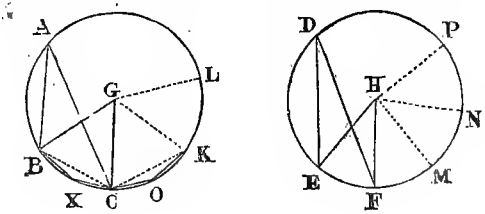
Def. 5, V. 20, III. The ∠ at the cen. is double of the ∠ at the ⊙ce upon the same base.

15, V. Magnitudes have the same R. to one another which their equimultiples have.

4, I. Def. 11, III. Sim. segs. of ⊙s are those in which the ∠s are equal.

24, III. Sim. segs. of ⊙s upon eq. st. lines are eq. to one another. Ax. 3, I.

E. 1	Hyp. 1.	Let ⊙ ABC = ⊙ DEF, having G & H as centres.
2	,, 2.	1°. On arcs BC, EF let there be at the centres ∠s BGC EHF, and at the ⊙ces ∠s BAC, EDF;
3	Conc. 1.	then arc BC : arc EF = ∠ BGC : ∠ EHF, and ∠ BAC : ∠ EDF.
4	Hyp3.	2°. Also on arcs BC, EF let there be sectors, BGC and EHF;
5	Conc.	then arc BC : arc EF = sect. BGC : sect. EHF.



CASE I, *Of angles at the centres or ⊙ces of equal circles.*

C. 1	1, IV.	In ⊙ ABC take any eq. arcs BC = CK = KL,
2		and in ⊙ DE any eq. arcs, EF = FM = MN;
3	Pst. 1, I.	and join GK, GL, HM, HN.
D. 1	28, III.	∵ arcs BC = CK = KL,
	27, III.	∴ ∠s BGC = CGK = KGL;
2		∴ the mult. which arc BL is of arc BC, that same mult. is ∠ BGL of BGC.
3	*Sim.*	So the mult. which arc EN is of arc EF, that same mult. is ∠ EHN of ∠ EHF;
4	27, III.	and if arc BL = > or < arc EN, ∠ BGL = > or < ∠ EHN.
5	H. 1.	Now ∵ there are 4 Ms, arcs BC, EF, and ∠s BGC and EHF;
6	D. 2.	and of arc BC and ∠ BGC equims, arc BL and ∠ BGL are taken;
7	D. 3.	and of arc EF and ∠ EHF equims, arc EN and ∠ EHN;
8	D. 4.	and ∵ if arc BL > = or < arc EN, ∠BGL > = < ∠ EHN;
9	Def. 5, V.	∴ arc BC : arc EF = ∠ BGC : ∠ EHF.
10	20, III.	and ∵ ∠ BGC = 2 ∠ BAC, and ∠ EHF = 2 ∠ EDF,
11	15, V.	∴ ∠ BGC : ∠ EHF = ∠ BAC : ∠ EDF;
12	Conc.	∴ arc BC : arc EF = ∠ BGC : ∠ EHF = ∠ BAC : ∠ EDF.

CASE II. *Of angles at the vertex of equal sectors.*

C. 1	Pst. 1, I.	Join BC, CK, and in arcs BC, CK take any .s X, O;
2	Pst. 1, I.	and join BX, XC, CO, OK,
D. 1	H.	∵ in △ GBC, GCK, BG, GC = CG, GK, and ∠ BGC = ∠ CGK.
2	4, I.	∴ base BC = base CK, and △ GBC = △ GCK.

3	C.1.Ax.3, I.	And arc BC = arc CK, ∴ rem. arc BLC = rem. arc CLK;
4	27, III, Def. 11, III.	∴ ∠ X = ∠ O, and seg. BXC is sim. to seg. CKO.
5	D. 2. 24, III.	and ∵ BC = CK, ∴ seg. BXC = seg. CKO.
6	D. 2.	And △ BGC = △ CGK, ∴ the whole sect. BGC = the whole sect. CGK.
7	*Sim.*	So sectors KGL = BGC = CGK,
8	*Sim.*	and sectors EHF = FHM = MHN;
9	Conc.	∴ the mult. which arc BL is of arc BC, that same mult. sect. BGL is of sect BGC;
10	*Sim.*	and the mult. which arc EN is of arc EF, that mult. sect. EHN is of sect EHF;
11		and if arc BL = > or < arc EN, the sect. BGL = > or < sect. EHN.
12	Hyp.	Now ∵ there are 4 Ms, the arcs, BC, EF, and the sects. BGC, EHF;
13	D. 9.	and of arc BC and sect. BGC, arc BL and sect. BGL are equims.;
14	D. 10.	and of arc EF and sect. EHF, arc EN and sect. EHN are equims.;
15	D. 11.	and ∵ if arc BL > = or < arc EN, the sect. BGL > = or < sect. EHN;
16	Def. 5, V.	∴ arc BC : arc EF = sect. BGC : sect. EHF.
17	Rec.	∵ *In equal circles, angles, &c.* Q. E. D.

Cor. 1. *The sectors are to each other as their angles.*

For, if arc BC : arc EF = ∠ BGC : ∠EHF;
and arc BC: arc EF = sect. BGC : sect. EHF,
then 11, V. sect. BGC : sect EHF = ∠ BGC : ∠ EHF.

Cor. II. *Similar sectors of the same or equal circles are equal.*

Cor. III. *An angle at the centre of a circle is to four rt. angles as the arc on which it stands to the circumference of the circle.*

For, the ∠ at the centre : one rt. ∠ = arc subtending central ∠ : arc subtending a rt. ∠ or quadrant;

Then, 4, V. ∠ at cen. : 4 rt. ∠s = arc of central ∠ : whole ⊙ce.

Cor. IV. *In different circles the arcs of equal angles at the centres or circumferences are similar.*

Cor. V. Hence, *similar segments are contained by similar arcs* and *vice versâ.*

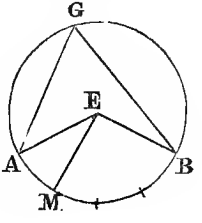
Sch. 1. If the ares and sector had all been in one circle the proof would have been the same,—for H would have coincided with G, and D,E,F,M,N would have been points in the circumference of circle ABL.

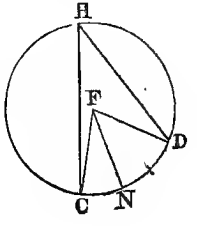
2. The second part of the Proposition was added by Theon, the Ptolemaist, and father of the renowned but unfortunate Hypatia, of Alexandria, in the time of the elder Theodosius ;—it is given in the Commentary on Ptolemy 's *Almagest.*

3. That the angles at the circumferences are as the arcs on which they stand, follows also as a Corollary from Prop. 20, bk. III.

Prop. 33 . *a. In the same or in equal circles AGB, CHD angles, whether at the centres, as ∠s AEB, CFD, or at the circumferences, ∠s AGB, CHD, have the same ratio as the arcs, AB, CD. on which they stand.*

E & C. 1 | Def. 1, V. | Let AM or CN be a com. meas. of arcs AB, CD ;
2 | | & arc AMB = 4 arc AM, and arc CND = 3 CN ;
3 | Pst. 1, I. | then join EM and FN.

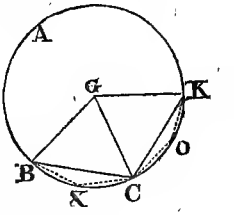




D.	1	27, III.	∵ ∠AEM = ∠CFN, & ∠AEB = 4 ∠AEM, & ∠CFD = 3 ∠CFN ;
	2	7 *a* V.	∴ ∠AEB : ∠CFD = 4 : 3; *i. e.*, arc AMB: arc CND = 4 : 3 ;
	3	11, V.	∴ ∠AEB : ∠ CFD = arc AMB : arc CND.
	4	20, III.	And ∵ ∠s AEB, CFD are mults. of ∠s G & H by 2,
	5	15, V.	∴ ∠G : ∠ H = arc AMB : arc CND Q. E. D.

COR. Since the ∠s and arcs are proportional when commensurable, ∴ *they are also proportional when incommensurable.* See Sch. 3—5 & Use 1, 16, VI.

PROP. 33 *b.* *In the same circle, ABK, or in equal circles, the sectors BGC, GCK, that stand on equal arcs, BC, CK, are equal.*



C. 1		Take any . s X,O, in the arcs BC, CK,
2	Pst. 1, I.	& join BX, XC, CO, OK.
D. 1	*Sub.*	From ⊙ ABK take separately arcs BC & CK,
2	Ax. 3, I.	∴ rem. arc. BAC = rem. arc CAK ;
3	27, III.	∴ ∠BXC is sim. to ∠COK, and seg. BXC to seg. CKO.
4	27, III.	Again ∵ BG = GC = GK, & ∠ BGC = ∠CGK,
5	4, I.	∴ △ BGC = △CGK, & base BC = base CK.
6	D. 3. 5.	And ∵ seg. BXC is sim. to seg. CKO, & their bases, BC, CK equal.
7	24, III.	∴ seg. BXC = seg CKO, & sect, BGC = sect CGK.
8	*Sim.*	So, if sect. CGK were in a ⊙ which is equal to ⊙ ABC, sect. CGK = sect. BGC.

PROP 33 *c.* *In the same circle ABK*, as in fig. to 33 *b. or in equal circles*, ABG, CHD, as in fig. to 33 *a, sectors* AEB, CFD, *have the same ratio as the arcs*, AMB, CND, *on which they stand.*

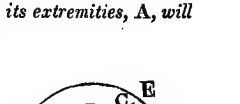
E. & C.	1	Fig. 33, *a.*	Let arc AM = arc CN, & be a com. measure of arcs AB, CD ;
	2		& let arc AB = 4 arc AM, & arc CD = 3 arc CN,
	3	Pst. 1, I.	then join EM and FN.
D.	1	33, *b.* VI.	∵ sect. AEM = sect CFN ;
	2	C. 2.	& ∵ sect. AEB = 4 sect. AEM, & sect. CFD = 3 sect. CFN ;
	3	7 *a* V.	∴ sect. AEB : sect. CFD = 4 : 3 ; and arc AB : arc CD = 4 : 3 ;
	4	11, V.	∴ sect. AEB : sect. CFD = arc AB ; arc CD.

Cor. Since the arcs and sectors are thus proportional when commensurable, ∴ *they must also be proportional when incommensurable.*

4. From this 33rd Proposition it results, that the angle at the centre of a circle is said to be measured by the arc on which it stands.

Use and Appl. 1. *If arcs* ACB, AEB, *of different circles have a common chord,* AB, *the lines,* AC, AD, *diverging from one of its extremities,* A, *will cut the arcs proportionally,* i. e. BF : FE = BD : DC.

D. 1	Ax. 8, I.	∵ ∠s BAF, BAD are identical, and also ∠s EAF, CAD ;
2	33, VI.	∴ arc BF : arc FE = ∠ BAF : ∠EAF,
3	„ „	also arc BD : arc DC = ∠BAF : ∠EAF ;
4	11, VI.	∴ arc BF : arc FE = arc BD : arc DC.



Q. E.D.

2. *The arcs,* A, A′ *of unequal circles are in a ratio compounded of their central angles a, a′ and their radii,* R, R′.

C. 1		With rad. = R desc. an ∠ = a',
2		and let the subtending arc be m.
D. 1	H. & C. 2	∵ arcs A and m have an eq. R, ∴ A : m = a : a' ;
2	C. & H.	and ∵ m and A′ have eq. central ∠s ;
3	Cor. 5, 20, VI.	∴ m : A′, = R : R′
4	D. 1 & 2.	But A : A′ is compounded of $\left\{\begin{matrix} A : m \\ m : A', \end{matrix}\right.$ or of the equivalent, ratios, a : a' and R : R′.

3. *Central angles a, a′, are in a ratio compounded of the direct ratio of their arcs* A, A′, *and the inverse ratio of their radii* R, R′.

D. 1	Use 2, 33, VI.	For A : A′ = $\left\{\begin{matrix} a & : & a' \\ R & : & R' \end{matrix}\right.$,
2		Let each of the equal ratios be compounded with R′ : R;
3	D. 1 and 2	then $\left\{\begin{matrix} A : A' \\ R' : R \end{matrix}\right\} = \left\{\begin{matrix} a & : & a' \\ R & : & R' \\ R' & : & R \end{matrix}\right.$
4	Sch. 1, 16, VI.	But $\left\{\begin{matrix} R : R' \\ R' : R \end{matrix}\right\}$ is a ratio of equality,
5	Conc.	∴ $\left\{\begin{matrix} A : A' \\ R' : R \end{matrix}\right\} = a : a'$

See Lardner's *Euclid,* p. 224.

OBS. "And herewith," remarks Captain Thomas Rudd, Chiefe Engineer to Charles I., "is the first six Books of EUCLIDE ended. There be hereafter added certain Propositions, which although they be not EUCLIDES, yet because they are both witty and usefull, I thought it good not to omit." RUDD'S *Euclides Elements*, A.D. 1651, p. 253.

SUBSIDIARY PROPOSITIONS.

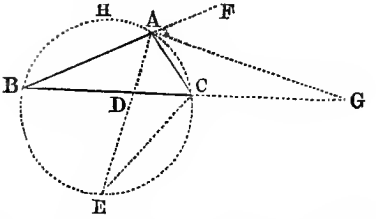
PROP. B.—THEOR.

If an angle of a triangle be bisected by a st. line which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the st. line which bisects the angle.

CON. 5, IV. To desc. a ⊙ about a given △. Pst 2, 1, I.

DEM. 21, III. The ∠s in the same seg. of a ⊙ are equal.
3, II. If a st. line be divided into any two pts, the ▭ contained by the whole and one of the pts is eq. to the ▭ contained by the two parts together with the square of the aforesaid part.
35, III. If two st. lines cut one another within a ⊙ the rect. contained by the segs. of one is eq. to the rect. contained by the segs. of the other.
32, I. 4, VI. 16, VI.

E. 1	Hyp.	In △ ABC let ∠ A be bisected by AD;
2	Conc.	then BA . AC = BD . DC + AD^2.
C. 1	5, IV.	About △ ABC describe ⊙ ACB;
2	Pst. 2 1, I.	prod. AD to ⊙ce E, and join EC;



D. 1	H. 21, III.	Then $\because \angle BAD = \angle CAE$, & $\angle ABD = \angle AEC$;
2	32, I. 4, VI.	$\therefore \triangle ABD$ is eq. ang. with $\triangle AEC$, & $\therefore$ BA : AD = EA : AC;
3	16, VI. 3, I.	& $\therefore$ BA . AC = EA . AD, *i. e.*, = ED . DA + AD^2.
4	35, III.	But ED . DA = BD . DC.
5	Conc.	$\therefore$ AB . AC = BD . DC + AD^2.
6	Rec.	$\therefore$ *If an angle of a triangle, &c.* Q. E. D.

Cor. From the hypothesis, that the exterior vertical angle, CAF, is bisected by GA, which also cuts the base produced BCG in G and the circle in H, the conclusion follows, that *the rectangle of the sides together with the square of the line which bisects the exterior angle is equal to the rectangle of the whole line produced and the extreme segment*; i. e BA . AC + AG^2 = BG . GC.

Sch. Prop B and its Corollary combined, may be thus enunciated; "if the vertical or ext. vertical $\angle$ of a $\triangle$ be bisected by a st. line, which cuts the base, or the base produced, the square of the st. line shall be equal to the difference of the rectangles under the two sides and under the segments of the base or of the base produced."

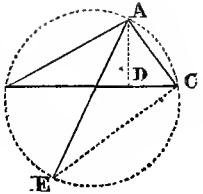
In the case of the bisection of the vert. $\angle$BAC, AD^2 = BA . AC − BD . DC.
And " ext. vert. $\angle$CAF, AG^2 = BG . GC ~ BA . AC

Prop. C.—Theor.

If from any angle of a triangle a st. line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

CON. 5, IV. DEM. 31, III. 21, III. 4, VI. 16, VI.

E. 1	Hyp.	In △ ABC, from ∠ A let AD be ⊥ base BC,
2	Conc.	then BA . AC = AD . AE.
C.	4, VI.	About △ ABC desc. ⊙ ACB and its diam. AE, & join EC.
D. 1	31, III. & 21 III.	∵ rt. ∠ BDA = ∠ ECA, & ∠ ABD = ∠ AEC;
2	4, VI.	∴ △ BDA is eq. ang. with △ ACE; & ∴ BA : AD = EA : AC;
3	16, VI.	∴ BA . AC = EA . AD.
4	Rec.	*If therefore from any angle, &c.* Q. E. D.



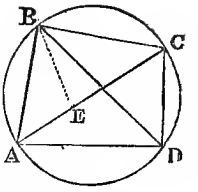
COR.—*If two* △*s*, ABC, ACE, *be inscribed in the same or in equal* ⊙*s, the rect. under the two sides of the one*, BA . AC, *shall be to the rect. under the two sides of the other*, EC . CA, *as the perp.* AD, *which is drawn from the vertex* A, *to the base*, BC, *of the one, is to the perp. which is drawn from the vertex* C *to the base*, AE, *of the other:* i. e. BA . AC : EC . CA = AD : *perp. from C.*

PROP. D.—THEOR.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

CON. 23, I. DEM. 21, III. 4, VI. 16, VI. 1, II. If there be two st. lines, one of which is divided into any number of pts.; the rect. contained by the two st. lines is equal to the rectangles contained by the undiv. line and the several pts. of the divided line.

E. 1	Hyp.	In a ⊙ let a qu. lat. ABCD be insc with diags. AC, BD;
	2 Conc.	then AC . BD = AB. CD + AD . BC.
C.	23, I.	Make ∠ ABE = ∠ DBC.
D. 1	*Add.*	Add ∠ EBD to the equals, ∴ ∠ ABD = ∠ EBC.
	2 21, III.	and ∵ ∠ BDA = ∠ BCE ∴ △ ABD is eq. ang. with △ EBC;
	3 4, VI.	∴ BC : CE = BD : DA,
	16, VI.	and ∴ BC . DA = BD . CE.
	4 C. 21, III	Again, ∵ ∠ ABE = ∠ DBC, and ∠ BAE = ∠ BDC,
	5 4, VI.	∴ △ ABE is eq. ang with △ BCD, ∴ BA : AE = BD : DC;
	6 16, VI.	∴ BA . DC = BD . AE.
	7 D. 3.	But BC . DA = BD . CE;
	8 1, II.	∴ the whole □ AC . BD = AB . CD + AD . BC.
	9 Rec.	Therefore, *the rectangle contained, &c.* Q. E. D.



Sch. This Proposition is named Ptolemy's Theorem, and is a Lemma at p. 9 of his Almagest, or Μεγαλη Συνταξις of which it is said, "Ptolemy appears as a splendid mathematician, and an (at least) indifferent observer."

It is curious to note how editor after editor of Euclid has followed the identical diagrams of the earliest printed editions. Ptolemy's thirteen books "*Magnæ Constructionis, Id est, Perfectæ cælestium motuum, pertractationis,*" published at Basle by Symon Grynoeus in the year 1538, contain several diagrams, and the one to Prop. D, bk. VI, among the number, which have scarcely undergone any change since that time. *Billingsley's* English Euclid also furnishes very many examples from which to prove the *imitativeness* of succeeding Editors, and by which to justify, if need be, the continuation of the practice.

PROP. E.—THEOR.

The diagonals, AB, CD, *of a quadrilateral* ACBD, *inscribed in a circle,* ABC *are to one another as the sums of the rectangles under the sides adjacent to the extremities of those diagonals:* i. e., AB : CD : : = CA . AD + CB . BD : AC . CB + AD . DB.

CON. 12, I. DEM. Cor. C, VI. Sch. 3, 24, V. In any number of magnitudes of the same kind forming two series, if the ratios of the 1st to the 2nd, of the 2nd to the 3rd, of the 3rd to the 4th, and so on, be the same in the two series; then any two combinations whatever, shall be to one another as two similar combinations of the corresponding magnitudes of the second series. 4, 5, VI. 11, V.

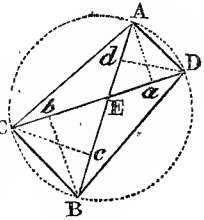
C. | *Sup.* | Let AB, CD, cut one another in the . E.

CASE I. *Suppose that AB cuts CD at rt.* $\angle$ *s.*

D. 1 | Hyp. | ∵ △s ACD, BCD, CAB, DAB are in the same ⊙;

2 | Cor. C, VI | ∴ perps. AE, BE CE and DE are to one another as the □s AC . AD, BC . BD, CA . CB, DA . DB;

3 | Sch. 3, 24, V. | ∴ AE + EB, or AB : CE + ED, or CD, = AC . AD + BC . BD : CA . CB + DA . DB.



CASE II. Or, *Suppose that AB cuts CD* not *at rt.* $\angle$ *s.*

C. | 12, I. | Draw the perps. A*a*, B*b*, C*c*, D*d*.

D. 1 | *Sim.* | As before, A*a* + B*b* : C*c* + D*d* = AC . AD + BC : BD : CA . CB + DA . DB;

2 | C. | but ∵ △s AE*a*, BE*b*, CE*c*, and DE*d* are eq. ang.;

D. 3	4, 5, VI.	∴ Aa, Bb, Cc, Dd, are to one another, as AE, BE, CE and DE;
4		∴ Aa, + Bb : Cc, + Dd = AE + BE : CE + DE = AB : CD.
5	11, V.	∴ AB : CD = AC . AD + BC . BD : CA . CB + DA . DB.
6	Rec.	Therefore, *the diagonals of a quadrilateral, &c.* Q. E. D.

USE AND APP. By aid of the last two Propositions, D & E, bk. VI, the following Problem may be solved.

Given four st. lines, any three of which are together greater than the fourth, to construct a quadrilateral, of which the sides shall be equal to those four given st. lines, in a given order, each to each, and of which also its angular points lie in the circumference of a circle.

By E, VI. The ratio of the diagonals is given; they are as the sums of the rectangles under the sides;

By D, VI. The rectangle of the diagonals is given, being equal to the sum of the rectangles under the opp. sides;

By Prob. 5 Use and App. 13, VI., we find the two st. lines; in this case, they are the two diagonals equal to a given rectangle;

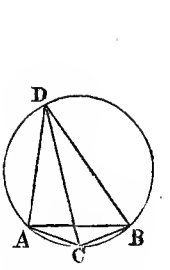
And by 22, I, having now the sides and diagonals of the quadrilateral, we construct two triangles the sum of which will equal the required quadrilateral.

PROP. F.—THEOR.

If AB, *a segment of a circle*, ABD, *be bisected in* C, *and from the extremities*, A, B, *of the base of the segment, and from* C *the point of bisection, st. lines be dawn to any point* D *in the circumference; the sum of the two lines* AD + DB *drawn from the extremities of the base, will have to the line* DC *drawn from the point of bisection the same ratio which the base* BA, *of the segment has to* AC *the base of half the segment.*

DEM. D, VI. 1, II. 14, VI.

D. 1	H.	∵ ADBC is a qu. lat. inscribed in a ⊙, of which AB and CD are diagonals;
2	D. VI.	∴ AD . CB + DB . AC = AB . CD.
3	H.	but ∵ AC = CB, ∴ AD . CB = AD, AC
4	D, 2, 3.	∴ AD . AC + BD . AC = AB . CD.
5	1, II.	But AD . AC, & BD . AC are the rectangles contained by AC & AD + DB;
6		∴ rect. AC . (AD + DB) = rect. AB C. D;
7	14, VI.	& ∵ the sides of eq. rectangles are reciprocally propl;
8	Conc.	∴ AD + DB : DC = AB : AC.
9	Rec.	Therefore, *if a segment of a circle be bisected &c.* Q. E. D.



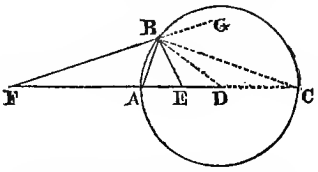
SCH. This and the following Subsidiary Propositions have been adopted, with some slight alterations from BELL'S *Plane Geometry*, p. 194 — 198.

PROP G.—THEOR.

If two points, E & F, *be taken in the diameter* AC *of a circle, or of the diameter produced*, CF, *such that the rectangle*, ED.DF, *contained by the segments intercepted between them and* D *the centre of the circle be equal to the square of* AD *the semidiameter*; *and if from these points two st. lines, EB, FB, be inflected to any point whatever*, B, *in the circumferenece of the circle, the ratio of the lines inflected*, FB : BE, *will be the same with the ratio of the segments*, FA : AE, *intercepted between the two first mentioned points and the circumference of the circle.*

DEM. 17, VI. 6, VI. 4, VI. 16, V. *altern.* 17, V, *dividendo.* 11, V.

C | Pst. 1 & 2, I. | Join DB, CB, AB & prod. FB to G.



D. 1	H.	∵ FD . DE = AD² = DB²;
2	17, VI.	∵ FD : DB = DB : DE;
3	D. 2.	& ∵ in △s FDB, BDE the sides about the common ∠D are propl.;
4	6, VI.	∴ △ FDB is eq. ang. with △ BDE, ∠DBF = ∠DEB, & DBE = ∠DFB.
5	4, VI.	∴ FB : BD = BE : ED,
6	16, V.	& *alt.* FB : BE = BD : ED, or AD : DE.
7	D, 2.	But ∵ FD : DB or DA = DA : DE;
8	17, V.	∴ *div.* FA : DA = AE : ED.
9	16, V.	& *alt.* FA : AE = DA : ED.
10	D. 6.	Now FB : BE = AD : DE,
11	11, V.	∴ FB : BE = FA : AE.
12	Rec.	∴ *If two points be taken in the diameter, &c.* Q. E. D.

COR. 1. ∵ FB : BE = FA : AE, ∴ *on joining* AB, *by* 3, VI., ∠FBE *is bisected by* AB.

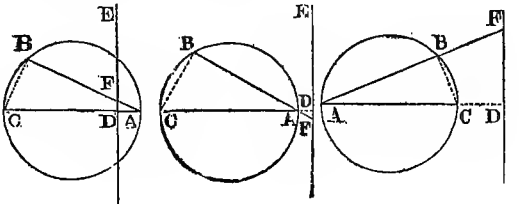
COR. 2. *Also on joining* BC, *the ext. vert.* ∠EBG *is bisected by* BC; for

D. 1	17, VI. & 18, V.	∵ FD : DB or DC = DC : DE, ∴ *comp.* FC : DC = CE : ED;
2	17, V.	& ∵ FA : AD or DC = AE : ED, ∴ *ex. æq.* FA : AE = FC : CE.
3	G, VI.	But FB : BE = FA : AE, ∴ FB : BE = FC : CE;
4	A, VI.	∴ ext, ∠EBG is bisected by BC. Q. E. D.

PROP. H.—THEOR.

If from one extremity, A, *of* AC *the diameter of a circle* ABC, *a chord* AB *be drawn, and a perpendicular* DE, *to the diameter, cut both the diameter and the chord either internally or externally in* D *and* F, *the rectangle*, CA . AD, *under the diameter and its segment reckoned from that extremity* A, *is equal to the rectangle* BA . AF, *under the chord and its corresponding segment.*

DEM. I, III. 15, I. 4, VI. 16, VI.



D. 1	H. 31, III.	∵ ∠ABC is in a semic, ∴ ∠ABC is a rt. ∠;
2	H. 15, I.	but ADF is also a rt. ∠, & ∠BAC = ∠DAF;
3		∴ △ABC is eq. ang. with △ADF;
4	4, VI.	∴ BA : AC = AD : AF :
5	16, VI.	∴ BA . AF = AC . AD.
6	Rec.	∵ *If from one extremity of the diameter, &c.* Q. E. D.

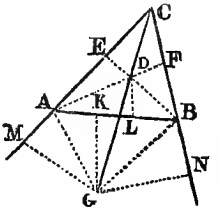
PROP. K.—THEOR.

If the angles, A *&* B, *at the base* AB *of a triangle* ABC, *be bisected by two lines*, AD, BD, *that meet, as in* D, *and the exterior angles at the base, formed by producing the two sides* CA, CB, *be similarly bisected by* AG & BG; *then the two points of concourse*, D & G, *and the vertex*, C, *shall be in one st. line, which shall bisect the vertical angle*, ACB.

PRELIMINARY THEOREM, that may be demonstrated by *superposition*, "If two △s have two sides of the one respectively equal to two sides of the other, and the ∠ opp. one of the sides in the first equal to the ∠ opp. to the equal side in the second, these △s are equal when they are of the same species or affection, *i. e.*, when they are both acute-angled, both right-angled, or both obtuse-angled. See Sch. P, 7, VI.

CON. 12, I. DEM. 26, I. & Prel. Theor.

C. 1	12, I.	Draw DE, DF, DL ⊥ to the sides AC, BC & AB;
2	,,	& GM, GN ⊥ to the sides produced AM, BN;
3	,,	& GK ⊥ to the side AB:



D. 1	Hyp.	∵ in △s ADE, ADL, ∠EAD = ∠DAL, ∠s at E & L also equal, & AD common.
2	26, I.	∴ AL = AE & DL = DE.
3	*Sim.*	So BL = BF, & DL = DF;
4	,,	also AM = AK, GM = GK, BN = BK, & GN = GK.
5	D, 2, 3.	& ∵ DE & DF each = DL, ∴ DE = DF, & *sim.* GM = GN
6	D. 5, C.	Again, ∵ in △s CED, CFD, CD, DE=CD, DF, & ∠s at E & F are rt. ∠s;
7	*Prel. Theor.*	∴ △ CED = △CFD, & ∠ECD = ∠FCD.
8	*Sim.*	So △CMG = △CNG, & ∠MCG = NCG;
9	Conc.	∴ the lines CG & CD coincide.
10	Rec.	∴ *If the angles at the base, &c.* Q. E. D.

PROP. L.—THEOR.

In a triangle ACB, *as in the last proposition, the segments,* CM *or* CN, *of each side produced that are intercepted between the vertex,* C *and the external perpendiculars,* GM, GN, *are each equal to* S, *the semiperimeter of the triangle; the segments,* CF *or* CE, *of these sides next the vertex,* C, *are equal to* S—AB, *the excess of the semiperimeter above the base* AB; *and the segment* AE *or* BF, *of each of these sides next the base is respectively equal to* S—BC, *or to* S—AC, *the excess of the semiperimeter above the other side.*

DEM. 26, I. Ax. 6 & 7, I. Ax. 2, I.

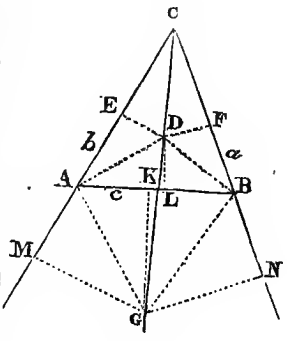
D. 1	26, 1.	∵ AK = AM, & BK = BN, ∴ CM + CN = perimeter;
2	26, I.	& CM = CN = S, the semiperimeter, & AM = S − AC.
3	Ax. 6, I.	Also 2 CE + 2 AE + 2 AM = perimeter = 2 CF + 2 AL + 2 BL.
4	D. 5, K. D. 2, K	Now CE = CF, & AE = AL;
5	Ax. 6 & 7, I.	∴ 2 AM = 2 BL, & AM or AK = BL;
6	*Add.* Ax. 2.	adding KL to both, ∴ AL = BK.
7	D. 5 & 4.	And ∵ AM = BL, & AE = AL,
8	Conc. 1	∴ ME = AB, & CE = S − AB, *i. e.*, CE = excess of semiperimeter above the base.
9	D. 7, 6 & K, 4	And ∵ AE = AL = BK = BN;
10	Conc. 2.	∴ AE = S − BC;
11	D. 7, & K 3. D. 2	& ∴ AM = AK = BL = BF, & AM = S − AC,
12	Conc. 3.	∴ BF = S − AC *i. e.*, each seg. of the side next the base is equal to the excess of the semiperimeter above the other side.
13	Rec.	∴ *the segment of each side, &c.* Q. E. D.

Prop. M.—Theor.

The area of a triangle is a mean proportional between two rectangles, the sides of one of which are equal to the semiperimeter and its excess above the base, and the sides of the other equal to the excesses of the semiperimeter above the other two sides.

Dem. K & L, VI. 4, VI. 29, I. Cor. 1, 23, VI. 41, I. 13, I. 32 .I 16, VI.

E. 1	Hyp. 1	Let ABC be a △, of which in the sides CA & CB produced CM = CN = S, the Semiperimeter;
2	„ 2	S—AB = CE; S—AC = AM & S—BC = AE, *i.e.* the excesses of S above AB, AC & BC;
3	Conc.	then CM . CE : △ ABC = △ ABC : AM . AE.
C	K & L, VI.	The fig. being constructed as in Pr. K & L, VI.
D. 1	29, I. 4, VI.	∵ ED ∥ MG, ∴ CE : ED = CM : MG;
2		but CM : CM = ED : ED;
3	Cor 1, 23, VI.	∴ CE . CM : ED . CM = CM . ED : MG . ED.
4	C.	But △ ABC = △s ADC, ADB, & CDB;
5	41, I.	& these △s = $\frac{AC \cdot DE}{2} + \frac{AB \cdot DL}{2} + \frac{BC \cdot DF}{2}$ = ED . CM.
6	13, I. 32, I.	And ∵ ∠s EAL + MAK = 2 rt. ∠s, ∴ ∠s EAD + MAG = 1 rt. ∠ = ∠s AGM + MAG;
7	Ax. 3, I.	∴ ∠ EAD = ∠ AGM;
8	C.	& ∵ ∠s at E & M are. rt. ∠s, ∴ △s AGM, EAD are eq. ang.;



9	4, VI. 16, VI.	∴ AE : ED = MG : AM, & ∴ AE . AM = MG . ED.
10	D 3.	But CM . CE : ED . CM = CM . ED : MG . ED becomes CM . CE : △ ABC = △ ABC : AE . AM;
11	H. 1, 2.	∴ S (S-AB) : ABC = ABC : (S-AC) (S—BC) Q. E. D.

Cor. Let the sides opp. ∠s A, B, C, be denoted by a, b, & c; then

$$s\,(s-c) : \triangle ABC = \triangle ABC : (s-a)\,(s-b);$$
$$\therefore \text{Area of } \triangle ABC = \sqrt{s\,(s-c)\,(s-a)\,(s-b)}.$$

Use and App. From this Prop. the Solution is obtained of the Problem *Given the three sides of a triangle to find the Area*; for, as in Cor. M. VI, the continual product of the Semiperimeter into the excesses of the semiper. above the three sides is equal to the square of the Area; whence the extraction of the square root gives the Area.

Ex. The sides of a triangle, ABC, are AB 221, BC 255 & AC 238 feet; required the Area.

Here $221 + 255 + 238 \times \frac{1}{2} = 357$ the Semiperimeter.

And $357-221 = 136$; $357-255 = 102$, & $357-238 = 119$, the Excesses.

Then $\text{Area}^2 = 357 \times 136 \times 102 \times 119 = 589324176$.

∴ Area of given $\triangle = \sqrt{589324176} = 24276$ square feet.

REMARKS ON BOOK VI.

1. In a general way it may be said that the Sixth Book, being an application of the Theory of Proportion propounded in the Fifth, treats chiefly of *similar* rectilineal and curvilinear figures, or of figures that differ in size, but not in form.

2. The Book contains 33 propositions by EUCLID, of which 10 are Problems, and 23 Theorems; to these have been added 13 Subsidiary Theorems.

3. The 11th, 12th, 13th, 18th and 25th Propositions are the most important among the Problems; and the 4th, 5th, 8th, 16th, 19th, and 31st, among the theorems.

4. As an *approximate* Classification for the Sixth Book, it may be divided, or rather arranged;

1°.—Into Propositions which treat of the Proportion existing between the *sides* of triangles; as Prop. 2, 4, 5, 6, 7, 8.

2°.—Into Propositions showing the Proportions between the *surfaces* of rectilineal figures; as Prop 1, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 26, 27 and 31.

3°.—Into those which relate to lines in Harmonical Progression; as Prop. 3 & A.

4°.—To the angles at the centres or circumferences of circles; as Pr. 33.

5°.—To Rectangles under the segments of chords; as Prop. B, C, D, E, F, G, H.

6°.—To certain Properties of the triangle when its angles are bisected, and its Semiperimeter compared with its sides; as Prop. K, L, & M.

5. In addition to the 10 Problems which EUCLID gives, there are several other very useful Problems evolved from the Principles established, as

1. To divide any rt. lined surface, Use P. 1.
2. To measure the height of an inaccessible object which casts an accessible shadow. Use 2, P. 2.
3. To divide a given line into proportional parts. Use 3, P. 2.
4.—To find a harmonic mean. Use 1, P. A. Use 2, P. 10.
5.—Eight Problems for ascertaining heights, distances, &c. Use P. 5.
6.—By aid of a square to measure inaccessible distances. Use 2, P. 8.
7.—To divide a given line into any number of equal parts. Use 1, P. 9.
8.—To divide a $\triangle$ into any number of equal parts. Use 2, P. 9.
9.—To divide a given line in a given ratio. Use 1, P. 10.
10.—To find a third harmonical progressional. Use 3, P, 10.
11.—To construct a triangle, one side, the angle opposite to it, and the ratio of the other sides being given. Use 4, P. 10.

12.—To draw a line which, if two other lines were produced, would pass through their point of intersection. Use 5, P. 10.

13.—To continue a series of Ratios in progression. Use 1, P. 11.

14.—From the two first terms in a series to obtain the sum of the series. Use 3, P. 11.

15.—Problems relating to the Use and Application of the Sector. Use P, 12

16.—Problems relating to a mean Proportional. Use 1—6, P. 13.

17.—To construct an isosceles triangle equal to a given scalene triangle and with the same vertical angle. Use P. 15.

18.—Application of the Theory of Limits. Use P. 16.

19.—Application of the Theory of Proportion in Prop. 14, 15, 16 & 17. Use 1—4, P. 19.

20.—The practical methods of drawing Plans, Maps, &c. Use P. 18.

21.—Methods for ascertaining the relative areas of similar figures. Use 1 2, P. 19.

22.—To increase or diminish a rt. lined, or a circular figure in any ratio. Use 1, 3, P. 20.

23.—From the areas and corresponding sides of similar figures to deduce the part not given. Use 4, P. 23.

24.—To describe a rhombus equal to a given rectilineal figure, and with an angle equal to a given angle. Use P. 27.

25.—In a given triangle to inscribe the greatest possible parallelogram. having an angle equal to a given angle. Use P. 27.

26.—Several Problems for applying rt. lined figures to a given line. Use P. 29.

27.—To divide a line, so that the rectangle under the whole line and one part shall bear a given ratio, as $m : n$, to the square of the other part. Sch. 3, P. 30.

28.— Problems depending on a given st. line cut in extreme & mean ratio, Use P. 30.

29.—The construction of a quadrilateral under certain conditions. Use P. E.

30—Given the three sides of a triangle to find the Area. Use P. M.

6.—As there are several useful Problems, so are there many important Theorems referred to, or contained in the Notes to Bk. VI.

1.—The Theory of Transversal Lines. Sch. 2, P. 2.

2.—The Equation of a rt. line in Analytical Geometry. Cor. 2, P. 6.

3.—The Criteria of similarity in Triangles. Use P. 7.

4.—The General Principle, if from the vertex of a triangle two lines be drawn to the base making the angles at the base or their supplements each equal to the vert. angle then the triangle formed by those lines and the segments shall be similar to the whole triangle and to one another. Sch. 1, P. 8.

5.—Series of Magnitudes in continued proportion. Use, 2, P. 11.

6.—Parallelograms are equiangular when their areas and sides are reciprocally proportional. Sch. 1, P. 14.

7.—Reciprocal ratios, and ratios of equality, Sch. II, P. 16.

8.—The conversion of the equality of two ratios into the equality of two rectangles. Sch. III, P. 16.

9.—The leading Theorems for the Doctrine of Limits. Sch. IV, P. 16.

10.—Various deductions and processes stated in other words than the original propositions, as Use III, P. 17. Sch. P. 22. Sch. P. 28. Use 2°, P. 28. Use 2° and 3°, P. 29.

11.—The General Theorem respecting incommensurable Magnitudes. Sch. 2, P. 30.

12.—The General Theorem, by PAPPUS, under given conditions, of the equality in Area of a parallelogram or a circle on the base of a triangle to the sum of the parallelograms or circles on the other two sides. Sch. 1, 2, P. 31.

13.—Three Propositions substituted for P. 33, VI. Sch. 4, P. 33.

14.—The cutting of arcs proportionally. Use, 1, P. 33.

15.—The ratio of arcs in unequal circles, and of central angles: Use 2, 3, P. 33.

N. B.—Additional Corollaries will be found under their respective Propositions.

7.—The importance of the sixth Book to the Mathematician can scarcely be over-estimated; it is the head and crown of Plane Geometry. Whoever has mastered it, has added no little to his amount of useful knowledge, and consequently to the power of his mind. Whether or not he adopts the quaint Motto round the Effigies of an old English Geometer,—"LIEFE IS DEATHE AND DEATH IS LIEFE," certain it is that he has planted his feet on one

of the summits of Human Wisdom, and as he looks around, either to survey the Country already traversed, or to take note of the Heights still rising before him, he may inscribe upon his work, in the spirit of devotion;

Laus Deo,

I thank God, and take courage.

SO NOW—SO EVER.

EUCLID'S PLANE GEOMETRY PRACTICALLY APPLIED.

Synoptical INDEX to Books I—VI.

Part I. containing Bks. I.—II.

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BOOK I.

The Geometry of Plane Triangles.

Definitions.	1—35	pp. 37—43	Explanatory Notes.
Definition.	A.	43—44	A Parallelogram. Observation on Magnitudes.
Postulates.	1—3	44	EUCLID added three other Postulates.
Axioms. I.	1—7	44—45	Applicable to number and quantity as well as to magnitude.
,, II.	8—12	45—46	Applicable especially to magnitude.
Notes.	8—12	45—46	Super-position. Illustration of Lines parallel and not parallel. Lines converging and diverging. Angles interior, exterior, opposite, adjacent, vertical and alternate.

FOURTEEN PROBLEMS :

Prop. 1, 2, 3 ;—9, 10, 11, 12 ;—22, 23 ;—31 ;—42 ;—44, 45, 46.

Prop 1. p. 47. To describe an equil. Δ on a given st. Line.

USE OR APP. 1°,—To solve 2, 3, 9, 10 & 11, I ; 2°. draw an isosc. Δ. ; 3°. approximate to an oval ; and 4° to measure an inaccessible distance.

2. 48. To draw from a . a st. Line = a given st. Line.

SCH.—Eight Solutions of this Problem.

3. 50. From the gr. of two Lines to cut off a part = the less.

Sch.—To lengthen the less to eq. the greater; Use.—to construct a Scale of eq. parts, and to apply the principle of *Representative values.*

9. 63. To bisect a rectil. ∠, *i. e.* to divide it into two eq. parts.

Sch.—To bisect an arc of a ⊙; & by successive bisections to divide an ∠ into any parts indicated by a power of 2. Use. To bisect the base of an isosc. △; and to construct the Mariner's Compass.

10. 66. To bisect a give st. Line.

Sch.—By successive bisections to divide a L. into parts, indicated by a power of 2.

11. 67. From a . *in* a L. to draw a L. at rt. ∠s to it.

Cor. Two lines cannot have a common segment.

Sch.—To draw a Perp. from the extremity of a L. Use 1.—To construct a square; 2. On a given Line to describe an isosc. △ of which the perp. height = the base.

12. 69. From a . *without* a L. to draw a Perpendicular.

Sch.—When the . is over the extremity of the L. Use. This Prop. indispensable to all artificers, &c.

22. 86. To make a △ of which the sides shall be eq. to three given st. Ls., any two being > than the third.

Sch.—Assumed that two ⊙s will have at least one . of intersection.

Use.—Of most extensive use,—to make one rect. fig. = or similar to another; And, on a given L. to describe an isosc. △ with sides each = twice the base; &c.

23. 88. At a given . to make a rectil. ∠ = a given rectil. ∠.

Use 1.— Of the widest use in Practical Mathematics; 2. To construct a Line of Chords; and by it to make an ∠ of a certain magnitude; At the end of a L. to draw a Perp.; to find the measure of an ∠; and to draw △s with certain parts given.

31. 103. Through a . to draw a st. L. parallel to a given st. L.

Sch.—Demonstration of the 12th Axiom. Use.—Prop. 31 required in all branches of Practical Mathematics;—it enables the Surveyor to ascertain inaccessible distances.

42. 125. To descr. a ▱ = a given △, and having one ∠ = a given ∠.

Sch.—Or, a △ = a given ▱, and having an ∠ = a given ∠.

44. 127. To a given L. to apply a ▱ = a given △, and having an ∠ = a given ∠.

Use.—Geometrical Division illustrated.

45, 130. To descr. a $\square$ = a given rectil. fig., and having an $\angle$ = a given $\angle$.

USE.—To measure the superficial content of any rectil. fig.; 2, To change any rectil. fig. into a $\triangle$, and then into a rect. of eq. Area; and 3. To straighten a crooked boundary without changing the dimension.

46. 132. To describe a Square on a given st. L.

COR. 1. The squares on eq. Lines are eq.; and *conversely*.
2. Every parallelogram with one rt. $\angle$ has all its $\angle$s rt $\angle$s.

SCH.—Given the diagonal to construct a square.

USE.—The Geometrical Square; its construction; its use in ascertaining inaccessible distances, as heights.

THIRTY FOUR THEOREMS.—ONE LEMMA;

Prop. 4, 5, 6, 7, 8;—13, 14, 15, 16, 17, 18, 19, 20, 21;—24, 25, 26;—Lemma;—27, 28, 29, 30,—32, 33, 34, 35, 36, 37, 38, 39, 40, 41,; —43; 47, 48.

4. 52. *Important.*—If two $\triangle$s have each two sides and their included $\angle$ eq. the bases and other $\angle$s are eq. and the $\triangle$s equal.

SCH.—The equality is perfect, not in Area only. USE.—The first *criterion* for establishing the *equality* of $\triangle$s; very frequently applied; and useful, along with the Theory of Representative Values, for ascertaining inaccessible distances.

5. 54. The $\angle$s at the base of an isosc $\triangle$ are eq.; and if the eq. sides are produced, the $\angle$s on the other side also are equal.

COR. Every equal triangle is also equiangular.

6. 57. *Conversely.* If two $\angle$s of a $\triangle$ are eq. the sides opp. the eq. $\angle$s also are equal.

COR. Every equiangular $\triangle$ shall be equilateral.

SCH. 1, 2.—Converse Theorems not universally true. 3. Two modes of Demonstration, *direct* and *indirect*. USE. To determine the Height of an object by its shadow.

7. 59. On the same side of the same base there cannot be two $\triangle$s, with sides terminated in one extremity equal, and also the sides equal terminated in the other extremity.

The Dilemma, or Double Antecedent, Prop. 7, used only to prove 8, I.

8. 61. *Important.*—If two △s have each the three sides eq. the ∠ contained by two equal sides in one △ equals the ∠ contained by the two corresponding eq. sides in the other △, and the △s are equal.

Sch.—The second *criterion* for equality of △s. Use 1. To determine without a theodolite the ∠ at a given ., made by Lines from two objects. 2. To measure and cut angles in a solid body.

13. 71. The ∠s made by one st. L. with another on one side of it, are either rt. ∠s, or together = two rt. ∠s.

Sch.—Any number of Lines converging to a , in a L on one side of it make the ∠s together = two rt. ∠s. *Supplement* and *complement* of an ∠ explained. Use Pr. 13 of frequent Use in Trigonometry, to determine the third ∠, when two ∠s are given.

14 73. *Conversely.*—If at a· in a st. L., two lines on the opp. sides of it, make the adj. ∠s together = 2 rt. ∠s, the two Lines form one and the same st. L.

15. 74. If two st. Lines cut one another, the opp. or vert. ∠s shall be equal; and *conversely.*

Cor. 1. The ∠s formed by two lines crossing each other are together = 4 rt. ∠s.

2. All the ∠s formed by any number of lines diverging from a com. centre are together = 4 rt. ∠s.

Sch.—A developement of the def. of an ∠. The *Converse* true.

Use 1.—To find the distance between two inaccessible objects; 2. To make one elastic ball strike another by reflection; and 3. to determine the number and kind of polygons which on being joined cover a given space.

16. 76. If one side of a △ be produced, the ext. ∠ is > either of the int. opp. ∠s.

Sch.—Each ∠ of a △ is < the supplement of either of the other ∠s.

Use 1 Among other conclusions,—only one perp. from a , to a given L. 2. Prop 16, of great use in reducing △s and other rectil. figures to rectangles.

17. 78. Any two ∠s of a △ are together < 2 rt. ∠s.

Explanatory of Ax. 12. Both Pr. 16 & 17 included in Prop. 32.

18. 79. The gr. side of every △ is opp. to the gr. ∠.

An instance of the argument "*à fortiori.*"

19. 79. *Conversely.*—The gr. side of every △ is subtended by the gr. ∠.

SCH.—Prop. 5, 6, 18 & 19, combined, prove, "One ∠ of a △ is, >, =, or < another ∠, as the side opposed is >, =, or < the other side opposed ; and "*vice versa.*" USE 1. The Perp. is the shortest L from a . to a given L. 2. From one , only two eq. lines to a given L. can be drawn. 3. All heavy bodies free to move seek the . nearest the earth's centre. 4. To construct a △, having the base, the less ∠ at the base, and the diff. of the sides given.

20. 81. Any two sides of a △ are together greater than the third side.

COR. The diff. of any two sides of a △ is less than the remaining side.

N.B.—More assumed in the Cor. than is expressed in AX. 5.

USE 1.—Of all lines from one . to another and reflected to a third ; those the shortest which make the ∠ of incidence = the ∠ of reflection. 2. Natural causes act by the shortest lines ; hence, by means of a mirror to construct a △ of which the Perp. is representative of the height of an object.

21. 84. If from the ends of a side of a △ two Ls be drawn to a . within the △, these lines are < the other two sides, but contain a greater ∠.

Applied in Optics, Astronomy, and Architecture.

24. 91. If two △s have two sides of one = two sides of the other, but the ∠ contained by the two sides of the one > the ∠ contained by the two sides of the other, the base of that which has the gr. ∠ shall be > the base of the other.

25. 92. *Conversely.*—If two △s have two sides of one = two sides of the other, each to each, but the base of one > the base of the other, the ∠ opp. the gr. base shall be > the ∠ opp. the less base.

Pr. 4, 8, 24 and 25, may be combined ; "If two △s have each two sides = two sides, the third side of the one will be >, <, or = the third side of the other, as the ∠ opposed in one is >, <, or = the ∠ opposed in the other ; and *vice versâ*.

26. 93. *Important.*—If two △s have two ∠s and a side of the one = two ∠s and a side of the other ; the other sides shall be eq. each to each, and the third ∠ of the one = the third ∠ of the other.

SCH.—The third *criterion* of the equality of △s. In two or more △s, any three parts of which one must be a side, being given equal, the equality of the other parts will follow.

Use 1.—Applied to measure inaccessible distances;—2 & 3, by the Theory of Representative Values to find the distance of two stations;—4, to construct an isosc. △, the vert. ∠ and perp. height of the △ being given.

96. Lemma. A L., perp. to one parallel, is also perp. to the other.

27. 97. If a L falling on two other lines makes the alternate ∠s equal, these two lines are parallel.

Sch.—Since some curved lines, though they never intersect, are not parallels, another demonstration is given.

28. 99. If a L falling on two other lines makes the ext. ∠ = the int. and opp. ∠ on the same side of the line; or the int. ∠s together on the same side = 2 rt. ∠s; the two lines shall be parallel.

Sch.—The principle in Ax. 12 really is,—that two st. lines intersecting cannot both be ‖ to the same L.

29. 100. If a L fall on two ‖ st. lines, it makes the alternate ∠s equal; and the ext. ∠ = the int. and opp. ∠ on the same side; and the two int. ∠s on the same side together = two rt. ∠s.

Converse of Pr. 27 & 28.

Sch.—Methods of expressing Ax. 12, Definition of Parallel Lines.

Use.—Pr. 27, 28 and 29 are applied to determine the earth's circumference.

30. 102. Lines ‖ the same L are parallel to each other.

Cor.—Two lines ‖ the same L cannot pass through the same point; *equivalent to* Ax. 12.

32. 105. *Very important.*—If a side of a △ be produced, the ext. ∠ = the two int. & opp. ∠s; and the three int. ∠s of every △ together = 2 rt. ∠s.

Cor. 1.—All the int ∠s of any rectil. fig., + 4 rt. ∠s = twice as many rt. ∠s as the fig. has sides. This Cor. is of universal extent.

2.—All the ext. ∠s of any rectil. fig. are together = 4 rt. ∠s,

Applicable only to *convex* figures; not to figures with re-entrant ∠s.

3.—If two △s have two ∠s of the one = two ∠s of the other; the third ∠ of the one = the third ∠ of the other.

Sch.—Lardner's *Euclid* gives twenty four corollaries.

Use.—This Theorem employed, 1. To determine the *Parallax* of a heavenly body; 2, To give the representative height of a mountain; and 3, to construct any regular right-lined figure.

33. 109. The lines joining the extremities of eq. and parallel lines, towards the same parts, are also eq. and parallel.

USE.—To ascertain the perp. height of a mountain, as well as the distance from the base to the foot of the perp.

34. 110. The opp. sides and $\angle$s of $\square$s are eq. to one another, and the diagonal bisects them; and *conversely*.

SCH.—If a quadril. fig. have any *two* of certain *ten* data, it will also have the others. By combining the ten, 360 questions are raised.

USE. 1.—The construction of the parallel ruler depends on this Prop. It is also useful, 2, to divide a line into any number of eq. parts; 3, to construct the Sliding Scale, called the *Vernier* or *Nonius* for measuring *minute* parts; 4, to obtain the distance between two objects; 5, to continue a st. line when an obstacle intervenes; 5, to divide a $\square$ into two eq. pts. from a · in one of the sides, &c.

35. 113. Parallelograms on the same base and between the same parallels are equal, or rather equivalent, to one another.

SCH.—The equality of $\square$s proved by the Method of Indivisibles.

USE.—The Prop. applied to convert a $\square$ into a rect. of eq. area. The linear units in the base multiplied by the linear units in the altitude of a $\square$ gives the Area.

36. 116. Parallelograms on eq. bases, and between the same ‖s are equal.

USE.—The Construction of the *Diagonal Scale*, and its application.

37. 118. Triangles on the same base and between the same ‖s, are equal.

Half the product of the base and altitude of a $\triangle$ gives the Area.

38. 119. Triangles on eq. bases and between the same ‖s, are eq. to one another.

SCH.—By dividing the base into eq. pts., and joining the .s of division to the vertex a $\triangle$ is divided into eq. parts.

USE.—This Prop. also enables us from any . in a side of a $\triangle$ to divide it into two eq. parts.

39. 120. Eq. $\triangle$s on the same base and on the same side of it are between the same parallels.

The *loci* of the vertices of eq. $\triangle$s on the same base, form a st. line

40. 121. Eq. $\triangle$s on eq. bases in the same st. line, and towards the same parts, are between the same parallels.

SCH. COR. 1.—A ‖ to the base of a $\triangle$ through the middle . of one side will bisect the other side.

2.—The lines joining the middle . s of the three sides divide the $\triangle$ into four eq. $\triangle$s.

3.—The L joining the points of bisection of each pair of sides is eq. to half the third side.

4.—A trapezium = a ▱ of the same alt. and of which the base is half the sum of the ∥ sides.

The Area of a trapezium = ½ the sum of the ∥ sides × the altitude.

The Area of a Square = the Square of the lineal units in one side.

Many other corollaries may be derived from Prop. 40.

41. 123. *Important.*—If a ▱ and a △ be on the same base and between the same ∥s, the ▱ shall be double of the △ ; and *conversely*.

Use.—The Area of any figure iresolvable into △s depends on this Proposition ; which enables us to find,—1, the area of a △ ; 2, the area of any rectilineal figure ; 3, the area of a polygon ; and 4th, the area even of a circle.

43. 126. The complements of the ▱s which are about the diam. of any ▱, are eq. to one another.

Cor.—The ▱s about the diag. and their complements are equiangular with the whole ▱.

Use.—To find a ▱ = a given ▱, and having one side = a given line.

47. 135. *Most important.*—In any rt. ∠d △, the square on the side opp. to the rt. ∠ is eq. to the squares on the sides containing the rt. ∠.

Cor. 1.—Hence, if the sides of a rt. ∠d △ be given in numbers the hypotenuse may be found.

2.—If the hyp. and one side be given, the other side may be found.

3.—If any number of squares be given, a square may be found = their sum ; or the multiple of a sq. may be ascertained ; or the difference of two squares ; or a sq. may be made = the ½, ¼ &c. of a given sq.

4.—If a perp. be drawn from the vert. of a △ to the base, the difference of the squares of the sides = the difference of the squares of the segments.

5.—If a perp. be drawn from the vertex to the base or base produced, the sums of the squares of the sides and alternate segments are equal.

138. Sch.—A Practical Illustration of Prop. 47. I.

139. Use 1.—Combined with other propositions, the 47. I. is applied 1°, to make a rectil. fig. similar to a given rectil. fig. ; 2°. to make a ⊙ double, or the half of another ⊙ ; 2. To construct the Chords, Natural Lines, Tangents and Secants of Trigonometrical Tables; 3. To find right triangular numbers ; & 4, 5, To ascertain heights and distances from the curvature of the earth.

48. 142. *Conversely.*—If the sq. on one of the sides of a △ be eq. to the squares on the other two sides of it, the ∠ contained by the two sides is a rt. ∠.

Extension of the Proposition,—the vert. $\angle$ of a $\triangle$ is $<, =$, or $>$ a rt. $\angle$ as the sq. on the base is $<, =$, or $>$ the sum of the squares of the sides.

REMARKS.

1. 143. The First Book founded entirely on the Defs. Posts. & Axioms.
2. 144. Only a few of the properties of a $\odot$ mentioned.
3. 144. A threefold Division of the Book ; 1°. from Pr. 1 to Prop. 26 the properties of $\triangle$s unfolded ; 2°. from Pr. 27 to Pr. 32, those of parallel lines ; and 3°. from Pr. 33 to Pr. 48, those of parallelograms.
4. 144. The most important Propositions are Prop. 4, 8, 26, 32, 41 and 47.

BOOK II.

THE PROPERTIES OF RT. $\angle$d. $\square$s, OR RECTANGLES.

145. A L may be cut *internally* or *externally*.
Magnitude the subject of Geometry,—Algebra and Arithmetic furnish, not proofs, but illustrations.

The Numerical Area of a Rectangle $= ab$, the altitude being represented by a, the base by b ; that of a $\triangle = \frac{1}{2}\, ab$.

Def. 1 & 2. p. 146. A rect is contained by any two conterminous sides; and in every $\square$, any $\square$ about a diam. + the complements is called the Gnomon.

Axiom. p. 146. The whole Area = the Areas of all the parts.

TWO PROBLEMS.—Prop. 11 and 14.

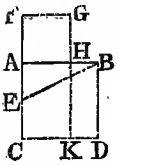
11. 171. To divide a L. into two parts, so that the rect. under the whole and one of the parts shall be eq. to the sq. of the other part; thus, in L

the L. is cut in the . H so that $AB \,.\, HB = AH^2$.

Or, as in 30, VI. to divide a L. in extreme and mean ratio.

The Algebraical and Arithmetical Solutions.

Cor. I. To cut a L. as AB, in extreme and mean ratio, it must first be produced in extreme and mean ratio, *i. e.* CF . FA must = AB^2



II.—When a L. as AB, or its equal AC, is cut in extreme and mean ratio, the rect. under the whole L. and its less segment = the sq. on the greater segments; thus, AC. (AC—AF) = AF^2, or AC . HB = AH^2

III.—Also the rect. under the whole L. as AC or AB and its greater segment = the difference between their squares, or AC . AF or AH = $AC^2 - AH^2$.

Sch.—Let L. be a line cut in extreme and mean ratio, g, the greater seg. l the less, and d the difference: then 1° $L^2 + l^2 = 3\,g^2$; 2°. $(L + l)^2 = 5\,g^2$; 3°. L . $d = g\,.\,l$; and 4 °. $l^2 = g\,.d$.

Use.—This Proposition is of frequent application, as for the construction of pentagons and the regular bodies called the Platonic Solids.

14. 179. To describe a square that shall be eq. to a given rectil figure A; *i. e.* EH^2 = rect. BD = A.

Use. 1 To find a mean propl. between two given lines; 2 to approximate to the sq. of curve-lined figures; 3. and to calculate the Areas of all plane figures.

TWELVE THEOREMS.

Prop. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; 12, 13.

1. 146. If there be two st. lines, one A, undivided, the other BC, divided into any number of pts. as BD, ED. EC, then A . BC = A . BD + A . DE + A . EC.

N.B. The Alg. and Arith. illustrations are attached to each proposition.

Use. Applied to the Demonstration of the Rule for the Multiplication of numbers.

2. 148. If a st. line AB, be divided into any two pts. AD, DB then the rectangles AB . AD, + AB . DB = the sq. on AB.

Numerical Multiplication may also be proved by this Prop.

3. 149. If a line, AB, be divided into any two pts. AD, DB, then the rectangle AB . DB = AD . DB + DB^2; or AB . AD = AD . DB + AD^2.

Cor. 1. $AB^2 - DB^2$ = (AB + DB) (AB—DB).
2.—$AD^2 - DB^2 >$ (AD—DB)^2 by 2 DB . (AD—DB).

Also applicable to the proof of numerical Multiplication.

4. 150. If a line AB, be divided into any two pts. AD, DB, then $AB^2 = AD^2 + DB^2 + 2\ AD\ .\ DB$.

Cor. 1. The parallelograms about the diam. of a sq. are also squares.

2.—The square of a line is four times the square of its half; thus $AB^2 = 4\left(\frac{AB}{2}\right)^2$

3.—Half the sq. of a line = 2 the sq. of half the line; thus $\frac{AB^2}{2} = 2\left(\frac{AB}{2}\right)^2$

4.—The sq. of a line will be equal to the sum of the squares of the parts + double the rect. under every distinct pair of parts.

Use 1.—In Algebra the *square of a binomial*, as $(x + y)^2 = x^2 + y^2 + 2xy$.

2.—This Prop. points out a practical way of extracting the Square Root of a number.

5. 154. If a line AB, A——|——|——B, be divided into two eq. parts, AC, CB, and two uneq. parts, AD, DB; then the rectangle $AD\ .\ DB + CD^2 = CB^2$.

Cor. The difference of the squares of two uneq. lines, AC, CD = the rect. under their sum and diff.; *i. e.* $AC^2 - CD^2 = (AC + CD)\ (AC - CD)$.

Lardner's; Corollaries to this Prop. are six.

Sch.—The principal properties connected with the eq. and uneq. division of a line.

Use.—We may apply this Prop. 1°. To find *the diff. between the squares of two uneq. numbers without squaring them*; 2°. To find *quantities in Arithmetical Progression*; 3°. To establish Prop. 35, III; and 4°. To find *the value of an Adfected Quadratic Equation* in Algebra.

6. 157. If a L. AB be bisected in C, and produced to any D; as A——C——B——D, then the rect.

$AD\ .\ DB + CB^2 = CD^2$.

Cor. If a line AD be drawn from the vert. A of an isosc. △ to the base or its production, the diff. between the squares of this line and the side of the △ is the rect. under the segments of the base; thus $AD^2 \sim AC^2 = BD\ .\ DC$.

Use.—By aid of this Prop. the diam. of the earth may be measured.

7. 160. If a L. AB, be divided into any two parts, as AD, DB then $AB^2 + DB^2 = 2\ AB\ .\ BD + AD^2$;
or $AB^2 + AD^2 = 2\ AB\ .\ AD + DB^2$.

Cor. 1. If AB and BD be considered as two independent lines, AD being their diff.; then $AB^2 + BD^2 = 2\ AB\ .\ BD + (AB—BD)^2$

2.—Also $(AB + BD)^2$; $(AB^2 + BD^2)$; and $(AB—BD)^2$ are in Arith. Progression; the com. diff. being 2 AB . BD.

8. 161. If a L. AB, be divided into any two parts, as AD, DB, four times the rect. under the L. and one of the pts. + the sq. of the other pt. = the sq. of the L. made up of the whole and that part; *i. e.* $4\ AB\ .\ BD + AD^2 = (AB + BD)^2$

Sch. 1. Otherwise the sq. of the sum of two lines = 4 lines the rect. under them + the sq. of their diff. *i. e.* $(AD + DB)^2 = 4\ AD\ .\ DB + (AD—DB)^2$

2.—Four times the sq. of half the sum = 4 times the rect. under the lines + 4 sq. of half the diff.; *i. e.* $4\left(\frac{AD + DB}{2}\right)^2 = 4\ AD\ .\ DB + 4\left(\frac{AD—DB}{2}\right)^2$

Use.—The above principles are applied to Algebra and to the extraction of the sq. root.

9. 164. If a L. be divided into two eq. pts, and also into two unequal parts, as A ——— C ——— D ——— B; then the squares of the two uneq. pts. together = double of the sq. of the half line, and of the sq. of the line between the points of section; *i. e.* $AD^2 + DB^2 = 2\ (AC^2 + CD^2)$

Sch.—Or, 1. $AD^2 + DB^2 = 2\left(\frac{AD + DB}{2}\right)^2 + 2\left(\frac{AD—DB}{2}\right)^2$

2. $AD^2 + DB^2 = \frac{(AD + DB)^2}{2} + \frac{(AD—DB)^2}{2}$

10 167. If a L. be bisected and produced to any point, as,

A ——— C ——— B ——— D,

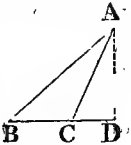
then the sq. of the whole line thus produced + the sq. of the part produced = double of the sq. of the half line + the sq. of the line made up of the half and the pt. produced; thus $AD^2 + DB^2 = 2\ (AC^2 + CD^2)$

Sch.—Props. 9 and 10 are applicable to Algebra.

Use.—Prop. 1—10 contain the whole theory of the relations of rectangles and squares formed by lines and their parts.

1°. Given the sum and diff. of two Ms. to find them. 2°, If the Area be divided by a side the quotient = the other side. 3°. Of the five quantities depending on a rect, any two being given, the sides can be found.

12. 173. *Important.*—In obtuse ∠d △s, if a perp. be drawn from either of the acute ∠s to the opp. side produced, the sq. of the side subtending the obt. ∠ is > the squares of the sides containing the obt. ∠ by twice the rect. under the side on which the perp. falls, and the line intercepted without the △ between the perp. and the obt. ∠.; as

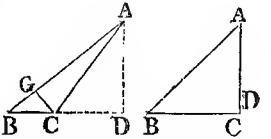


$AB^2 = BC^2 + AC^2 + 2\,BC.CD.$

or $AB^2 > BC^2 + AC^2$ by $2\,BC.CD.$

Use.—By this Prop. the Area of a △ may be ascertained when the three sides are known.

13. 175. In every △, the sq. of the side subtending either of the acute ∠s is < the squares containing that acute ∠ by twice the rect. under either of these sides, and the L. intercepted between the perp. let fall upon it from the opp ∠, and the acute ∠; *i. e.*,



1°. $AC^2 < (AB^2 + BC^2)$ by $2\,BC.BD$; 2°. or, by $2\,BD.BC$; and 3°. by $2\,BC.BC$.

Thus is obtained the measure of the sq. of the side subtending an acute ∠.

Cor. If in the fig. to Case 2°. a perp. CG be drawn from ∠ C to AB, then the rect. $AB.GB$ = the rect. $BC.DB$

Sch.—Prop. 12 and 13 contain the Elements of Trigonometrical Analysis.

Use.—To obtain the perp, when the three sides of a △ are given; 1°. when the perp. falls *within* the base; 2°. *without* the base

The *Area* of the $\triangle = \frac{BD + DC}{2} \times AD$; or $\frac{BD - DC}{2} \times AD$.

Remarks.

1. 181. Of the fourteen Propositions, the *ten* first contain the theory of the relations of the rectangles and squares on divided lines; the *twelfth* and *thirteenth* the theory of the relation between the sq. of any one side of a △, and the squares of the other two sides.

2. 181. Lines cut into *any two parts*, in Prop. 2, 3, 4, 7 and 8.
3. 181. Lines cut into *two eq.* and *two uneq.* parts, in Prop. 5, 6, 9 and 10.

Synopsis of Book II.

CASE I—VIII. Pages 182 — 186.

PRACTICAL RESULTS.

I. 188—197 Problems 1—31 for the Construction of Geom. Figures bks. I and II.
II. 197—199 Problems 1—10 „ „ bk. III.
200—203 Problems 1—18 „ „ bk. IV.
204—207 Problems 1—11 „ „
III. 207—211 Principles of Construction; 1°. For Geom. Instruments to measure Distances and Angles; 2°. For Geom. Figures to exhibit the representative values of actual magnitude and space.
IV. 211—213 Principles which, without requiring that we should measure all the boundaries of a Surface, enable us accurately to calculate 1°, Lines or Distances; 2°. Angles; and 3°. Magnitudes, or Areas.

APPENDIX.

I. 215—219 Geometrical Analysis, Rules for Conducting, and Examples.
II. 219—227 Geometrical Exercises; *Series* I Problems and Theorems in Bk. I; Problems and Theorems in Bk. II. *Series* II Propositions, including Problems and Theorems, not fully proved, or not inserted in Bks. I. and II.

PART II.

Containing Books III, IV, V, & VI.

Preface
Symbolical Notation and Abbreviations.

BOOK III.

PROPERTIES OF THE CIRCLE AND OF LINES IN AND ABOUT IT.

1 The word circle employed in two senses; and certain Properties *assumed* from experimental knowledge.
2 The foundations of Trigonometry, Plane and Spherical.
3 Summary of Book III from Billingsley's Euclid.

Def. 1—12 pp. 3—6. Explanatory Notes.
Ax. A, 6.

SIX PROBLEMS.

Prop. 1;—17;—25;—30;—33, 34.

Prop. 1. p. 7. To find the centre of a circle.

Cor. If in a ⊙ one L. bisects another at rt. ∠s, the cen. of the ⊙ is in the bisecting line.

Sch.—The rigour of the reasoning requires a previous proof of the conditions on which a point is *within* or *without* a circle.

17. 39. To draw a L. from a given point. either without or in the ⊙ce, which shall touch a given circle.

Sch.—From the same . two eq. tangents may be drawn.

Use.—Tangent Lines, how drawn practically;—of frequent use in Trigonometry.

25. 54. A segment of a circle being given, to describe the circle of which it is the segment.

Sch. This problem is equivalent to the following;—to inscribe a △ in a ⊙; or, to make a ⊙ pass through three given . s not in the same line.

Use.—Applied for constructing an arch,—for cutting stone, wood, metals &c., and for finding the apogee of the moon, and the eccentricity of the earth's orbit.

30. 64. To bisect a given arc of a circle.

Cor, To divide an arc into any number of eq. pts. that are powers of 2.

Sch. 1.—Except in the case of a quadrant an arc cannot be cut into 3, 5, &c., eq. pts.

2. To trisect a quadrant.

The trisection of an ∠ not effected by *Euclid's* Geometry;—what required for the trisection of an arc;—the *trisectrix* one of the trochoidal curves.

33. 72. On a given L. to describe a segment of a circle, which shall contain an ∠ = a given rectil. ∠.

Sch.—To divide a ⊙ into any number of eq, pts. of which the perimeters also are equal.

Use—Of extensive Application, as, 1. Given the distances of three landmarks to find their distances from the place of observation. 2. To ascertain the distance from two stations; 3. Given the base and vert. ∠ to find the locus of the vertex; 4. Given the vert. ∠, the base, and the area to construct the △; 5. Through three . s to draw lines so as to make an equil. △; 6. Given the ∠ = the vert. ∠, and the base, to find the locus of the vertex. 7. Given the base, the vert. ∠ and the perp. from one end of the base to the opp. side, to construct the △.

34. 76. From a given circle to cut off a segment, which shall contain an ∠ = a given rectil. ∠.

Use 1.—By Prop. 33 and 34, if three observations be taken, the eccentricity of the earth's annual orbit, and its aphelion may be found. 2. Also in optics, to ascertain the , where two uneq. lines will appear equal.

THIRTY-ONE THEOREMS.

Prop. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. 16;—18, 19, 20, 21, 22, 23, 24;—26, 27, 28, 29;—31, 32;—35, 36, 37.

2. 9. If any two . s be taken in the ⊙ce of a circle, the st. line which joins them shall fall within the circle.

Cor 1. A st. line cannot cut the ⊙ce of a ⊙ in more than two points.
2. A st. line which touches a circle meets it only in one point.
3. A circle is concave towards its centre.

Sch.—Commandine's *direct* proof of Prop. 2.

Use.—A globe on a plane surface touches the plane in only a single point.

3. 11. If a st. line through the cen. of a ⊙ bisect a st. line not through the centre, it shall cut it at rt. ∠s; and *conversely*.

Cor. 1. A L. bisecting a chord at rt. ∠s passes through the cen. of the circle,
2. All chords ‖ to the tang., at either extremity of the diam., are bisected by the diam.
3. The Line bisecting the com. chord of two ⊙s at rt. ∠s passes through the centres of both ⊙s.
4. When in a ⊙ there are several chords ‖ to each other, the *locus* of their . s of bisection is in that diam. which is at rt ∠s to them ; and if the L. which bisects one chord be a perp., that line bisects all the par. chords at rt ∠s.

Use. Various, as 1. Given a ⊙ to find its centre; 2. Given an arc, to find the cen. of the ⊙ of which it is an arc; 3. Through three . s not in a st. line to draw a ⊙; and 4. In Trigonometry.

4. 14. If in a ⊙ two st. lines cut one another, which do not pass through the centre, they do not bisect each other.

Cor. No ▱ except a rectangle can be inscribed in a circle.

Use.—Applied to determine the eccentricity of the Sun's apparent path; and the centre of rotation of an eccentric wheel.

5. 15. If two circles cut one another, they shall not have the same centre.

Sch, Or, concentric ⊙s cannot meet,—and that with the less rad. is within the other.

6. 16. If one circle touch another internally, they shall not have the same centre.

Sch.—Props. and 6 may be combined into one.

7. 17. If any . which is not the cen. be taken in the diam. of a ⊙, then, 1st, of all st. lines from it to the ⊙ce, the *greatest* is that in which the cen. is, and the other pt. of that diam. is the *least*; and 2nd, of any other st. lines, that which is nearer to the line through the centre is always *greater* than the one more remote; also 3rd, those lines which make eq. ∠s with the diam. are *equal*; and, 4th, from the same . there can be drawn *only two equal* st. lines, one on each side of the diam.

Sch.—*Maximum* and *minimum* of a revolving line cut by a ⊙ce.

Use.—Of the arcs of great ⊙s from the pole to the horizon the *greatest* is that part of the meridian which passes through the zenith. The Prop. may be used for showing aphelion and perihelion distance.

8. 20. If any . be taken without a ⊙, and st. lines be drawn from it to the ⊙ce, whereof one passes through the cen.; 1st, those which make eq. ∠s with the line through the cen, are *equal*; 2nd, of those which fall upon the *concave* ⊙ce, the *greatest* is that which passes through the cen.; and of the rest that nearer to the one passing through the cen. is always *greater* than one more remote; but, 3rd, of those which fall upon the *convex* ⊙ce, *the least* is that between the given . without the ⊙ and the diam., and of the rest, that nearer to the least is always *less* than one more remote; and 4th, only *two eq. lines* can be drawn from the same . to the ⊙ce, one upon each side of the line passing through the centre.

Sch. 1. The *concave* and *convex* pts of the ⊙ce determined by tangents.

2. What is meant by the *distance* of a . from or to a line.

Of all st. lines from a . out of a line to a given line, the perp. is the *least*, and the lines nearer to the perp. are *less* than those more remote; and to a line on one side of the perp. there is only one eq. line on the other side of the perp.

Use.—If a tang. and sec. be drawn to the same., the tang. is < sec., but > ext. pt. of sec.

2. By Pr. 7 & 8, and Ax. A, III, 1st, When one ⊙ is within another, without touching, the distance of the centres, are < the diff. of the radii; and *conversely*. 2nd, When two ⊙s lie each without the other and do not meet the distance of the centres is > the sum of the radii; and *conversely*.

9. 24. If a . be taken *within* a ⊙, from which there fall more than two eq. st. lines to the ⊙ce, that point is the centre of the ⊙.

Cor. 1. From any other . than the cen. only two eq. st. lines can be drawn to the ⊙ce, whether the . be within, or without the ⊙.

2. From three . s given not in the same st. line the ⊙ce of the ⊙ may be found.

Sch.—This Prop. gives the *criterion* for determining the cen. of a ⊙.

Uses. Applied, 1st, To draw a ⊙ through three given . s ; 2nd, To find the cen. of a given ⊙ ; and 3rd, To determine the centre of an arc of a ⊙.

10. 25. One ⊙ce of a ⊙ cannot cut another in more than two parts.

Sch.—If the ⊙ces of ⊙s coincide in three . s they will coincide in every point.

11. 26. If one ⊙ touch another *internally* in any . , the st. line joining their centres, being produced shall pass through the . of contact.

Sch.—When the distance between the centres = the diff. of the radii the ⊙s touch *internally*.

Use 1.—A *practically* useful method for drawing an *oval* on any given major axis may be derived from this Prop.

2. On the same principle a *Spiral* is described.

A *Spiral* defined and some of its varieties.

12. 29. If two ⊙s touch *externally* in any . , the st. line joining their centres shall pass through that . of contact.

Use.—The drawing of a *Serpentine Line*, or *cima recta*, depends on this Prop.

13. 30. One ⊙ cannot touch another in more points than one, whether it touches it on the inside or the outside.

Sch. 1, 2.—A *direct* method of demonstration substituted for the *indirect*.

Use.—Prop. 10, 11, 12 and 13, explain the motion of the Planets in Epicycles.

14. 32. Eq. st. lines in a ⊙ are equally distant from the centre ; and *conversely*.

Sch.—A principle employed in Prop. 14 & 15 is this ; if $A + B = C + D$, then if $A = C$, $B = D$; if $A > C$, $B < D$; if $A < C$, $B > D$.

15. 33. The diam. is the greatest st. line in a ⊙ ; and of all others that nearer to the cen. is always greater than one more remote : and the greater is nearer to the cen. than the less.

Sch. The *longest* chord is the diam. ; the shortest through a given . , that perp. to the diam.

USE. Prop. 14 & 15 employed, 1°. To show that ⊙s of latitude diminish towards the poles; 2°. In constructing the *Astrolabe*. 3°. In determining the position of greatest effect for a given lever.

16. 36. The st. line at rt. ∠s to the diam. of a ⊙, from the extremity of it, falls *without* the ⊙; and no st. line can be drawn from the extremity between that st. line and the ⊙ce, so as not to cut the ⊙; *or*, which is the same thing, no st. line can make so great an acute ∠ with the diam. at its extremity, or so small an ∠ with the st. line at rt. ∠s to it, as not to cut the ⊙.

COR. 1. If a st. line be drawn at rt ∠s to any diam. of a ⊙, from its extremity, it shall touch the ⊙ at the extremity; and a st. L. touching the ⊙ at one . shall touch it at no other point.

2. By 28, I. st lines at rt. ∠s to the extremities of the same diam. are parallel.

3. Tangents to a ⊙ from the same . are equal.

SCH. The 16th Prop. may be proved *directly*.

USE. By this Prop. we can prove, 1°. The *infinite divisibility* of linear magnitudes; 2°. The distances and heights of objects on the earth's surface, when they are situated on the verge of the natural horizon.

18. 41. If a st. L. touches a ⊙, the st. L. from the cen. to the . of contact shall be perp. to the line touching the ⊙.

SCH. Prop. 16 & 18 may be regarded as the *converse* of each other.

USE. To draw a tang. to each of two given circles.

19. 42. If a st. L. touches a ⊙, and from the . of contact a st. L. be drawn at rt. ∠s to the touching line, the cen. of the ⊙ shall be in that line.

SCH. In concentric ⊙s all chords of the greater ⊙ touching the less are eq. and bisected at the . of contact.

USE. Tangent lines are employed, 1°. in Optics to determine the part of the earth enlightened by a meteor, &c. 2°. To ascertain the earth's diam.; 3°. To explain the Phases of the Moon; 4°. To trace the ⊙ of the physical horizon, and 5°. in Dialling to calculate the Hour Lines.

20. 44. The ∠ at the cen. of a ⊙ is double of the ∠ at the ⊙ce on the same base, *.i e.*, on the same part of the ⊙ce.

COR. Any ∠ at the ⊙ce is measured by half the arc on which it stands.

SCH. 1. The reasoning assumes, that among 4 Ms, A, B, C, D, if A = 2 B, and C = 2 D, then A + C = 2 (B + D); also that if M = 2 M′, and (M − *a*) = 2 (M − *a*) then the *remainder* of M = 2 the remainder of M′. Another method of proof given for Case 3, Prop. 20.

2. If *Euclid's* def. of an ∠ be strictly adhered to, Prop. 20 not universally true.

3. Demonstrations may sometimes be arranged as a Simple Equation.

Use. This Prop. applicable to Trigonometry and Astronomy.

21. 47. The $\angle$s in the same segment of a $\odot$ are eq. to one another.

Cor. If on the base of a $\triangle$, there be described a seg of a $\odot$, the vertex of the $\triangle$ shall fall *without*, or *within*, or *upon the arc*, as the vert. $\angle$ is $<$, $>$, or $=$ the $\angle$ in the segment.

Sch. The $\angle$ at the $\odot$ce has for its measure one half of the arc on which it stands.

Use. Applied, 1°. To construct a building so that the spectators may all see an object under the same $\angle$; 2°. To bisect an $\angle$; 3°. To construct a fig, representative of the distance of the place of observation from an object. 4°. To draw the arc. of any $\odot$ on a large scale.

22. 49. The opp. $\angle$s of any quadril. figure inscribed in a $\odot$ are together eq. to two rt. $\angle$s; and Cor. 1. *conversely.*

Cor. 2. If any side of a quadril. in a $\odot$ be produced the ext. $\angle$ = int. opp. $\angle$.

3. If two chords cut off sim. segments from the same or different $\odot$s, the other segments will also be similar.

4. If opp. $\angle$s of a quadril. be equal they must be both rt. $\angle$s.

Use. Applicable to the Construction of the Tables of Cords, and in Trigonometry.

23. 51. On the same st. L. and on the same side of it there cannot be two sim. segments of $\odot$s, not coinciding with one another.

Sch. This Prop. the same in principle as the 7th bk. I.

24. 52. Similar segments of $\odot$s on eq. st. lines are equal to one another.

Cor. 1. Sim. segs. having eq chords have also eq. arcs.

2. Sim. segs. having eq. chords are parts of eq. circles.

3. If the radii, and $\angle$s of sectors are eq., the sectors themselves are equal.

Use. By this Prop. curved lined figures are often reduced to rectilineals.

26. 56. In eq. $\odot$s, eq. $\angle$s stand upon eq. arcs, whether they be at the centres or the $\odot$ces.

Cor. 1. If the opp. $\angle$s be eq. their opp. diagonal must be a diam., and the seg a semicircle.

2. In the same or eq. $\odot$s one central or circumferential $\angle$ is $<$, $=$, or $>$ another, as the arc of the one is $<$, $=$, or $>$ the arc of the other.

3. The diameters intersecting at rt. $\angle$s divide the $\odot$ce into four eq. arcs.

4. When the sum of the central $\angle$s = 4 rt. $\angle$s, the sum of their arcs = the whole $\odot$ce.

5. When the sum of the ∠s at the ⊙ce = 2 rt. ∠s, the sum of their arcs also = the whole ⊙ce.

6. Sim. arcs. of eq. ⊙s are equal.

7. Par. chords of a ⊙ intercept eq. arcs.

8. If two chords intersect *within* a ⊙, the sum of the intercepted arcs = the arc which the ∠ would intercept at the ⊙ce, that is eq. to the ∠ under the chords.

9. If two chords intersect at a . *without* a ⊙, the difference of the arcs which they intercept is = to the arc, which an ∠ would intercept at the ⊙ce that is eq. to the ∠ under the chords.

USE. Applied to find the true central ∠ of an imperfect theodolite.

27. 59. In eq. ⊙s the ∠s which stand upon eq. arcs are eq. to one another, whether they be at the centres or the ⊙ces.

COR. 1. In the same or eq. ⊙s, the sectors on eq. arcs are equal; and *conversely*.

2. If the chords of a ⊙ are parallel, they intercept eq. arcs; and *vice versâ*.

SCH. 1. What is true of eq. ⊙s, is true of eq. arcs in the same ⊙.

2. The sum of the ∠s at the cen. of a ⊙ = 4 rt ∠s; and the sum of the ∠s at the circumference = 2 rt ∠s.

USE. By this Prop. a . parallel through a given . may be drawn.

2. To ascertain the Area of a Sector.

28. 61. In eq. ⊙s, eq. st. lines cut off eq. arcs, the greater = the greater, and the less to the less.

29. 62. In eq. ⊙s, eq. arcs are subtended by eq. st. lines.

COR. 1. In the same or eq. ⊙s, eq. sectors stand on eq. arcs; and *conversely*.

2. St. lines which intercept eq. arcs are parallel; and par. st. lines intercept eq. arcs.

USE. In Spherical Trigonometry Prop. 26, 27, 28 and 29, are of continual use.

31. 66. In a ⊙, the ∠ in a semicircle is a rt. ∠, but the ∠ in a seg. > a semicircle is < a rt. ∠; and the ∠ in a seg. < a semicircle is > a rt. ∠.

COR. If one ∠ of a △ be eq. to the other two, it is a rt ∠.

SCH. 1. *Conversely*, The seg. containing an acute ∠ is > a semicircle; and the seg. containing an obtuse ∠ is < a semicircle.

2. *Lardner's* elegant and brief demonstration of 31. III.

68. USE. Problems derived; 1°. From a . *in* a line or at its extremity to draw a perp.; 2°. From a . *without* a line to draw a perp.; 3°. From a . without a ⊙ to draw a tangent; 4°. By means of a Square to find the centre of a ⊙; 5°. To try if Squares are exact.

32. 69. If a st. L touches a ⊙, and from the . of contact a st L. be drawn cutting the ⊙; the ∠s which this line makes with the line touching the ⊙, shall be eq. to the ∠s in the alternate segs of the ⊙.

Cor. 1. *Conversely*, If from the end of a line cutting the ⊙, &c.

2. If two or more ⊙s touch each other, and through the . of contact two st. lines be drawn meeting their ⊙ces, the chords of the intercepted arcs will be parallel.

3. If two or more ⊙s touch each other, at a com. point of contact, any line passing through the . of contact will cut off sim. segs. from each.

4. In an equil. △, if the sides be bisected, and st. lines be drawn joining the . s of bisection, of those lines, two will be tangents to the ⊙, which passes through the ends of the other line, and the ang. point opp. to that line.

5. Tangents through the extremities of the same chord, make the ∠s on the same side equal.

6. If tangents are par., the line joining the . s of contact is a diam.

Use. 1. This Prop. preliminary to the 33rd, and required for various Theorems.

35. 77. *Very important.* If two st. lines cut one another within a ⊙, the rect. under the segments of one of them = the rect. under the segments of the other.

Cor. *Conversely*, If the rectangles be equal &c.

Sch. Or, If two chords of a ⊙ cut one another, the rectangles under their segs. terminating in the . of section shall be eq.

The terms *ordinate* and *abscissa* explained.

Use. 1. If of two eq. ⊙s, the centres be each on the ⊙ce of the other, and a com. chord be drawn ‖ to the line joining the centres, then the lines joining the . s where the com. chord cuts the ⊙s and the extremities of the lines joining the centres, form ▱s; 2. To find a line which is a fourth propl. to three given lines, or a third propl. to two given lines.

36. 81. *Important.* If from any . *without a* ⊙ two st. lines be drawn, one of which cuts the ⊙, and the other touches it; the rect. under the whole line which cuts the ⊙, and the part of it without the ⊙, shall be = the sq. of the L. which touches it.

Cor. 1. If from any . *without* a ⊙ there be drawn two st. lines cutting it; then the rectangles under the whole lines and the parts of them without the ⊙ shall be = to one another.

Cor. 2. If from the same . two tangents be drawn to the same ⊙, they are eqnal.

Sch, Or, If any chord of a ⊙ be prod. to cut a tang. to the same ⊙, the sq. of the tang. shall be = the rect. under the segs. of the chord produced.

USE 1. In any semicircle if from the extremities of the diam. chords be drawn intersecting within the semicircle; then the sum of the rectangles under any two such intersecting chords and the sections of the chords between the extremities of the diam. and the intersecting., shall = the sq. on the diam.

USE. 2. The *Art of taking a true Level* is deduced from this proposition; 1° Plan of a Field Book; 2° The measuring of an Ascent. 3° Correction for curvature. 4° Deviation of the horizontal from the true level. 5° Distances of the horizontal boundary. 6° Sum of the horizons when taken. 7° The earth's diam. ascertained.

37. 85. If from a . *without* a ⊙ there be drawn two lines, one of which cuts the ⊙, and the other meets it; if the rect. contained by the whole L. which cuts the ⊙, and the part of it without the ⊙, be eq. to the sq. of the line which meets it, the L. which meets shall touch the ⊙.

COR. Tangents from the same . without a ⊙ are equal.

USE 1. Through two given . s to describe a ⊙ touching a given ⊙.

2. Prop. 35, 36 and 37 are amongst the most important in Plane Geometry. The earth's diam. calculated from them by *Maurolico* in the 16th century.

REMARKS.

1. 87. The Propositions of Book III, classified under *five* general heads.
2. 87. Of the 37 Propositions six only are Problems.
3. 88. Twelve others have been deduced.
4. 88. Problems might be given for drawing ⊙s which are tangents to two or three given ⊙s, &c.
5. 89. Several of the Principles of Levelling and Surveying, and of making Geographical and Astronomical Observations laid down.

BOOK IV.

METHODS OF CONSTRUCTING REGULAR STRAIGHT-LINED FIGURES IN AND ABOUT A CIRCLE &C.

91. Excepting Pr. A, Theor., the fourth book consists entirely of Problems.

This book of essential service in Astronomy, and in Civil and Military Engineering.

Def. 1—7 pp. 92, 93. Euclid's Definitions.
8—10 97. Definitions additional to those of *Euclid.*

SIXTEEN PROBLEMS AND ONE THEOREM.

Prop. 1, 2, 3, 4, 5;—Prop. A;—Prop. 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16.

1. 94. In a given ⊙ to fit exactly a rt. line = a given rt. line, which is not greater than the diam of the ⊙,

Use 1°. Within a given ⊙ to place a line of a given length, not greater than the diam. of the given ⊙, which line shall pass through a given . in the ⊙cc.

2°. To draw that diam. of a ⊙ which shall pass at a given distance from a given point.

2. 95. In a given ⊙ to inscribe a △ equiangular to a given △.

Sch. The *analysis* given.

Use. An equil. △, being inscribed in a ⊙, and through the angular . s tangents being drawn, these tangents will also form an equil. △, the area of which is four times that of the inscribed equil. △.

3. 97. About a given ⊙ to circumscribe a △ equiangular to a given △.

Sch. The analysis of the Problem.

4. 99. To inscribe a ⊙ in a given triangle.

Sch. 1. Or, to describe a ⊙ to touch three given lines not parallels. *Analysis.*

Use. 1. If three ∠s of a △ be bisected by st. lines, these lines will intersect in the same point.

2°. An expression deduced for *the Area of a* △; and for *the Rad. of the inscribed* ⊙.

3° The Properties of *Circles exscribed* to a triangle. 1. The bisectors of any int. ∠ and of the remaining two ex. ∠s; 2. The rad. of the exscribed ⊙ may be found in numbers; 3. A Formula for the *Area* of a △ in the terms of the sides; 4. Expressions *for the radii of the three exscribed* ⊙*s*; 5. Area of $\triangle = \sqrt{r\,r'\,r''\,r'''}$; 6. In a rt. ∠d △ the *diam.* of the *inscribed* ⊙ = the diff. of the sum of the sides and the hyp.; and the *diam.* of the ⊙ *exscribed* to the hyp. = the perimeter of the △.

5. 102. To circumscribe a circle about a given triangle.

Cor. 1. When the cen. of the ⊙ falls *within* the △, each ∠ is an acute ∠; when *on* a side of the △, the ∠ opp. that side is a rt. ∠; and when *without* the △, the ∠ opp. the side nearest the cen. is an obt. ∠; and *conversely.*

Cor. 2. The perps. bisecting the sides of a △ meet at the cen. of the circumscribing ⊙.

Cor. 3. Perps. from each ∠ on the opp. side intersect in the same point.

Sch. With what this proposition is identical.

USE. If one ⊙ be inscribed in an equil. △ and another circumscribed about it, the ⊙s are *concentric*, and the rad. or the diam of one is *double* the rad. or the diam. of the other.

A. 105. THEOR. A circle may be described about any reg. polygon, or inscribed within it; and *conversely*.

USE. 1. The construction of a reg. polygon; 2. To inscribe a polygon in a given ⊙; 3. To circumscribe a polygon; 4. The Area of a reg. polygon; 5. The Area of a ⊙.

6. 106. PROB. To inscribe a square in a given circle.

COR. The Sq. on the rad. of an inscribed ⊙ is ½ the sq. inscribed in a ⊙, and ¼ the sq. on its diameter.

USE. By bisecting the arcs, and joining the .s of bisection an *octagon* may be formed, &c.

7. 107. To circumscribe a square about a given circle.

COR. In the same ⊙ the circumscribed square = twice the inscribed square.

USE 1. To inscribe and to circumscribe a reg. *octagon*.

2. A reg octagon inscribed in a ⊙ = rect. under the sides of the inscribed and circumscribing squares. 3. If a quadril. be circumscribed about a ⊙, any two of its opp. sides = ½ the perimeter.

8. 109. To inscribe a circle in a given square.

SCH. To inscribe a circle in a given quadrant.

9. 111. To circumscribe a circle about a given square.

10. 112. To construct an isosc. △, having each of the ∠s at the base double of the third, or vertical angle.

USE 1. In the fig. constructed for Prob. 10, the side AC inscribed in the smaller ⊙ACD = the side of a *pentagon* in that ⊙, and also = the side of a reg. *decagon* in the larger ⊙ BDE.

2. On the side DC being produced to meet the ⊙ BDE in F, and FB being joined, the ∠ ABF = *three* times ∠BFD.

3. To quinquisect, *i. e.*, divide a quadrant into five equal parts, &c.

11. 115. To inscribe an equil. and equiangular pentagon in a given circle.

SCH. 1—4. Remarks on reg. polygons. 5. Formulæ for determining the relative magnitudes of the ∠s of isosc △s to be used in the construction of reg. polygons.

USE. 1. To draw a △ = in area to a given polygon.

2. The lines joining the alternate ∠s of a reg. pentagon, will form another reg. pentagon; and the .s of intersection of the alternate sides produced will also form another reg. pentagon.

12. 118. To circumscribe an equil. and equiang. pentagon about a given circle.

Sch. If the ⊙ce of a ⊙ be divided into any number of parts, the chord joining the . s of division shall include a reg. polygon, inscribed in the ⊙; and the tangents through those . s shall include a reg. polygon of the same number of sides circumscribed about the ⊙.

13. 120. To inscribe a circle in a given equil. and equian. pentagon.

14. 121. To circumscribe a circle about a given equil. and equiang. pentagon.

Sch. 1. To circumscribe a reg. polygon about a given circle.
2. To inscribe a circle in a reg. polygon.
3. To circumscribe a circle about a given polygon.

Addenda to 14, IV. 1. To analyze the conditions on which the drawing of a reg. *decagon* and a reg. *pentagon* depends.

2. To demonstrate, that the sq. on the side of a reg. pentagon inscribed in a ⊙ = the sum of the squares of the rad., and of the side of the inscribed decagon.

15. 124. To inscribe an equil. and equiang. hexagon in a given circle.

Cor. 1. The side of a reg. hexagon inscribed in a ⊙ is = the rad., or semidiam, of the circle; or the chord of 60° = the rad.
2. An equil. △ would be inscribed by joining the alternate . s in the hexagon.
3. Every equil. fig. inscribed in a ⊙ is equiangular.

Sch. 1. The opp. sides of a hexagon are parallel.

Use 1. On a given L. to describe a reg. hexagon.
2. The inscribed hexagon in a ⊙ is three-fourths of the area of the circumscribed hexagon.
4. Half the rad. = sine of 30°.
5. The inscribed hex. and the successive bisections of its arcs, the ground-work for finding the *approximate ratio of the* ⊙; *i. e., of a* ⊙ *to its diameter.* Or a Square may be taken as the ground-work of the process.

16. 128. To inscribe an equil and equiang. quindecagon in a given circle.

Cor. The only reg. st. lined figures which can be placed, side by side, so as to make a continuous plane surface, are the equil. △, the square, and the hexagon.

Sch. 1. To circumscribe a reg. quindecagon about a circle.
2. To find the arc subtending a side of a *reg. thirty-sided* figure.

Use. This Prop. opens the way for the *construction of other polygons.*

Observations on Polygons.

1. 130. I. The *four* known ways of dividing a ⊙ geometrically.
2. 130. II. To many polygons we must apply an *approximate process.*

1. In a given ⊙ to inscribe any reg. rt. lined fig.; or to divide the ⊙ce of a ⊙ into any assigned number of eq. parts.
2. Approximative Method,— for the Heptagon, Nonagon, &c.

3. 132. III. 1st. To find the magnitude of an ∠ at the centre of a reg. polygon.
2nd. To find the magnitude of an ∠ formed by two adjacent sides of a polygon.

4. 133. IV. On a given rt. Line AB to construct a reg. polygon.
1. The Formula, $\pi - \frac{2\pi}{n}$ = one of the eq. ∠ s of the polygon.
2. The Line × tabular rad. = units of length in the rad. of the circumscribing ⊙.

5. 134. V. The Area of a reg. polygon = $\frac{n\,AB}{2}$ × the perp. from the centre.

By using the Table, AB^2 × tabular Area = Area of reg. polygon.

6. 135. VI. Dodson's Tables for calculating and constructing Polygons of not more than 12 sides.
1°. When the length of the side = 1.
2°. When the radius of circumscribed ⊙ = 1.
3°. When the radius of inscribed ⊙ = 1.
4°. When the area = 1.

7. 137. VII. Polygonal numbers, their nature and construction.
To find the numbers which bear the name of an n*—sided figure.*

8. 138. VIII. The nature and Construction of *Star-shaped Polygons, i. e.* reg. polygons with *re-entrant* angles.

Remarks.

1. 139. Classification of the 16 Problems under *five* general heads.
2. 139. Theoretical Reasoning in Geometry guides to most important Practical results.

BOOK V.

The Theory of Proportion, or of the Comparative Magnitudes of Plane Figures.

141. Book V. independent of the preceding books,—its subjects, *ratio* and *proportion.*
Illustrations to be derived from Algebra and Arithmetic.

The Theory of Proportion an application of Numbers to the purposes of Plane Geometry.

SOME PROPERTIES OF PROPORTIONAL NUMBERS.

142. Numbers compared either by their *difference* or their *quotient*.
Identity in the *diff.* of three or more numbers constitutes Arithmetical Proportion ; in the *quotient*, Geometrical Proportion.
Extremes, means, antecedent, consequent ; Ratio how expressed.
Various Rules, 1—6 modifying a Proportion, all depending on the principle, "*resulting equations are equally true, whenever the thing which is done on one side of an equation is also done on the other side.*"

143. *Essential Property*, or *Criterion* of numbers in true Proportion.
When three terms of a Proportion are given the fourth may be found.

144. Other important Properties of numbers in proportion,—variations and combinations ; 1°. *Multiplicando, & Dividendo* ; 2°. *Alternando, & Invertendo* ; 3°. *Componendo, & Dividendo*; 4°. *Addendo, & Subtrahendo* ;

146. COMPOUND RATIO ; resulting products of corresponding antecedents and consequents will be in proportion.
Like Powers and Roots of Proportionals are also in Proportion.

147. Irrational numbers and Incommensurables.

EUCLID'S THEORY OF GEOMETRICAL PROPORTION.

148. Named the Elements of Mathematical Logic.
The Definitions and Propositions of Bk. V. may be extended to every species of quantity and magnitude.

DEFINITIONS, POSTULATES, AXIOMS.

1. 148, A part,—*i. e.*, aliquot part, or submultiple. Magnitudes compared must be of the same kind.

2. 149. A multiple. Equimultiples. Commensurable and Incommensurable magnitudes.

3. 150. Ratio, what. The *how great* the hinge on which the def. turns.

A Ratio expressed by two terms,—antecedent, consequent. Numerical Ratio. Measure of a Ratio. Inverse or Reciprocal Ratio.

4. 151. Magnitudes, when said to have a ratio to one another.
Multiplication and Division in Geometry, what they are.

5. 151. Definition of Proportion,—magnitudes having the *same ratio*.

The *Criterion* for determining the equality of Ratios.
Applicable to all *magnitudes*, incommensurables and commensurables.
Ratios when they are the same.

6. 152. Magnitudes which have the same ratio are called *proportionals*.

7. 153. When a Magnitude has a *greater* ratio,—when a *less*.

8. 153. Analogy or Proportion is the similitude of ratios.

9. 153. Proportion consists in three terms at least.

10. 153. The *Duplicate Ratio*,—when three magnitudes are proportionals.

The Duplicate Ratio expressed algebraically and arithmetically.
Such magnitudes in *continued* proportion. A mean Proportional.
Double Ratio and *Duplicate* Ratio not to be confounded.

11. 154. Triplicate Ratio, quadruplicate, &c.

A 155. *Of Compound Ratio*,—with examples.

B. 155. Progression on the lengths of chords producing musical sounds.

The 1st : 3rd = 1st ~ 2nd : 2nd ~ 3rd.
Three st. lines when in *Harmonical* Progression. Harmonical Mean.

12. 156. Terms when *homologous*, or corresponding.

TECHNICAL WORDS TO DENOTE CHANGES IN THE ORDER OF PROPORTIONALS.

1°.—*For Four Proportionals.*

156—8. Def. 13. *Permutando*, or *alternando*; Def. 14. *Invertendo*; Def. 15. *Componendo*; Def. 16. *Dividendo*; Def. 17. *Convertendo*. Conjungendo.

2°.—*For any number of Proportionals above Two.*

158—160. Def. 18. *Ex æquali* (sc. *distantia*) or *ex æquo*; Def. 19. *Ex æquali*, or *ex æquo ordinate*; Def. 20. *Ex æquali in proportione perturbatâ seu inordinatâ*, or *ex æquo pertubate*.

POSTULATES 1 and 2 p. 161.

3. 173. Three Ms being given, A,B,C, there is a 4th M., as x, to which C has the same ratio, as A to B, *i. e.*, A : B = C : x.

Axioms.—1—4 p. 161.
Algebraical Expressions, &c., p. 161.

ONE PROBLEM.—PROP. N.

N. 237. To find a common measure of two lines.

Cor. 1. The greatest com. meas. of the rem. and lesser M. is also the greatest com. measure of the two Ms.

2. Any aliquot part or subm. of a com. meas. is a com. measure.

3. By repeating the process with the rem. and the lesser M., and again with the new rem. (if there be one) and the preceding, and so on, *the greatest com.* meas. of two given commensurable Ms. may be found.

4. Any two commensurable Lines are to one another as the numbers denoting the no. of times that they respectively contain their com. measure.

239. Sch. When Ms are incommensurable.

Use. To find the greatest com. measure of two numbers.

EUCLID'S THEOREMS ARE TWENTY-FIVE,—THE SUBSIDIARY FOURTEEN.

Prop. 1, 2, 3, 4, 5, 6 ;—A, B, C, D ;—7 A ;—7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 ;—E ;—20, 21, 22, 23, 24, 25 ;—F, G, H, K, L, M ;—O, P.

1. 162. If any no. of Ms be equims. of as many, each of each; what m soever any one of them is of its part, the same shall all the first Ms be of all the other.

Cor. The sum of the equimultiples = the equimultiple of the sum.

Sch. If to a m of a M. by any number a m of the same M. by any number be added, the sum will be the same m of that M that the sum of the two numbers is of unity.

Con. 1, Thus, as $A = m\ E$, $B = n\ E$, $C = p\ E$, &c., $A + B + C = (m + n + p)\ E$.

2. Also $m\ E + n\ E + p\ E = (m + n + p)\ E$.

2 164. If the first M be the same m of the second that the third is of the fourth, and the fifth M the same m of the second that the sixth is of the fourth; then shall the first + the fifth be the same m of the second, that the third + the sixth is of the fourth.

Cor. If AB, BG, GH, &c., be mults. of C. and as many, DE, EK, KL, &c., the same *ms* of F, each of each, then AH, *i. e.*, (AB + BG + GH &c.) the same *ms* of C, that DL, *i. e.*, (DE + EK + KL) is of F.

Sch. If the 1st of three Ms contain the 2nd as often as there are units in a certain number;—and if the 2nd contain the 3rd also as often as there are units in a certain number, the 1st will contain the third as often as there are units in the product of these numbers.

3. 165. If the 1st be the same m of the 2nd, which the 3rd is of the 4th; and if of the 1st and 3rd there be taken equims; these shall be equims, the one of the 2nd, and the other of the 4th.

Cor. If A, A′ be equims of B, B′, and also of C, C′; and if B be a m of C, then B′ is the same m of C′.

Sch. If $A : B = C : D$, and m A, m C, n B, n D be taken, then $m\ A : n\,B = m\ C : n\ D$.

Cor. When $n = 1$, then $m\ A : B = m\ C : D$.

4. 167. If the 1st of four Ms has the same ratio to the 2nd which the 3rd has to the 4th; then any equal *ms* whatever of the 1st and 3rd shall have the same ratio to any equims of the 2nd and 4th.

Cor. 1. If the 1st has the same ratio to the 2nd, which the 3rd has to the 4th, then also any equims whatever of the 1st and 3rd shall have the same ratio to the 2nd and 4th; *so*, the 1st and 3rd shall have the same ratio to any equims. whatever of the 2nd and 4th.

2. If 4 Ms are propls, they will be propls by *inversion*.

3. If $A : B = C : D$, then $\frac{A}{2} : \frac{C}{2} = \frac{B}{3} : \frac{D}{3}$.

Appl. Hence in the rule for Simple Proportion in Arith.,—divide the 1st and 2nd terms by any com. measure and make use of the results.

5. 170. If one M. be the same m of another, which a M. taken from the 1st is of a M. taken from the other; the rem. is the same m of the rem., that the whole is of the whole.

Sch. If from a m of a M. by any number, a m of the same M. by a less number be taken away, the *rem.* will be the same m of that M. that the diff. of the numbers is of unity.

Cor. When $m - n = 1$, then $m\ A = n\ A =$; or $2\ A - A = A$.

6. 171. If two Ms be equims of two others, and if equims of these be taken from the first two; the rems are either = these others, or equims of them.

Sch. Pr. 1—6 chiefly useful to establish, by the method of equims. the Props. which follow.

Postulate. Three Ms, A, B, C, being given, there is a 4th M., as x, to which C has the same ratio as A to B; *i. e.*, $A : B = C : x$.

A. 173. If the 1st of four Ms has the same ratio to the 2nd which the 3rd has to the 4th; then if the 1st be > the 2nd, the 3rd is > the 4th; and if =, = ; if < <.

Use. For 25, V ; 21, VI ; 34, XI, and 15, XII. Simson added Props. A, B, C, V.

B. 174. *Invertendo.* If 4 Ms are propls. they are propls. also when taken *inversely.*

Sch. Or the Pr. may be stated, "The reciprocals of eq. ratios are eq. to one another."

C. 175. If the 1st be the same *m* of the 2nd, or the same part, *i. e.*, *subm.* of it that the 3rd is of the 4th, the 1st : 2nd = 3rd : 4th.

Sch. Four Ms in proportion by Def. 5, V. are also in proportion by Def. 20, Bk. VII.

D. 178. If the 1st be to the 2nd as the 3rd to the 4th, and if the 1st be a *m*, or pt of the 2nd, the 3rd is the same *m* or the same pt of the 4th.

7. A. 179. The ratio of two lines is the same as that of the numbers which express the number of times that any third line is contained in them respectively.

7. 181. Equal magnitudes have the same ratio to the same M., and *conversely.*

Cor. If a rat. A : C, compound of two ratios A : B, and B : C, be a rat. of equality one of them must be the inverse or reciprocal of the other ; *i. e.* A : B is the inverse or reciprocal of B : C.

8. 183. Of two uneq. Ms, the gr. has a gr. ratio to another M. than the less has; and *conversely.*

9. 185. Ms which have the same ratio to the same M. are eq. to one another; and those to which the same M. has the same ratio are eq. to one another.

Cor. A ratio compounded of two ratios, of which one is the reciprocal of the other is a ratio of equality.

10. 187. That M. which has a gr. ratio than another has unto the same M. is the gr. of the two; and that M. to which the same ratio has a gr. ratio than it has unto another M. is the less of the two.

11. 188. Ratios that are the same to the same ratio, are the same to one another.

Cor. 1. If A : B = C : D, but C : D > or < E : F, then A : B > or < E : F.

2. If A : B > or < C : D, and C : D = E : F, then A : B > or < E : F.

12. 189. If any number of Ms be propls, as one of the antecs is to its conseq., so shall all the anteceds taken together be to all the conseqs.

13. 191. If the 1st : 2nd = 3rd : 4th, but the 3rd : 4th > 5th : 6th; the 1st : 2nd > 5th : 6th.

Cor. If A : B > or < B : E, but C : D = E : F, then A : B > or < E : F.

14. 193. If the 1st : 2nd = 3rd : 4th; then if the 1st > 3rd, the 2nd > 4th; and if =, =, and if <, <.

Cor. If A : B = C : D, and if B > = or < D, then A > = or < C.

15. 194. Ms have the same ratio to one another which their equims have.

Cor. 1. Ms have the same ratio to one another which their eq. submults. have.

2. If A : B = C : D, then m A : m B = n C : n D.

3. Also $\frac{A}{2} : \frac{B}{2} = \frac{C}{3} : \frac{D}{3}$.

16. 196. *Alternando.* If 4 Ms of *the same kind* be propls, they shall also be propls when taken alternately.

197. Use 1. If to the terms of a rat. the same M. be added, the rat. will be unchanged, increased, or diminished, according as it is a rat. of equality, of less inequality, or of greater inequality.

2. If all the terms, or any two homol. terms, or the terms of either of the ratios of proportion, be multiplied or divided by the same number, the resulting Ms will remain propl.

17. 199. *Dividendo.* If Ms taken jointly, be propls, they shall also be propls when taken separately; *i. e.*, if two Ms. together have to one another the same ratio which two others have to one of these, the rem. one of the first two shall have to the other the same ratio which the rem. one of the last has to the other of these.

Cor. 1. *Convertendo.* If A : B = C : D, then A : A ~ B = C : C ~ D.

2. The greatest of four propls + the least > the other two.

3. If A : B : C, then A + C > 2 B; and $\frac{A + C}{2} > B$.

Use. The arith. mean between 2 Ms is > the geom. mean.

18. 203. *Componendo.* If Ms taken separately be propls, they shall also be propls when taken jointly, by *composition*; *i. e.* if A : B = C : D, then A + B : B = C + D : D.

Use. *Addendo.* If A : B = C : D, then A : A + B = C : C + D.

19. 207. If a whole M. be to a whole, as a M. taken from the first is to a M. taken from the other; the rem. shall be to the rem. as the whole to the whole.

Cor. 1. Also the rem : rem. = M. from the 1st : M. from the other.

2. If A : B : C : D &c. then A ∼ B : B ∼ C : C ∼ D &c. will form a geom. progression, the successive terms of which have the same ratio with the successive terms of the former; and Cor. 3, *conversely.*

E. 210. *Convertendo.* If four Ms be propls, they are also propls by conversion; *i. e.* the 1st *is* to its excess above the 2nd *as* the 3rd *to* its excess above the 4th.

Use 1. If any no. of Ms be in contd. proportion, the diff. between the 1st and 2nd terms *is* to the 1st, *as* the diff. between the 1st and last *is* to the sum of all the terms, except the last.

Use 2. In a series of contd. propls., the differences of the successive terms are also in contd. proportion.

3. In an infinitely decreasing series of Ms in contd. proportion, the 1st term is a mean propl. between its excess above the 2nd, and the sum of the series.

20. 212. If there be three Ms, and other three, which, taken two and two, have the same ratio; then if the 1st be >, =, or < the 3rd, the 4th shall be >, =, or < the 6th.

Sch. Also, if A : B = C : D; and B : E = D : F; then C >, = or < F, as A >, =, or < E.

21. 214. If there be three Ms and other three which have the same ratio taken two and two, but in a cross order, i. e. *in proportione perturbatâ*, in disturbed proportion; then if the 1st M be >, =, or < the 3rd, the 4th shall be >, =, or < the 6th.

Sch. A variation of the Prop. If A : B = C : D; and B : E = F : C; then F shall be >, =, or < D, *as* A is >, = or < E.

22. 216. *Ex æquali*, or *ex æquo*, by equality. If there be any no. of Ms and as many others, which taken two and two in order, have the same R.; the first shall have to the last of the first Ms the same R. which the first has to the last of the others.

Cor. Rs. compd. of any no. of eq. Rs. in the same order are equal to one another.

Sch. Varied thus;—If the 1st M be to to the 2nd, *as* the 3rd to the 4th; and if the second M be to the 5th, *as* the 4th to the 6th; then the 1st shall he to the 5th, as the 3rd to the 6th.

Use. Proportionals remain propl, *miscendo*, by mixing; *i. e.*, by using their sum and difference.

23. 219. *Ex æquo perturbato.* If there be any no. of Ms, and as many others, which taken two and two in a cross order, have the same R.; the first shall have to the last of the first Ms the same R. which the first has to the last of the others.

Sch. Different ways of announcing Prop. 23. 1. Rs compd of any no. of eq. Rs, but in reverse order, are eq. to one another; 2. Also for two or more series of Ms.

24. 222. If the first has to the second the same R. which the third has to the fourth, and the fifth to the second the same R. which the sixth has to the fourth; the first and fifth together shall have to the second, the same R. which the third and sixth together have to the fourth.

Cor. 1. The excess of the 1st above the 5th, shall he to the 2nd, *as* the excess of the 3rd above the 6th is to the 4th.

2. In any no. of Proportions,—if the 2nd is the same throughout and also the 4th; then the sum of all the first terms is to the sum 2nd term, *as* the sum of all the third terms *is* to the sum. 4th term.

Sch. Or, "If two series of Propls. have the same conseqs., the sum of the first antecs: sum. eonseq. = sum of the second antecs: their sum. eonseq."

Cor. 1. If two proportions have the same consequents, as A : B = C : D, and E : B = F : D, then A − E : B = C − F : D.

2. If four Ms form a proportion, A : B = C : D, *miscendo*, A + B : A − B = C + D : C − D.

3. In two series A, B, C, D, E, F, &c., and G, H, I, K, L M, &c.,—if the ratios, A : B, or G : H; B : C, or, H : I, &c., be the same in the two series, then any two combinations of the first series shall be to one another as any two similar combinations of the second series.

25. 226. If four Ms of the same kind are propls, the greatest + the least are > the other two together.

Cor. If three Ms be propls, the sum of the extremes > twice the mean, and half the sum > the mean.

The *Arith.* mean is > the *geom.* mean.

F. 227. Ratios compounded of the same ratios are the same to one another.

SCH. Ratios compounded in any order whatever are the same with one another.

G. 229. If several Rs. be the same to several Rs., each to each; the R. compounded of Rs. which are the same to the *first* Ratios, shall be the same to the R. compd. of Rs. which are the same to the *other* Rs., each to each.

H. 230. If a R. compd. of several Rs. be the same to a R. compd. of several other Rs, and if one of the *first* Rs, or the R. compd. of several of them, be the same to one of the *last* Rs., or to the R. compd. of several of them; then the rem. R. of the *first*, or if there be more than one, the R. compd. of the rem. Rs., shall be the same to the rem. R. of the *last*, or, if there be more than one, to the R. compd. of these rem. Rs.

K. 231. If there be any no. of Rs., and any no. of other Rs., such, that the R. compd. of Rs. which are the same to the *first* Rs., each to each, is the same to the R. compd. of Rs. which are the same, each to each, to the *last* Rs.; and if one of the *first* Rs., or the R. compd. of Rs. which are the same to several of the *first* Rs., each to each, be the same to one of the *last* Rs, or to the R. compd. of Rs which are the same, each to each, to several of the *last* Rs.; then the rem. R. of the first, or, if there be more than one, the R. compd. of Rs. which are the same, each to each, to the rem. Rs. of the *first*, shall be the same to the rem. R. of the *last*, or if there be more than one to the R. compd. of Rs., which are the same each to each, to these rem. Rs.

SCH. Propositions F, G, H, & K are frequently made use of;—SIMSON'S remark.

L. 234. A compound R. is eq. to the product of its component simple Rs.

M. 235. If there be two fixed Ms., A and B, which are the limits of two others, P and Q, (that is, to which P and Q, by increasing together, or by diminishing together, may be made to approach more nearly than by any the same given diff.), and if P be to Q always in the same given R. of C to D; then A shall be to B in the same R.

USE. This Prop. is extensively applied, and is one of the first steps to the higher Geometry.

O. 240. The diagonal and side of a Square are incommensurable.

P. 241. If four st. Ls., A, B, C, D, be propls, (whether commensurable or incommensurable,) the rect. under the extremes, A · D, will be = the rect. under the means, B . C.

USE The Theory of Proportion in Arith. and Alg. is founded on this truth.

243. *Examples of Reasoning by Proportion* 1°. —11°.

244 Remarks on Book V. by BILLINGSLEY.

BOOK VI.

THE THEORY OF PROPORTION APPLIED FOR COMPARING THE SIDES AND AREAS OF PLANE RECTILINEAL FIGURES.

247. Billingsley's Summary.
248. Identity of Form, not of size, the basis of the Comparison. The extension given to other truths.
249. What the sixth Book establishes stated in general terms.

DEFINITIONS.

1. 249. Similar rectil. figures,—conditions to be fulfilled.
2. 250. Reciprocal figures,—in what way the sides propl.
3. 250. A st. L. cut in extreme and mean ratio.
   A L. divided medially, or in *medial ratio.*
4. 251. The altitude of a figure.

SUBSIDIARY DEF. A — F.

A. 251. A L. *harmonically* divided.
B. 251. A fig. given in species.
C. 251. A fig. given in magnitude.
D—F. 252,3. A $\square$ applied to a st. L.

TEN PROBLEMS.

Prob. 9, 10, 11. 12, 13 ;—18 ;—25 ; 28, 29, 30.

9. 285. From a given st. L. to cut off any measure, or sub-multiple.

Sch. Pr. 10, I. a particular case of this Prob.

Use 1. To divide a given L. into any no. of eq. parts.

2. To divide a △ into any no. of eq. pts., by lines from a given . in one of the sides.

3. Given the *n*th part of a L to find the (n + 1)th part.

10. 288. To divide a given st. L. similarly to a given divided st. L. ; or into parts that shall have the rat. to one another which the pts. of the divided given st. line have.

Use 1. To divide a given st. L. internally or externally, in a given ratio, as of M : N.

2. To find a harmonical mean between two given st. lines.

3. To find a third harmonical progressional to two given st. lines.

4. To construct a △ of which one side, the ∠ opp. and the R. of the other sides are given.

5. Through a given . to draw a L., which, on being produced, would pass through the . s of intersection of two given lines without their being produced to meet. The Centrolinead.

11. 292. To find a third proportional to two given st. lines.

Sch. Construction requiring the compasses alone.

Use. 1. To continue a series of Rs. in progression, AB : BC, being the given antec. and conseq.

2. *Theorems* allied to the last Prob. ; 1°, If a series of Ms. be in cont. proportion, their successive differences are also in cont. proportion, and in the same R.

2°. If a series in cont. proportion be an increasing one, there is no limit to the increase of its terms.

3°. If a series in cont. proportion be a decreasing one, there is no limit to the diminution of its terms.

4°. If a series of Ms decreasing in cont. proportion, be continued, or imagined to be continued, to an infinite no. of terms, the sum of all the terms, or the Sum of the Series, will be a finite and determinate M.

3. *Problems* from the foregoing principles. 1°. Gregory's Prob., from the two first terms in a series to obtain the sum of the terms. 2°. Of three quantities, the 1st and 2nd terms, and the Sum of the Series, if any two be given, the rem. one may be found.

12. 298. To find a fourth proportional to three given st. lines.

Sch. Solution of the Problem by the Compasses alone.

Use 1. The *Sector*,—"A large number of pairs of compasses packed up one."

2. *Problems* by aid of the compasses and sector; 1°. To find a fourth propl. to three given lines.
2°. To find a chord to a rad., from a Sector the rad. of the chord of which equals 4 inches.
3°. To divide a given L. into two pts, x and y, which shall be to each other as two lines or numbers.
3. The *various* Lines on the Sector, and on GUNTER's Scale.
4. Examples of the use of the Sector.
5. Other applications, and caution.

13. 302. To find a mean proportional between two given st. lines.

SCH. Other constructions;—formula for numerical calculations.
USE. 1. Any rectangular parallelogram may be reduced to an equivalent square.
2. Of three lines in cont. proportion, any two being given, to find the unknown.
3. Given one of three terms and the sum of the other two, to find the two unknown terms.
4. Of three lines in cont. proportion, if one be given and the diff. of the other two, those other two may be found.
5. To find two st. lines to contain a rect. = a given rect. and to have a given ratio one to the other.
6. To find any number of means represented by a power of 2, *minus* 1.

ADDENDA. I. To obtain two mean propls. between two given st. lines;
1°. PLATO's method of a perp. moveable along a side of a sq.
2°. PHILO's method of a graduated ruler revolving round the vertex of a rt. $\angle$.
3°. The method of DES CARTES, with a collection of rulers.
308. II. The Trisection of a rectilineal $\angle$. 1°. The Trammel of NICOMEDES, a T square with a moveable ruler.
2°. By this instrument to trisect a given $\angle$.
3°. COOLEY's Tentative Method.

18. 324. Upon a given st. L. to describe a rectilineal figure, similar and similarly situated to a given rectilineal figure.

SCH. 1. A more simple way for making a rectil. fig. sim. to a given rectil. fig.
2. To construct a rectil. fig. sim. to a given rectil. fig., and having its perimeter = a given st. L.
3. Figures of the same species with different areas on the same rt. L.
4. Sim. $\triangle$s and polygons are to one another as the squares of their homol. sides.
USE. Nearly all the *practical methods of taking a Plan or Map*, founded on this Prop.

25. 344. *Of extensive use.* To describe a rectil. figure which shall be sim. to one and eq. to another given rectil. fig.; *i. e.* eq. in area.

Sch. The chief point in this Prob. is to find a mean propl.

Use. While we keep the *same* area, we can *change* the form.

28. 350. To a given st. L. to apply a parallelogram = a given rectil. fig., and *deficient* by a parallelogram sim. to a given ▱, but the given rectil. fig. to which the ▱ to be applied is to be eq., must not be greater than the ▱ applied to half of the given L, having its defect sim. to the defect of that which is to be applied; *i. e.* to the given ▱.

Sch. 1. To divide a given st. L., so that the rect. contained by the segs. may be eq. to a given space, as the sq. on C; but that given space must not be gr. than the sq. of half the given L.

2. To inscribe in a given △ a ▱ eq. to a given fig. not gr. than the maximum inscribed ▱, and having an ∠ in com. with the △.

Use 1. To a given st. L. to apply a rect., deficient by a Sq., which rect. shall be eq. to a given square, that on line C; but the given sq. on C must not be gr. than the sq. on the half of the given L.

2. To a given st. L. to apply a rect., which shall be eq. to a given rect., and be deficient by a square; but the given rect. must not be gr. than the sq. upon half the given line. Other enunciations.

29. 354. To a given st. L. to apply a ▱ eq. to a given rectil. fig., and exceeding by a ▱ sim. to another given ▱.

Sch. The algebraical signification of Pr. 27, 28, & 29, bk. VI.

Use 1°. To exscribe to a given △, a ▱ eq. to a given rectil. fig., and having an ∠ = to one of the ∠s of the given △.

2°. To a given st. L. to apply a rect. which shall be eq. to a given square, and exceeding by a square. *Or*, To produce a given st. L. so that the rect. under its external segs may be eq. to a given space, as C^2.

3°. To a given L. to apply a rect. that shall be eq. to a given rect., and exceeding by a square, as on BN. Variations of the Prob.

30. 358. To cut a given st. L. in extreme and mean ratio; *i. e.* so that the whole L. shall be to the gr. seg. as the gr. seg. to the less.

Sch. 1. A L. thus divided, divided *medially*; *medial ratio.*

2. This Prob. belongs to a class which relate to *incommensurable* Ms.

Theor. Let there be two Ms of the same kind, P & Q; and let P be contained in Q a certain no. of times which is to P as P is to Q; then the Ms, P and Q shall be incommensurable.

3. Or this 30th Prob. may be considered as a particular case of the

Prob. To divide a line so that the rect. under the whole L. and one part shall bear a given ratio, as $m : n$, to the square of the other part.

4. Two lines in ext. and mean R. are cut similarly; and *conversely.*

5. Numerically to approximate to the ratio of incommensurables.

USE. This Prop. employed in EUCLID'S 13th Bk. on the Platonic Solids.
PROB. 1°. On a given L. to construct a rt. ∠d △, the sides of which shall be in cont. or geom. progression.
PROB. 2°. The altitude of a rt. ∠d △ being given, of which the sides are in a given R., to find the sides.
PROB. 3°. Given, *a*, *b*, two sides of a △, and the diam. *d*, of the circumscribing ⊙, to find the other side, *c*.

EUCLID'S THEOREMS ARE TWENTY-THREE ; THE SUBSIDIARY TWELVE.

Prop. 1, 2, 3 ;—A, 4, 5, 6, 7, 8 ;—14, 15, 16, 17 ;—19, 20, 21, 22 ;—Lemma. 23, 24, 25, 26, 27 ;—31, 32, 33 ;—B, C. D ;—E, F, G, H, K, L, M.

1. 252. Triangles and parallelograms of the same altitude are to one another as their bases.

COR. 1. Triangles & □s with eq. altitudes are one to another as their bases ; & *conversely*.
2. Any two △s or □s, are to one another in the R. compounded of the Rs of their altitudes and of their bases.
3. The rect. under two lines is a mean propl. between their squares.
4. If two △s or two □s be as their bases, they have eq. altitudes ; and if they have eq. altitudes they have eq. bases.

SCH. 1. This Proportion might have been directly inferred.
2. Propositions distinguished by the name *Variant*.
3. One quantity does not vary *as* another, because it varies *with* it.

USE. From a trapezium to cut off a third part.

2. 257. If a st. L. be drawn ‖ to one of the sides of a △, it shall cut the other sides, or these produced, proportionally ; and *conversely*.

COR. If the sides of an ∠ be cut by any no. of parallels, any two pts. of the one will have the same R. to one another, as the corresponding parts of the other.

SCH. 1. The Enunciation not sufficiently explicit.
2. The theory of Transversal lines is connected with this Prop.
3. The ways in which a st. L. may be cut in a given Ratio.
4. Parallel lines cut diverging lines proportionally.

USE 1. For the measurement of the height of an inaccessible object which casts an accessible shadow.
2. To divide a given L. into parts propl. to those of another L.

3. 260. If the ∠ of a △ be divided into two eq. ∠s by a st. L. which also cuts the base ; the segs. of the base shall have the same R. which the other sides of the △ have to one another ; and *conversely*.

A. 262. If the outward ∠ of a △, made by producing one of its sides, be divided into two eq. ∠s, by a st. L., which also cuts the base produced; the segs. between the dividing line and the extremities of the base, have the same R. which the other sides of the △ have to one another; and *conversely*.

Cor. 1. The segs. of the base produced made by the external bisector are propl. to the segs. of the base made by the internal bisector.

2. The two lines bisecting the vert. ∠ and its adj. ext. ∠, cut the base produced *harmonically*.

3. Also the two sides of a △, and the lines which bisect the vert. & ext. vert ∠s are *harmonicals*.

4. If BG, BC & BD, in the same st. L. be in harmonical progression, DC, DG & DB, will also be in harmonical progression.

Sch. Case in which there is no point of external bisection &c.

Use 1. The harmonic mean obtained from the harmonic proportion.

2. Applied to Optics and Acoustics.

4. 265. The sides about the eq. ∠s of eq. ang. △s are proportionals; and those which are opp. to the eq. ∠s are homol. sides.

Cor. 1. If diverging Ls, cut par. Ls, the par. Ls will be cut proportionally.

2. If two par. st. Ls be cut by any number of diverging Ls, the parallels will be similarly cut in the s of section.

3. In a △, a L from the vertex, bisecting the base also bisects the parallel to the base.

4. A par. to the base of a △ cuts off a sim. △.

Sch. 1. Homologous sides,—homologous terms.

2. Only in △s, if their ∠s are eq. their sides about the eq. ∠s are propl.

5. 268. *Important.* Conversely,—If the sides of two △s, about each of their ∠s, be propls, the △s shall be eq. ang.; and the eq. ∠s shall be those which are opp. to the homol. sides.

269. Sch. Pr 47, 47, I, and 4, 5, VI contain the principles of every kind of rectilineal Measurement.

Use 1. The Theory of Representative Value,—the Practice of Triangulation, and various other methods of Practical Geom. depend on **Pr. 4** & 5, bk. VI.

2. For Practical Purposes Eight Problems and Formulas deduced;

1°. Given the observed length of the shadows of two perp. objects, and the alt. of one, to find *the alt.* of the other.

2°. By means of a mirror placed horizontally, the ∠ of incidence being = the ∠ of reflection, to ascertain the *height* of a perp. object.
3°. To find the *height* of a perp. object, by means of two uneq. rods or poles, placed perpendicularly on a horizontal line.
4°. By means of a pole placed perp., to ascertain the *alt.* of a perp. object.
5°. By a Geom. Square to measure the *height* of an object.
6°. By the same means, with an index to point to the extremities of two objects, of which one is in the vertex of a rt. ∠; to measure the *distance*.
7° To find, by aid of the cross staff, or theodolite, the *distance* between two objects.
8°. By means of a L. of which the length is known, to find the *length* of its parallel, one end of which only can be approached.

The Proportional Compasses, Pentagraph, and Eidograph.

6. 275. If two △s have one ∠ of the one = one ∠ of the other, and the sides about the eq. ∠s propl., the △s. shall be eq. ang., and shall have those ∠s. eq. which are opp. to the homol. sides.

Cor. 1. The sides also about each pair of eq. ∠s shall be propl.
2. If through any . s of a rt. L. parallels be drawn, proportional to distances from any . A in that st. L., then their extremities will be on the rt. L. passing through A.

Use 1. Conditions of similarity between one rectil. figure and another.
2. Sim. rectil. figures may be divided into the same no. of sim. △s.

7. 278. If two △s, have one ∠ of the one = one ∠ of the other, and the sides about two other ∠s propls.; then, if each of the rem. ∠s be either < or ∢ a rt. ∠, or if one of them be a rt. ∠, the △s shall be eq. ang. and shall have those ∠s. eq. about which the sides are proportionals.

Sch. Angles when of the *same affection*.

Use. The *criteria* of the similarity of two △s; 1°. Equality of the three ∠s. 4, VI. 2°. Identity of the Rs of the respective sides, 5, VI. 3°. Equality of two ∠s, one in each △, and the identity of the Rs of the containing sides, 6, VI; 4°. Identity of the Rs of two sides in each △; the equality of an ∠ in each opp. one pair of homol. sides; and each of the rem. ∠s opp. the other pair of homol. sides < a rt. ∠, or one of them a rt. ∠.

8. 281. *Important.*—In a rt. ∠d △, if a perp. be drawn from the rt. ∠ to the base; the △s on each side of it are similar to the whole △ and to one another.

Cor. 1. The perp. from the vert. of the rt. $\angle$ to the opp. side, is a *mean* propl. between the segs. of this side ; and also each of the sides, including the rt. $\angle$ is a mean propl. between the opp. side and the seg. of it adj. to that side.

2. The segs. of the hyp. made by the perp. are to one another as the squares on the sides of the rt. $\angle$.

3. The squares on the sides about the rt. $\angle$ and on the hyp. are to each other as the segments of the hyp. made by the perp. and the hyp. itself.

4. If the base of a $\triangle$, the two sides, and the perp. be four proportionals, the $\triangle$ must be rt. $\angle$d.

Sch. Prop. 8 and its deductions are particular cases of the general principle ; "If from the vert of a $\triangle$, two Ls be drawn to the base, making the $\angle$s at the base or their supplements each eq. to the vert. $\angle$; then the $\triangle$s formed by those lines, and by the seg. which each cuts off, shall be sim. to the whole $\triangle$ and to one another.

Use 1. A very clear and brief dem. of Pr. 47, Bk. I.

2. By this Prop. and a Square *inaccessible* distances may be measured.

3. In a $\odot$ any chord is a mean propl. between the diam. and the seg. of the diam. which is drawn from one extr. of the ch., and cut off by a perp. let fall from the other extr. of the chord.

14. 310. Eq. $\square$s, which have one $\angle$ of the one = one $\angle$ of the other, have their sides about eq. $\angle$s reciprocally propl. ; and *conversely*.

Sch. 1. When two $\square$s have eq. areas and their sides reciprocally propl, they will be eq. ang.

2. The Prop. applicable also to $\triangle$s.

15. 311. Eq. $\triangle$s which have one $\angle$ of the one = one $\angle$ of the other, have their sides about the eq. $\angle$s reciprocally propl. ; and *conversely*.

Use. To construct an isosc. $\triangle$ = a given scalene $\triangle$, and with the same vert. $\angle$.

16. 314. *Very important.* If four st. Ls. be propl. the rect. contained by the extremes = the rect. contained by the means ; and *conversely*.

Cor. Rectangles which have their sides about the rt. $\angle$s. reciprocally propl. are equal ; and *conversely*.

Sch. 1. Definition of Rs *reciprocally* proportional.

2. Prop. 16 may be deduced from this Definition ; and 1°. A R. compounded of recip. Rs. is a R. of equality. 2°. If a R of equality be compd. of two Rs. they must be reciprocals ; 3°. The sides of eq. rects. are four proportionals.

3. The equality of the two Rs. converted into the equality of two rectangles.

4. The doctrine of *Limits* needed for incommensurable Ms.

5. THEOR. 1°. If there be two fixed Ms, A & B, which are the limits of two others, P and Q, and if P be to Q always in the same given R. of C to D; then A shall be to B in the same R.

THEOR. 2°. If four st. Ls. be propls, (whether commensurable or otherwise) the rect. under the extremes will be = rect. under the means.

USE. The Theory of Limits applied; 1°. Rectangles having the same alt. are to each other as their bases; 2°. Rectangles are to each other as the product of their bases by their altitudes; 3°. Among other theorems the rect. contained by any two sides of a △ is eq. to the rect. contained by its alt., or perp. to the third side from the opp. ∠ and by the diam. of the circumscribing ⊙.

17. 321. COR. to Pr. 16. If three st. Ls. be propls, the rect. contd. by the extremes = the Sq. of the mean; and *conversely*.

USE. 1. The Demonstration of Proportion in Arithmetic, or *the Rule of* Three.

2. A clear dem. of 47, I.; In a rt. ∠d △, $Hyp.^2 = side^2 + side^2$.

3. Deductions stated in other words.

4. Instances in which Props. 16 and 17 shorten the demonstrations.

1°. If from a . there be drawn two st. Ls, one a tang. to a ⊙, the other a secant, then the tang. will be a mean propl. between the whole secant and its ext. seg.; 2°. If from a . there be drawn several st. Ls., cutting the ⊙, then the whole secants will be one to another *inversely* as their ext. segment.

19, 328. *Very important.* Sim. triangles are to one another in the duplicate R. of their homol. sides; *i. e.*, as the squares of their like sides.

COR. If three st. Ls. are propls., *as* the 1st *to* the 3rd, *so* is any △ on the 1st *to* a sim. and sim described △ on the 2nd.

SCH. The duplicate R. of two st. Ls. is the same with the R. of their squares.

USE. *A general method of reasoning* thus supplied; for, the areas of sim. △s are to another as the squares for the corresponding sides.

In sim. △s, when any side of one △ is double that of the other, the area of one △ = 4 times that of the other.

20. 331. Sim. Polygons may be divided into the same no. of sim △s, having the same R. to one another that the polygons have; and the polygons have to one another the duplicate R. of that which their homol. sides have.

COR. 1. Sim figures of four, or of any no. of sides, are to one another in the duplicate R. of their homol. sides.

COR. 2. If three Ls. be propls., then, the first shall be to the 3rd *as* any pol. on the 1st *to* the sim. and similarly described pol. on the 2nd.

Cor. 3. The R. of any two squares to one another is the same with the duplicate R. of the sides.

Cor. 4. In sim. figures their perimeters are to one another *as* the R. of the homol. sides.

Cor. 5. Perimeters of sim. figures are as their homol. diagonals.

Cor. 6. Circles are to each other *as* the squares of their respective diams. radii and ⊙ces.

Cor. 7. If on the three sides of a rt ∠d △, sim. figures be described, as semicircles, the fig. on the hyp. = the sum of the sim. figures on the two sides.

Cor. 8. Hence, Lunes on the sides of a rt. ∠d ∠ are equal in area to the rt. ∠d △.

Use 1. A rt. lined figure may be increased or diminished in any R.

2. The Proportion of one sim. figure to another is found by obtaining the 3rd propl. to any two corresponding sides.

3. The increase or diminution of ⊙s effected in the same way.

4. Of the areas and corresponding sides of sim. figures, any three being given the fourth may be readily found.

21. 337. Rectil. figures which are sim. to the same rectil. fig. are also sim. to one another.

Sch. This Prop. similar to 30, I. and 11, VI.

22. 338. If four st. Ls. be propls., the sim. rectil. figures similarly described upon them shall also be propls.; and *conversely*.

Cor. If four st. Ls be propls, their squares shall be propls; and *conversely*.

Sch. If two Rs be eq., their duplicates and subduplicates, or powers and roots shall be equal.

Use. This Pr. is often employed in Arith. and Alg.; If four quantities or numbers are in proportion their like powers or roots are also propls.

340. Lemma. If rectil. figures be eq. and sim. their homol. sides are equal.

23. 340. Eq. ang. parallelograms have to one another the R. compounded of the R. of their sides.

Cor. 1. If the terms of two analogies are Ls, the rectangles under their corresponding terms are propl.

2. Hence, Rectangles whose bases are propl, and also their alts, are propl.

Use. To describe a rhombus eq. to a given rectil. figure, and having an ∠ = a given ∠.

24. 343. Parallelograms about the diam. of any ▱ are sim. to the whole and to one another.

Use. This Prop. is available in Perspective.

26. 346. If two sim. parallelograms have a com. $\angle$, and be similarly situated, they are about the same diam.

Sch. This Prop. the converse of 24. VI.

27. 347. Of all $\square$s applied to the same st. line, and *deficient* by $\square$s sim. and similarly situated to that which is described upon the half of the L.; that which is applied to the half, and is sim. to the defect, is the greatest.

Or, Of all the rectangles contd. by the segs. of a given st. L., the greatest is the square described on half the L.

Use. In a given $\triangle$ to inscribe the greatest parallelogram possible, having an $\angle$ in common with the $\triangle$.

31. 363. *Important.* In rt. $\angle$d $\triangle$s, the rectil. fig. described on the side opp. to the rt. $\angle$, is = the sim. and similarly described figures on the sides containing the right $\angle$.

Sch. 1. A very comprehensive Prop., but one still more general is,—If any $\square$s be described on the two sides of any $\triangle$, and if the sides of the $\square$s be produced to meet, and if that . of intersection and the vertex of the $\triangle$ be joined, and the L. produced; then these $\square$s are eq. in area to a $\square$ described on the base, and having two of its sides parallel to the L. produced through the . of intersection and the vertex, and limited by the sides of the two $\square$s.

2. A circle with the hyp. of a rt. $\angle$d $\triangle$ for diam. is equal in area to the two $\odot$s, having the other two sides for diameters.

32. 366. If two $\triangle$s which have two sides of the one propl. to two sides of the other, be joined at one $\angle$, so as to have their homol. sides par. to one another; the rem. sides shall be in a st. L.

Sch. The position of the given sides, not homol., must form an $\angle$ at the . of junction.

33. 367. In eq. $\odot$s, $\angle$s, whether at the centres, or $\odot$ces, have the same R. which the $\odot$ces, on which they stand, have to one another; so also have the sectors.

Cor. 1. The sectors are to each other as their $\angle$s.

2. Sim. sectors of the same or eq. $\odot$s are equal.

3. An $\angle$ at the centre of a $\odot$ is to four rt $\angle$s *as* the arc on which it stands to the $\odot$ce of the $\odot$.

4. In different $\odot$s the arcs of eq. $\angle$s at the centres or $\odot$ces are similar.

5. Hence, sim. segments are contained by sim. arcs, and *vice versâ*.

Sch. Some Editors of Euclid substitute the three following Propositions applicable *to the same* or *equal circles*.

PROP. 33. *a.* ∠s, whether at the cent. or at the ⊙ces, have the same R. as the arcs on which they stand.

COR. The arcs are also propl. when incommensurable.

PROP. 33. *b.* The sectors on eq. arcs are equal.

PROP. 33. *c.* Sectors have the same R. as the arcs on which they stand.

COR. They must also be propl. when incommensurable.

The ∠ at the cen. of a ⊙ is measured by the arc on which it stands.

USE 1. If arcs of different ⊙s have a com. chord, the Ls diverging from one of its extremities will cut the arcs proportionally.

2. The arcs of uneq. ⊙s are in a R, compd. of their central ∠s and their radii.

3. Central ∠s are in a R. compd. of the direct R. of their arcs, and the inverse R. of their radii.

SUBSIDIARY PROPOSITIONS.

B. 373. If an ∠ of a Δ be bisected by a st. L., which likewise cuts the base; the rect. contained by the sides of the Δ is eq. to the rect. contained by the segs. of the base + the sq. of the st. L. which bisects the base.

COR. The rect. of the sides + the sq. of the L. which bisects the ext. ∠ = the rect. of the whole L. produced and the ext. seg.

SCH. Pr. B and its corollary may be combined.

C. 374. If from any ∠ of a Δ a st. L. be drawn perp. to the base; the rect. contained by the sides of the Δ = the rect. contained by the perp. and the diam. of the ⊙ described about the Δ.

COR. If two Δs be inscribed in the same or in eq. ⊙s, the rect. under the two sides of the one shall be to the rect. under the two sides of the other, *as* the perp. from the vertex to the base of the one *is* to the perp. from the vertex to the base of the other.

D. 375. The rect. contained by the diags. of a quadril. fig. inscribed in a ⊙ is eq. to both the rectangles contained by its opp. sides.

SCH. PTOLEMY'S Theorem. Imitativeness of EUCLID'S editors.

E. 377. The diagonals of a quadril. inscribed in a ⊙ are to one another *as* the sums of the rectangles under the sides adj. to the extremities of those diagonals.

USE. To solve the Prob.;—Given four st. Ls, any three of which are together > the fourth, to construct a quadril., of which the sides shall be = those four given st. Ls., in a given order, each to each, and of which also its angular .s lie in the ⊙ce of a ⊙.

F. 378. If a seg. of a ⊙ be bisected, and from the extremities of the base of the seg., and from the . of bisection, st Ls be drawn to any . in the ⊙ce; the sum of the two Ls from the extremities of the base, will have to the L. from the . of bisection the same R. which the base of the seg. has to the base of half the seg.

G. 379. If two . s be taken in the diam. of a ⊙, or of the diam. produced, such that the rect. contained by the segs. intercepted between them and the cen. be eq. to the sq. of the semidiam.; and if from these . s two st. Ls be inflected to any . whatever in the ⊙ce of the ⊙, then the R. of the Ls inflected will be the same with the R. of the segs. intercepted between the two first-mentioned . s and the ⊙ce of the ⊙.

COR. 1. In the fig. ∠ FBE is bisected by AB.
2. Also the ext. vert. ∠ EBG is bisected by BC.

H. 381. If from one extr. of the diam. of a ⊙ a chord be drawn, and a perp. cut both the diam. and the chord, either intr. or extr., the rect. under the diam. and its seg. reckoned from the extremity is eq. to the rect. under the chord and its corresponding seg.

K. 381. If the ∠s at the base of a △ be bisected by two Ls that meet, and the ext. ∠s at the base, formed by producing the two sides, be sim. bisected; then the two . s of concourse, and the vertex shall be in one st. L. which shall bis. the vert. ∠.

L. 383. In a △, as with last Pr., K, the segs. of each side produced that are intercepted between the vertex and the extn. perps. are each = the semiperimeter of the △; the segs. of these sides next the rect. are = the excess of the semiperimeter above the base; and the seg. of each of these sides next the base is respectively = the excess of the semiperimeter above the other side.

M. 384. The area of a Δ is a mean propl. between two rectangles, the sides of which are eq. to the semiperimeter and its excess above the base, and the sides of the other eq. to the excesses of the semiperimeter above the other two sides.

Cor. Let S denote semiperimeter, and a, b, c the sides opp. $\angle$s A, B, C; the Area of $\Delta = \sqrt{s\,(s-c)\,(s-a)\,(s-b.)}$

Use. The Sol. of the Prob. Given the three sides of a Δ to find the Area.

Remarks.

1. 385. The 6th Book treats chiefly of *similar* rectil. and curvilineal figures.
2. 386. The Book contains 33 Prs. by Euclid, of which 10 are Probs. & 23 Theors; there are 13 Subsidiary Theorems A—M.
3. The most important Problems and Theorems.
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7. 388. The sixth Book the Head and Crown of Plane Geometry.

Corrigenda.

The errors to be found in Mathematical works, even in those of high repute prove the difficulty of avoiding them, especially when signs and abbreviations are freely used. Besides, the printing was undertaken by those who were not accustomed to mathematical work; and there was also an unavoidable change of workmen; hence without much real blame to any one, the mistakes are more than they would have been; few, however, extending to any error in the reasoning.

Along with the remark in the Preface on the same subject,—will the Reader accept the quaint apology of an old Editor of Euclid? "Some mistakes will still remain, which when thou chancest to meet with correct with the pen, so shalt thou do right to the Anthor, and supply the defects of the Revisor, and in both doe good to thyself."—Rudd's *Euclides Elements*.

N.B.—In Column 1, the page is given, P; in Col. 2, the line, L, the arabic numeral on the *left* hand denoting the line from the *top* of the page,—that on the *right*, the line from the *bottom*; in Col. 3, the correction, corr.; and in Col, 4, the printed error, err.

P.	L.	corr.	*for* err.
8	20	Pst. 2.	3
11	8	16, I.	10, I.
17	2	FC;	FG;
39	19	AC–CB;	=CB;
44	1	∴	∵
82	4	A	AD
85	1	FD	ED
99	12	DEB	DEF
104	17	*rad.*	*diam.*
108	6	in L;	in I;
110	2	□;	□;
115	19	∠CAD	∠CAG
	22	DB	CB
120	6	∴	∵
126	4	arc E &c. F&c.	
128	6	30, III.	28, III.
172	15	KC	KH
176	20	∴ B	∴
178	7	contains =	
		GH	GA
179	11	C=F	C=E
182	11	or <	or >
187	14	>	<
188	1	: C,	: A.
	12	F,	E,
192	7	C : D,	B : E,
194	3	D >	B >
197	20	$\frac{2}{3}$	$\frac{2}{4}$
198	2	$> n$ D	$< n$ D
201	10	KX,	KH.
202	14	C:D	C:C

P.	L.	corr.	*for* err.
205	3	or <	or >
206	8	DF;	DE;
207	13	CF	CE
208	6	A′,	A,
209	12	*ay*	*by*
210	8	DF(*bis*)	EF,
211	7	$\frac{a+b}{b}$	$\frac{a+b}{d}$
214	8	& 4th,	& 5th,
219	5	22	32
221	8	$\frac{a'}{c'}$	$\frac{a'}{b'}$
222	14	A 12	H 12
230	10	is compd. and	
235	7	continued con-	tained
236	11	≠B	≠B′
	11	B′;	B;
	10	B	B′
	8	A′	A
238	5	B	R
	4	of B	of R
250	8	DE,	EF,
255	1	are as	are
261	14	∵	∴
264	21	GD,	CD,
272	3,4	CB	EB
	6	C	E
289	2	HC,	AC,
302	5,11	BC;	AC;
	12	ADC,	ABC,

P.	L.	corr.	*for* err.
311	2	triangles	angles
320	6	Cor. 3.	Cor. 33.
325	14	∵	∴
329	11	*omit* EF or	
330	3	6480	3240
331	1	6480	3240
	6	.969	.973
335	12	AB	CB
	15	AC	AB
336	11	$\frac{2\times2}{1}$	$\frac{2\times2}{2}$
348	12	∴	∵
353	18	∵	∴
355	4	EB	EL
	8	36, I.	31, I.
356	9	*To exscribe*	
357	21	10, I.	31, I.
358	8	46, I.	45, I.
362	6	EC	EB
	15	CE	AE
364	12	∵	∴
365	9	33, I.	31, I.
375	7	5, IV.	4, VI.
393	11	equil.	equal
402	13	4 times	4 lines
406	1	5 and	and
413	1	93	97
421	8	=A;	=;
423	10	C : D	B : E
428	2	up into	up
432	8	47, 48,	47, 47.

ERRATA,

Involving an Error of Reasoning.

P.	L.	
46	1 & 3	remove the bracket.
208	16	B∼C, C∼D, *for* ∼ B : C CD
255	1	*add* ∴ $A^2 : A.B = A.B : B^2$.
259	14	BG : AG *for* AG : BG
270	17 & 20	GB *for* CB
276	7	Ac, Ad for *bc, cd*

P.	L.	
332	3	*read* in the duplicate ratio of AB : FG.
354	6	The full argument not given.
359, 430	16, 7	(*read*) with a remainder.

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